Computing Singularity Dimension

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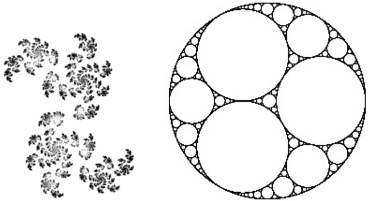
Overview

In this talk I want to do three things:

1. Recall some familiar examples (which everybody knows);
2. Describe some classic results of Falconer and Hueter-Lalley (which everyone who knows them likes);
3. Present a result on estimating Hausdorff Dimension (which at least I like).

General question

Assume that we given some compact set $X \subset \mathbb{R}^2$ in the plane.



Basic Question

What is the Hausdorff Dimension $\dim_H(X)$ of the set X?

Even for the most regular of fractals it can be impossible to give an explicit closed form for the Hausdorff Dimension.

A More Practical Question

How do we estimate its Hausdorff Dimension $\dim_H(X)$?
How well can we approximate $\dim_H(X)$?

Self-similar sets

We call maps $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, \cdots, k)$ of the plane (contracting) *similarities* if

$$T_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_i \cos\theta_i & a_i \sin\theta_i \\ -a_i \sin\theta_i & a_i \cos\theta_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where $0 \le \theta_i < 2\pi$ and $0 < a_i < 1$ and $b_1, b_2 \in \mathbb{R}$, i.e.,

1. rotate by θ_i,
2. scale down by a_i, and
3. translate by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Definition

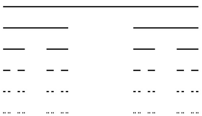
We call a set $X \subset \mathbb{R}^2$ *self-similar* if there are similarities $T_1, \cdots, T_k : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T_1(X) \cup \cdots \cup T_k(X) = X$$

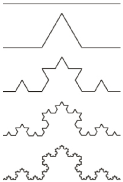
Self-similar sets are particularly nice to deal with (especially if they also satisfy some extra conditions, e.g., open set condition, strong separation condition, etc).

Self-similar sets

Some examples of self-similar sets have simple expressions for their dimension.
(i) *Middle third Cantor set*. Let $T_1(x, y) = (\frac{x}{3}, \frac{y}{3})$ and $T_2(x, y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3})$.



(ii) *von Koch curve*. Let $T_1(x, y) = (\frac{x}{3}, \frac{y}{3})$, $T_2(x, y) = (\frac{x}{6} - \frac{\sqrt{3}y}{6}, \frac{\sqrt{3}x}{6} + \frac{y}{6}) + (\frac{1}{3}, 0)$, $T_3(x, y) = (\frac{x}{6} + \frac{\sqrt{3}y}{6}, -\frac{\sqrt{3}x}{6} + \frac{y}{6}) + (\frac{1}{2}, +\frac{\sqrt{3}}{6})$ and $T_4(x, y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3})$.



Self-affine sets

We say $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1, \cdots, k)$ are affine if

$$T_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

(which we assume to be contractions). i.e.,

1. apply the linear transformation $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and
2. translate by $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Definition

We call a set X self-affine if there are affine maps if $T_1, \cdots, T_k : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$T_1(X) \cup \cdots \cup T_k(X)$$

After self-similar sets, one would hope self-affine sets are the next easiest to deal with.

Example 1: Barnsley Fern

Consider the four affine maps:

$$T_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.00 & 0.00 \\ 0.00 & 0.16 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.85 & 0.04 \\ -0.04 & 0.85 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix}$$

$$T_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.20 & -0.26 \\ 0.23 & 0.22 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 1.60 \end{pmatrix}$$

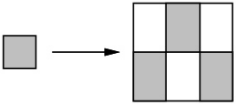
$$T_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.15 & 0.28 \\ 0.26 & 0.24 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.44 \end{pmatrix}$$

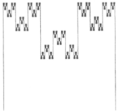
The limit set is a fern:

Example 2: Bedford-McMullen sets

This is an standard construction of a self-affine set.
Consider for simplicity a s particular special case, called the Hironaka curve, which is the limit set of

