Some progresses on Lipschitz equivalence of self-similar sets

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Part I. Lipschitz equivalence of dust-like self-similar sets

Definition

Let E, F be compact sets in $\mathbb{R}^d$. We say that E and F are Lipschitz equivalent, and denote it by $E \sim F$, if there exists a bijection $g : E \longrightarrow F$ which is bi-Lipschitz, i.e. there exists a constant $C > 0$ such that for all $x, y \in E$,

$$C^{-1}|x - y| \leq |g(x) - g(y)| \leq C|x - y|.$$

Question

Under what conditions, two self-similar sets are Lipschitz equivalent?

- Necessary condition: same Hausdorff dimension.
- The condition is not sufficient even for dust-like case. (The generating IFS satisfies the strong separation condition.)

Example

Let E be the Cantor middle-third set. Let $s = \log 2 / \log 3$ and $3 \cdot r^s = 1$. Let F be the dust-like self-similar set generated as the following figure. Then $E \not\sim F$.

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- Let E, F be dust-like self-similar sets generated by the IFS $\{\Phi_j\}_{j=1}^n$, $\{\Psi_j\}_{j=1}^m$ on $\mathbb{R}^d$, respectively.
- ρ_j (resp. τ_j) is the contraction ratio of Φ_j (resp. Ψ_j).
- $\mathbb{Q}(a_1, \dots, a_m)$: subfield of $\mathbb{R}$ generated by $\mathbb{Q}$ and $a_1, \dots, a_m$.
- $\mathrm{sgp}(a_1, \dots, a_m)$: subsemigroup of $(\mathbb{R}^+, \times)$ generated by $a_1, \dots, a_m$.

Theorem (Falconer-Marsh, 1992)

Assume that $E \sim F$. Let $s = \dim_H E = \dim_H F$. Then

(1) $\mathbb{Q}(\rho_1^s, \dots, \rho_m^s) = \mathbb{Q}(\tau_1^s, \dots, \tau_n^s)$;

(2) $\exists p, q \in \mathbb{Z}^+$, s.t. $\mathrm{sgp}(\rho_1^p, \dots, \rho_m^p) \subset \mathrm{sgp}(\tau_1, \dots, \tau_n)$ and $\mathrm{sgp}(\tau_1^q, \dots, \tau_n^q) \subset \mathrm{sgp}(\rho_1, \dots, \rho_m)$.

- Using (2), we can show that $E \not\sim F$ in the above example.

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Question

What's the necessary and sufficient condition? How about for two branches case?

- WLOG, we may assume that $\rho_1 \leq \rho_2$, $\tau_1 \leq \tau_2$ and $\rho_1 \leq \tau_1$.
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Some Notations

- K: self-similar set determined by the IFS $\{\mathbb{R}^d; f_1, \ldots, f_m\}$.
- ρ_j: contraction ratio of f_j, $\forall j$.
- $(\rho_1, \ldots, \rho_m)$ is called a contraction vector (c.v.) of K.
- For any c.v. $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_m)$ with $\sum \rho_j^d < 1$, we define $\mathcal{D}(\overrightarrow{\rho})$ to be all dust-like self-similar sets with c.v. $\overrightarrow{\rho}$ in $\mathbb{R}^d$.
- Throughout the talk, the dimension d will be implicit.
- Define $\dim_H \mathcal{D}(\overrightarrow{\rho}) = \dim_H E$, for some (then for all) $E \in \mathcal{D}(\overrightarrow{\rho})$.
- $E \sim F$ for any $E, F \in \mathcal{D}(\overrightarrow{\rho})$.
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Step 1 to solve the Question on two branches case

Assume that $\mathcal{D}(\rho_1, \rho_2) \sim \mathcal{D}(\tau_1, \tau_2)$. By FM' theorem, one of followings must happen:

(1). $\log \rho_1 / \log \rho_2 \notin \mathbb{Q}$.

(2). $\exists \lambda \in (0, 1)$, and $p_1, q_1, p_2, q_2 \in \mathbb{Z}^+$ such that

$$\rho_1 = \lambda^{p_1}, \quad \rho_2 = \lambda^{p_2}, \quad \tau_1 = \lambda^{q_1}, \quad \tau_2 = \lambda^{q_2}.$$

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Step 2 to solve the Question

Let's study case (2) first.