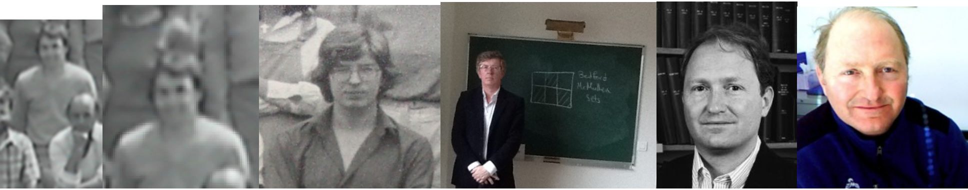
$$T_1(x,y) = \left(\frac{x}{3}, \frac{y}{2}\right)$$
$$T_2(x,y) = \left(\frac{x}{3} + \frac{1}{3}, \frac{y}{2} + \frac{1}{2}\right)$$
$$T_3(x,y) = \left(\frac{x}{3} + \frac{2}{3}, \frac{y}{2}\right)$$





In the limit one gets the “Hironaka curve” .
These results were contained in the first published paper of Curt McMullen in 1984.

Aside: Bedford, McMullen and me



Tim Bedford was a PhD student of Caroline Series at Warwick, and an exact contemporary of mine. One day, in Warwick in 1984 he told me about some result in his thesis on Hausdorff Dimension.
Later that year I met Curt McMullen, then a PhD student of Dennis Sullivan, in the tea room at IHES (France) and he told me about some results he recently obtained on Hausdorff Dimension. They sounded vaguely familiar. I wrote to Bedford who didn't know about McMullen's proof of the same results (who immediately panicked since he hadn't submitted his PhD yet). Bedford wrote to McMullen (who never panics, although he hadn't submitted his PhD either). McMullen went on to win a Fields medal and has a chair at Harvard, and Bedford is now an Associate Deputy Principal at the University of Strathclyde.

Explicit and Implicit expressions

Sometimes it is possible to give **explicit** expressions for the Hausdorff Dimension when the limit set X is particularly simple.

- Middle third Cantor set ($\dim_H X = \frac{\log 2}{\log 3}$)
- von Koch Curve ($\dim_H X = \frac{\log 4}{\log 3}$)
- Hironaka curve ($\dim_H X = \log_2(1 + 2^{\log_3 2})$)

Sometimes it is possible to give **implicit** expressions for the Hausdorff dimension.

- For some self-similar sets (open set condition, etc.)
- some self-conformal sets, (e.g., limit sets of Julia sets, via pressure and the dynamical viewpoint)
- some special affine sets (e.g., Bedford-McMullen sets)

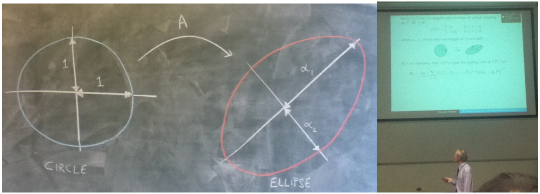
Question

How can we (implicitly) describe the Hausdorff dimension of typical limit sets for self-affine maps?

Matrices and their singular values

Let $A_1, \cdots, A_k \in GL(2, \mathbb{R})$ be 2×2 matrices.

- Given $n \geq 1$ and $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$ we denote the product of matrices $A_{\underline{i}} = A_{i_1} A_{i_2} \cdots A_{i_n}$.
- We denote their singular values $\alpha_1(A_{\underline{i}}) \geq \alpha_2(A_{\underline{i}})$.



These are the major and minor axes of the ellipse which is the image of the unit circle under $A_{\underline{i}}$. Equivalently, these are the eigenvalues of the 2×2-matrix $\sqrt{A_{\underline{i}}^* A_{\underline{i}}}$.

(As explained in the talk of Kenneth Falconer.)

Definition

We denote

$$\phi^s(A_{\underline{i}}) = \begin{cases} \alpha_1(A_{\underline{i}})^s & \text{if } 0 < s \leq 1 \\ \alpha_1(A_{\underline{i}})\alpha_1(A_{\underline{i}})^{1-s} & \text{if } 1 \leq s < 2. \end{cases}$$

Singularity dimension of limit sets

Let $b_1, \cdots, b_k \in \mathbb{R}^2$ we vectors and can consider affine maps $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_i(x) = A_i x + b_i$ $(i = 1, \cdots, k)$.

Definition

The *limit set* $\Lambda \subset \mathbb{R}^2$ is the unique smallest closed set such that $\Lambda = T_1\Lambda \cup \cdots \cup T_k\Lambda$.

Finally, we have the following definition.

Definition

We define the *singularity dimension* of Λ by

$$\dim_S(\Lambda) = \inf \left\{ s > 0 : \sum_{n=1}^{\infty} \sum_{|\underline{i}|=n} \phi^s(A_{\underline{i}}) < +\infty \right\}.$$

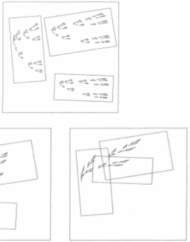
where for $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$ we write $|\underline{i}| = n$.

Falconer's theorem

We now recall the elegant theorem of Falconer.

Theorem (Falconer, Solomyak)

Assume that $\|A_1\|, \cdots, \|A_k\| < \frac{1}{2}$. *For a.e.* $(b_1, \cdots, b_k) \in \mathbb{R}^{2k}$, *we have* $\dim_H(\Lambda) = \dim_S(\Lambda)$.



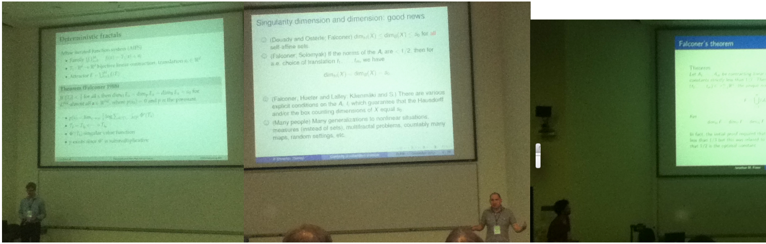


Figure: Three limit sets corresponding to the same affine contractions A_1, A_2, A_3, but different translations b_1, b_2, b_3.

As explained in the talks of Esa Järvenpää, and Pablo Shmerkin and Jonathan Fraser.

Kenneth Falconer and Friends

Figure: Karoly Simon, M.P. and Kenneth Falconer

Hueter-Lalley theorem: Four assumptions

Question

How can we remove the "a.e." hypothesis?

We want to assume the following assumptions:

Additional assumptions

1. $\|A_i\| < 1$ for $i = 1, \cdots, k$;
2. $\alpha_1(A_i)^2 < \alpha_2(A_i)$ for $i = 1, \cdots, k$;
3. Let $Q_2 = \{(x, y) : x \leq 0, y \geq 0\}$ then $A_1^{-1}Q_2, \cdots, A_k^{-1}Q_2$ are pairwise disjoint subsets of $\text{int}(Q_2)$;and
4. there is a bounded open set V such that $\overline{T_i V}$ are disjoint, $i = 1, \cdots, k$.

(1)-(3) depend on the A_i; (4) also depends on the b_i.

Theorem (Hueter-Lalley)

Under the above hypotheses we have that

$$0 < \dim_H(\Lambda) = \dim_S(\Lambda) < 1.$$

- Thus at the cost of the additional hypotheses, we have avoided the "a.e." part.
- The hypotheses also automatically force that $\dim_S(\Lambda) < 1$.

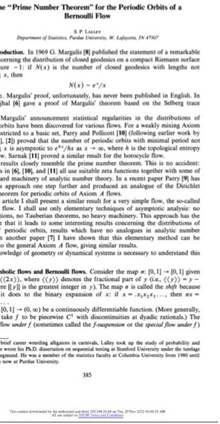
Hueter and Lalley

Figure: Steven Lalley and Irene Hueter

I actually know Lalley from his earlier work on closed orbits for suspension flows.