- From $s = \dim_H \mathcal{D}(\rho_1, \rho_2) = \dim_H \mathcal{D}(\tau_1, \tau_2)$, we have

$$(\lambda^{p_1})^s + (\lambda^{p_2})^s = (\lambda^{q_1})^s + (\lambda^{q_2}) = 1.$$

- Denote $x = \lambda^s$, then

$$x^{p_1} + x^{p_2} = x^{q_1} + x^{q_2} = 1.$$

- That is,

$$x^{p_1} + x^{p_2} - 1 = 0 \quad \text{and} \quad x^{q_1} + x^{q_2} - 1 = 0$$

have same root in $(0, 1)$, where $p_1 \geq p_2, q_1 \geq q_2, p_1 \geq q_1$.
- Using Ljunggren's result on the irreducibility of trinomials $x^n \pm x^m \pm 1$, we proved that the above happens iff
  - $(p_1, p_2) = (q_1, q_2)$ or
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Step 3 to solve the Question

Let's study Case (1) now.

- Given a c.v. $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_m)$. Define

$$\langle \overrightarrow{\rho} \rangle := \{\rho_1^{\alpha_1} \cdots \rho_m^{\alpha_m} : \ \alpha_1, \ldots, \alpha_m \in \mathbb{Z}\}.$$

- $\langle \overrightarrow{\rho} \rangle$ is an abelian group and has a nonempty basis.
- Define $\mathrm{rank}\langle \overrightarrow{\rho} \rangle$ to be the cardinality of the basis.
- Clearly, $1 \leq \mathrm{rank}\langle \overrightarrow{\rho} \rangle \leq \mathrm{m}$.
- If $\mathrm{rank}\langle \overrightarrow{\rho} \rangle = \mathrm{m}$, we say $\overrightarrow{\rho}$ has full rank.
- By FM' theorem, $\mathrm{rank}\langle \overrightarrow{\rho} \rangle = \mathrm{rank}\langle \overrightarrow{\tau} \rangle$ if $\mathcal{D}(\overrightarrow{\rho}) \sim \mathcal{D}(\overrightarrow{\tau})$.

Theorem (Rao-R-Wang, 2012)

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$\mathcal{D}(\rho_1, \rho_2) \sim \mathcal{D}(\tau_1, \tau_2)$ iff $(\rho_1, \rho_2) = (\tau_1, \tau_2)$ or there exists $\lambda \in (0, 1)$, s.t.

$$(\rho_1, \rho_2, \tau_1, \tau_2) = (\lambda^5, \lambda, \lambda^3, \lambda^2).$$

Related and further Works

- In case that $\mathrm{rank}\langle\overrightarrow{\rho}\rangle = \mathrm{rank}\langle\overrightarrow{\tau}\rangle = 1$,
  - Xi and Xiong have a very nice result.
  - Rao and his collaborators also have some progresses.
- In case that $1 < \mathrm{rank}\langle\overrightarrow{\rho}\rangle = \mathrm{rank}\langle\overrightarrow{\tau}\rangle < \mathrm{m}$, everything remains open!

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Part II. Lipschitz equivalence of self-similar sets with touching structures

A problem posed by David and Semmes, 1997

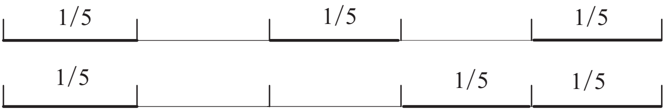

Figure: Initial construction of M and M'

- David and Semmes conjectured that $M \not\sim M'$.
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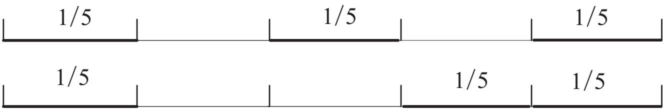

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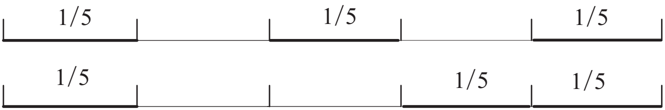