Aside: Lalley's eariler life

S. P. Lalley, Amer. Math. Monthly 95 (1988), no. 5, 385-398:



he "Prime Number Theorem" for the Periodic Orbits of a Bernoulli Flow

S. P. Lalley
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oduction. In 1969 G. Margulis [8] published the statement of a remarkable
ncerning the distribution of closed geodesics on a compact Riemann surface
ure −1: if $N(x)$ is the number of closed geodesics with lengths not
; x, then

$$N(x) \sim e^x/x$$

. Margulis' proof, unfortunately, has never been published in English. In
jhal [6] gave a proof of Margulis' theorem based on the Selberg trace

Margulis' announcement statistical regularities in the distributions of
orbits have been discovered for various flows. For a weakly mixing Axiom
estricted to a basic set, Parry and Pollicott [10] (following earlier work by
], [2]) proved that the number of periodic orbits with minimal period not
; x is asymptotic to e^{hx}/hx as $x \to \infty$, where h is the topological entropy
w. Sarnak [11] proved a similar result for the horocycle flow.
results closely resemble the prime number theorem. This is no accident:
s in [6], [10], and [11] all use suitable zeta functions together with some of
ard machinery of analytic number theory. In a recent paper Parry [9] has
s approach one step further and produced an analogue of the Dirichlet
neorem for periodic orbits of Axiom A flows.
article I shall present a similar result for a very simple flow, the so-called
flow. I shall use only elementary techniques of asymptotic analysis: no
tions, no Tauberian theorems, no heavy machinery. This approach has the
e that it leads to some interesting results concerning the distributions of
periodic orbits, results which have no analogues in analytic number
n another paper [7] I have shown that this elementary method can be
o the general Axiom A flow, giving similar results.
owledge of geometry or dynamical systems is necessary to understand this

nbolic flows and Bernoulli flows. Consider the map σ: $[0,1] \to [0,1]$ given
$\langle\langle 2x \rangle\rangle$, where $\langle\langle y \rangle\rangle$ denotes the fractional part of y (i.e., $\langle\langle y \rangle\rangle = y -$
ere $[[y]]$ is the greatest integer in y). The map σ is called the *shift* because
it does to the binary expansion of x: if $x = .x_1x_2x_3\ldots$, then $\sigma x =$
...
$[0,1] \to (0,\infty)$ be a continuously differentiable function. (More generally,
take f to be piecewise C^1 with discontinuities at dyadic rationals.) The
flow under f (sometimes called the *f-suspension* or the *special flow under f*)

brief career wrestling alligators in carnivals, Lalley took up the study of probability and
e wrote his Ph.D. dissertation on sequential testing at Stanford University under the tutelage
egmund. He was a member of the statistics faculty at Columbia University from 1980 until
now at Purdue University.

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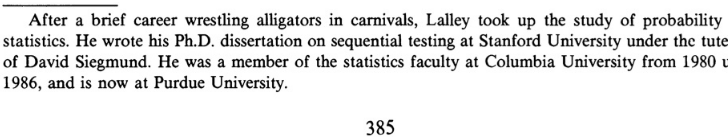

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1986, and is now at Purdue University.

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Presumably he no longer wrestles alligators in carnivals.

Example of Heuter and Lalley

It is nice to know some examples do exist satisfying the assumptions:

Heuter and Lalley proposed the matrices

$$A_1 = \begin{pmatrix} \frac{1}{30} & \frac{1}{120} \\ \frac{1}{30} & \frac{1}{60} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{30} & \frac{1}{40} \\ \frac{1}{30} & \frac{1}{30} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{40} & \frac{1}{30} \\ \frac{1}{60} & \frac{1}{30} \end{pmatrix}.$$

It is easy to check that for these A_1, A_2, A_3 for (1)-(3) hold, and it is then easy to find b_1, b_2, b_3 such that (4) holds.

Question

How do we actually estimate the singularity dimension ?

Working from the definition itself isn't the most efficient way.

Statement of Main Theorem

Our main result is the following (which was suggested by Karoly Simon).

Theorem (Main Theorem)

Let us assume (1)-(4) above. Then there exists $0 < \theta < 1$ such that we can define a sequence δ_N using the k^n values $\{\alpha_1(A_{\underline{i}}) : |\underline{i}| = N\}$ so that

$$|dim_S(\Lambda) - \delta_N| = O\left(\theta^{N^2}\right) \text{ for } N \geq 1.$$

In particular, in the theorem speed of convergence of the nth approximation is super exponential, whereas the number of values needed to compute it only grows exponentially.

Remark

If one wanted to approximate the dimension by working from the definition we could try to solve for t_N, $N \geq 1$, such that

$$\sum_{|\underline{i}|=N} \phi^{t_N}(A_{\underline{i}}) = 1.$$

This would "only" lead to exponentially fast approximations

$$|\mathrm{dim}_S(\Lambda) - t_N| = O\left(\theta^N\right) \text{ for } N \geq 1.$$

Example 1

Recall that Heuter and Lalley proposed the matrices

$$A_1 = \begin{pmatrix} \frac{1}{30} & \frac{1}{120} \\ \frac{1}{30} & \frac{1}{60} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{30} & \frac{1}{40} \\ \frac{1}{30} & \frac{1}{30} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{40} & \frac{1}{30} \\ \frac{1}{60} & \frac{1}{30} \end{pmatrix}.$$

N	δ_N	t_N
1	0.410717582765210	0.373123313880933
2	0.375211732460593	0.375566771742160
3	0.375799107164494	0.375775898884967
4	0.375797703892749	0.375795619644123
5	0.375797704495199	0.375797504758157
6	0.375797704495199	0.375797685359066
7	0.375797704495199	0.375797702683667
8	0.375797704495199	0.375797704340403
9	0.375797704495199	0.375797704507750
10	0.375797704495199	0.375797704514025

In particular, we see that for $N = 5$ the theorem gives a solution $\delta = 0.375797704495199\cdots$ which is accurate to 15 decimal places. However, even when $N = 10$ the direct method is only accurate to 9 decimal places.

Example 2

Consider the matrices

$$A_1 = \frac{1}{2^6}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, A_2 = \frac{1}{2^6}\begin{pmatrix} 5 & 3 \\ 5 & 6 \end{pmatrix} \text{ and } A_3 = \frac{1}{2^6}\begin{pmatrix} 4 & 5 \\ 2 & 9 \end{pmatrix}.$$

N	δ_N	t_N
1	0.609325221387553	0.514374159566069
2	0.502335263611167	0.508602279690240
3	0.507406976235507	0.507597431583781
4	0.507371544351918	0.507413527612153
5	0.507371616545424	0.507379412950468
6	0.507371616478486	0.507373067887602
7	0.507371616478486	0.507371886819237
8	0.507371616478486	0.507371666879226
9	0.507371616478486	0.507371625895939
10	0.507371616478486	0.507371618256548

In particular, we see that for $N = 6$ the determinant method gives a solution $\delta = 0.507371616478486\cdots$ which is accurate to 15 decimal places. However, even when $N = 10$ the Matrix method is only accurate to 8 decimal places.

The hypotheses are rather strong

The hypotheses are rather strong. Moreover, those examples which do exist typically have singularities α_1, α_2 which are quite small.

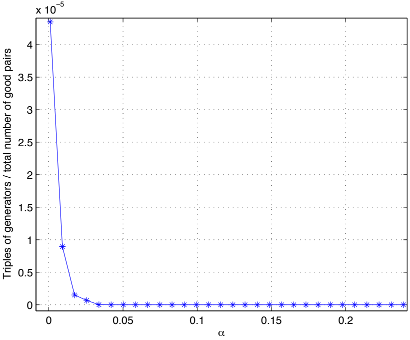


Figure: For each $\alpha > 0$ we consider the number of triples (A_1, A_2, A_3) of $360,000$ systematically chosen matrices with $\alpha < \alpha_1, \alpha_2 < 1$ satisfying the hypotheses

As α increases the number of triples satisfying the hypotheses decreases rapidly.