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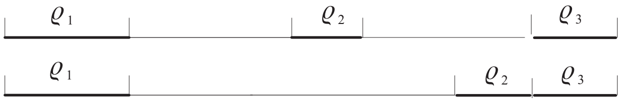

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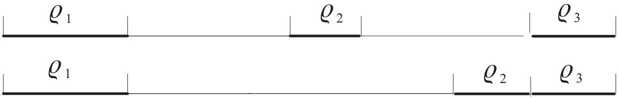

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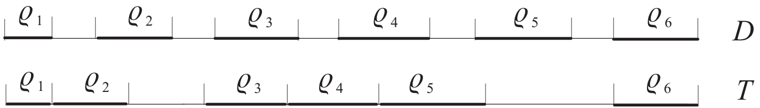

Figure: Initial construction of D and T, where $n = 6$

- $\overrightarrow{\rho} = (\rho_1, \ldots, \rho_n)$ is a c.v. in $\mathbb{R}$ with $n \geq 3$.
- $D \in \mathcal{D}(\overrightarrow{\rho})$.
- T: attractor of IFS $\{\Psi_j(x) = \rho_j x + t_j\}_{j=1}^n$ satisfying
  - The subintervals $\Psi_1([0,1]), \ldots, \Psi_n([0,1])$ are spaced from left to right without overlapping.
  - Left endpoint of $\Psi_1[0,1]$ is 0; right endpoint of $\Psi_n[0,1]$ is 1.
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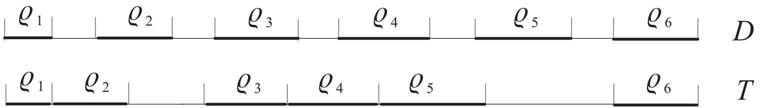

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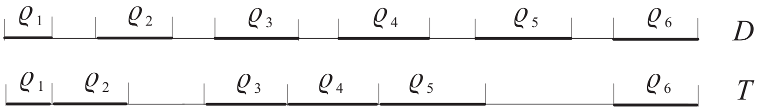

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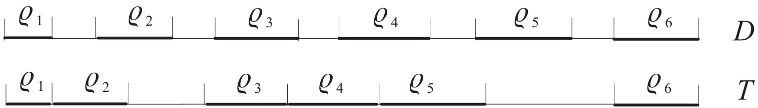

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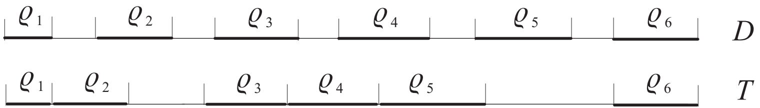

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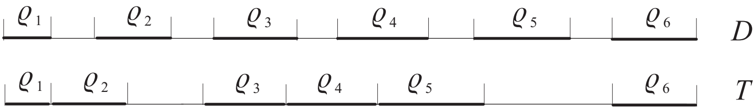

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Assume that $D \sim T$. Then $\log \rho_1 / \log \rho_n \in \mathbb{Q}$.

- A letter $j \in \{1, \ldots, n\}$ is a (left) touching letter if $\Psi_j([0,1])$ and $\Psi_{j+1}([0,1])$ are touching, i.e. $\Psi_j(1) = \Psi_{j+1}(0)$.
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Let $n = 4$, $\rho_1 = \rho_4$, and $\Sigma_T = \{2\}$. Assume that $D \sim T$. Let $s = \dim_H D = \dim_H T$ and $\mu_j = \rho_j^s$ for $1 \le j \le 4$. Then μ_2 and μ_3 must be algebraically dependent, namely there exists a nonzero rational polynomial $P(x, y)$ such that $P(\mu_2, \mu_3) = 0$.

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Corollary

Let $M_{\vec{\rho}}$ and $M'_{\vec{\rho}}$ be sets defined in generalized {1,3,5}-{1,4,5} problem. Then $M_{\vec{\rho}} \sim M'_{\vec{\rho}}$ iff $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$.

Note: If $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$, the unique touching letter $\{2\}$ is substitutable.

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