Computational algorithm: Step 1

It remains to explain how the δ_n are defined. Consider matrices A_i, $i = 1, \ldots, k$ satisfying the hypotheses (1)-(3)

Step 1. For each $n \geq 1$ we can consider one of the k^n strings $\underline{i} = (i_0, \cdots, i_{n-1})$ and associate the product matrix

$$A_{\underline{i}} = A_{i_0} A_{i_1} \cdots A_{i_{n-1}} = \begin{pmatrix} a_{\underline{i}} & b_{\underline{i}} \\ c_{\underline{i}} & d_{\underline{i}} \end{pmatrix}, \text{say,}$$

and the corresponding linear fractional maps $\overline{A}_{\underline{i}} : [0,1] \to [0,1]$ given by

$$\overline{A}_{\underline{i}}(x) = \frac{(a_{\underline{i}} - b_{\underline{i}})x + b_{\underline{i}}}{(a_{\underline{i}} + c_{\underline{i}} - b_{\underline{i}} - d_{\underline{i}})x + (b_{\underline{i}} + d_{\underline{i}})}.$$

We can then associate to each string $\underline{i} = (i_0, \cdots, i_{n-1})$:

1. the (unique) fixed point $\overline{A}_{\underline{i}}(x_{\underline{i}}) = x_{\underline{i}}$;
2. the derivative $D\overline{A}_{\underline{i}}(x_{\underline{i}})$ of the map at the fixed point;add
3. for each $t > 0$ the weight

$$\Phi_n(\underline{i}, t) = \left(\frac{\det(A_{\underline{i}})}{D\overline{A}_{\underline{i}}(x_{\underline{i}})} \right)^{t/2} \frac{1}{1 - D\overline{A}_{\underline{i}}(x_{\underline{i}})}$$

Computational algorithm: Step 2

Step 2. Fix $N \geq 1$. We can introduce a formal expression in z:

$$D_N(z,t) := \exp\left(-\sum_{n=1}^{N} \frac{z^n}{n} \sum_{|\underline{i}|=n} \Phi_n(\underline{i},t)\right).$$

Expanding the exponential as $\exp(y) = 1 + y + y^2/2 + \cdots + y^N/N! + O(y^{N+1})$ (first year calculus) we can rewrite this as

$$D_N(z,t) = 1 + \sum_{n=1}^{N} a_n(t) z^n + O(z^{N+1}).$$

Step 3. Setting $z = 1$ we can define

$$\eta_N(t) := D_N(1,t) = 1 + \sum_{k=1}^{N} a_k(t).$$

Let $\delta_N > 0$ be the largest zero for $\eta_N(t)$ (i.e., $\eta_N(\delta_N) = 0$) then

$$\delta_N = \dim_H(\Lambda) + O\left(\theta^{N^2}\right)$$

Idea of the proof

Let us denote

$$\eta_\infty(t) := 1 + \sum_{n=1}^{\infty} a_n(t) = \underbrace{\sum_{n=1}^{N} a_n(t)}_{\eta_N(t)} + \sum_{n=N+1}^{\infty} a_n(t).$$

It suffices to show that:

- If $\delta_\infty > 0$ is the largest zero for $\eta_\infty(t)$ then $\delta_\infty = \dim_H(\Lambda)$ *(Easy)*
- If $\delta_N > 0$ is the smallest solution to $\eta_N(\delta_N) = 0$ then $\delta_N = \dim_H(\Lambda) + O(\theta^{N^2})$.

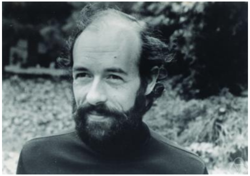
To achieve this:

- If we know that $\eta_\infty(t) = \det(I - L_t)$ for some suitable trace class operator then there exists $0 < \theta < 1$ with

  $$\sum_{n=N+1}^{\infty} a_n(t) = O\left(\theta^{N^2}\right),$$

  by a result of *A. Grothendieck, "Produits tensoriels topologiques et espaces nuclaires" Mem. Amer. Math. Soc. (1955), no. 16.*
- But the the appropriate trace class "Ruelle-Perron-Frobenius transfer" operator appears in the work of *D. Ruelle, "Zeta-Functions for Expanding Maps and Anosov Flows" Invent. math, 34, 231-242 (1976).*

Grothendeick and Ruelle


Grothendieck made major contributions to the modern theory of Algebraic Geometry but his earlier work was in Functional Analysis.

Ruelle is a theoretical physicist who has made major contributions to Dynamical Systems.

Ruelle and Grothendieck were both permanent professors together at IHES (Bures-sur-Yvette) in the 1960s.

Final silde

Thank you for your time.

Figure: Mathematics Department, Warwick University