# Infinite iterated function systems with overlaps

Sze-Man Ngai

**Georgia Southern University** and **Hunan Normal University**

*International Conference on Advances in Fractals and Related Topics*

*The Chinese University of Hong Kong*

*December 14, 2012*

Joint with Jixi Tong, Hunan Normal University

# IFSs and limit set:

- $\emptyset \neq X \subset \mathbb{R}^d$ compact
- $I$ finite or countably infinite index set
- $\{S_i\}_{i \in I}$ an *iterated function system (IFS)* if $S_i : X \to X$ are injective contractions that satisfy the *uniform contractivity condition*: $\exists\, 0 < \rho < 1$ such that

$$|S_i(x) - S_i(y)| \leq \rho |x - y| \quad \forall i \in I \text{ and } x, y \in X.$$

- *Limit set:*

$$K := \bigcup_{\mathbf{i} \in I^\infty} \bigcap_{n=1}^{\infty} S_{\mathbf{i}|_n}(X) \subseteq \bigcap_{n=1}^{\infty} \bigcup_{\mathbf{i} \in I^n} S_{\mathbf{i}}(X). \quad (K \text{ is Souslin})$$

- c.f. *attractor* or *fixed point:* $F = \overline{\bigcup_{i\in I} S_i(F)}$.
- $K$ satisfies

$$K = \bigcup_{i\in I} S_i(K),$$

but $K$ is not the unique set satisfying this equality, unless $K$ is compact.

Problem: Compute $\dim_{\mathrm{H}}(K)$.

# Motivations for studying IIFSs

Fernau (1994): IIFSs have strictly more powerful descriptive power than FIFSs:

- In a separable metric space, every closed set is a fixed point of an IIFS and,
- there is a closed and bounded subset of a complete metric space that is a fixed point of an IIFS but not of any FIFS.

# Conformal IIFS

**Definition**

IFS of injective $C^1$ conformal contractions: *if each $S_i$ can be extended to a $C^1$ injective conformal contraction on some bounded open connected neighborhood $V$ of $X$ and*

$$0 < \inf_{x \in V} \|S_i'(x)\| \leq \sup_{x \in V} \|S_i'(x)\| < 1 \quad \textit{for all} \quad i \in I.$$

*Define*

$$r_{\mathbf{i}} := \inf_{x \in V} \|S_{\mathbf{i}}'(x)\|, \quad R_{\mathbf{i}} := \sup_{x \in V} \|S_{\mathbf{i}}'(x)\|, \quad \forall\, \mathbf{i} \in I^* := \bigcup_{n=0}^{\infty} I^n.$$

# Bounded distortion property

## Definition

*Bounded distortion property (BDP):* $\exists\, c_1 > 0$ *such*

$$\frac{\|S_{\mathbf{i}}'(x)\|}{\|S_{\mathbf{i}}'(y)\|} \le c_1 \quad \forall\, \mathbf{i} \in I^* \text{ and } x, y \in V.$$

*In particular,*

$$r_{\mathbf{i}} \le R_{\mathbf{i}} \le c_1 r_{\mathbf{i}} \quad \forall\, \mathbf{i} \in I^*.$$

A sufficient condition for BDP: $\exists$ constants $C \ge 1$ and $\alpha > 0$ s.t.

$$\Big| \|S_i'(y)\| - \|S_i'(x)\| \Big| \le C \|(S_i')^{-1}\|^{-1} |y - x|^{\alpha}, \quad \forall i \in I,\ x, y \in V.$$

# Open set condition

*Open set condition (OSC):* $\exists$ bounded open $\emptyset \neq U \subset X$ such that

$$S_i(U) \subseteq U \;\; \forall\, i \quad \text{and} \quad S_i(U) \cap S_j(U) = \emptyset \;\; \forall\, i \neq j.$$

*Cone condition (CC) for $E \subset \mathbb{R}^d$:* $\exists \beta, h > 0$ s.t. $\forall x \in \partial E$, $\exists$ open cone $C(x, u_x, \beta, h) \subset E^\circ$ with vertex $x$, direction vector $u_x$, central angle of Lebesgue measure $\beta$, and altitude $h$.

*Topological pressure:*

$$\widetilde{P}(s) = \lim_{n \to \infty} \frac{1}{n} \ln \sum_{\mathbf{i} \in I^n} R_{\mathbf{i}}^s.$$

# Dimension result for IIFS under BDP and OSC

Theorem

*(Mauldin-Urbánski, 1996) Assume BDP, OSC and CC, and let* $\xi := \inf\{t \geq 0 : \widetilde{P}(s) < 0\}$. *Then*

$$\dim_{\mathrm{H}}(K) = \xi.$$

*In particular, if* $\widetilde{P}(\xi) = 0$, *then* $\dim_{\mathrm{H}}(K) = \xi$.

# Anomalous phenomena for IIFSs

- M. Moran (1996): Even for similitudes satisfying OSC, it is possible to have

$$\mathcal{H}^{\alpha}(K) = 0, \quad \text{where} \quad \alpha = \dim_{\mathrm{H}}(K).$$

(Nevertheless, for such IIFSs, $\mathcal{H}^{\alpha}(K) < \infty$. )

- Mauldin-Urbánski (1996): Under BDP and OSC, its possible to have

$$\dim_{\mathrm{H}}(K) < \underline{\dim}_{\mathrm{B}}(K) \leq \dim_{\mathrm{P}}(K).$$

- Szarek-Wedrychowicz (2004): OSC $\not\Rightarrow$ SOSC.

- Topological pressure functions need not have a zero. In fact, domain of various topological pressures could be empty.

# Weak separation condition for IIFSs

For $0 < b < 1$, let

$$\mathcal{I}_b = \{\mathbf{i} = (i_1, \dots, i_n) : R_{\mathbf{i}} \le b < R_{i_1 \cdots i_{n-1}}\} \quad \text{and} \quad \mathcal{A}_b = \{S_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}_b\}.$$

## Definition

(a) Weak separation condition (WSC): $\exists$ *invariant subset* $D \subseteq X$ *with* $D^\circ \neq \emptyset$, *called a* WSC set, *and a constant* $\gamma \in \mathbb{N}$ *such that*

$$\sup_{x \in X} \#\{\tau \in \mathcal{A}_b : x \in \tau(D)\} \le \gamma \quad \text{for all } b \in (0,1). \qquad (2.1)$$

(b) *If* $E \subseteq X$ *is an invariant set and (2.1) holds with* $E$ *replacing* $D$, *we call* $E$ *a* pre-WSC set. *Thus, any pre-WSC set that has a nonempty interior is a WSC set.*

# Example for WSC

## Example

*Let* $X = [0,1]$, $0 < r < (2-\sqrt{2})/2 \approx 0.292893\ldots$, $r(2-r)/(1-r) < t < 1-r$, *and*

$$S_1(x) = rx + (1-r), \qquad S_{2k}(x) = r^k x + t(1-r^{k-1}),$$
$$S_{2k+1}(x) = r^k x + t(1-r^{k-1}) + r^k(1-r), \qquad k \geq 1.$$

*Then the IIFS does not satisfy OSC, but BDP holds and WSC holds with* $D = X$.

# Figure for the example

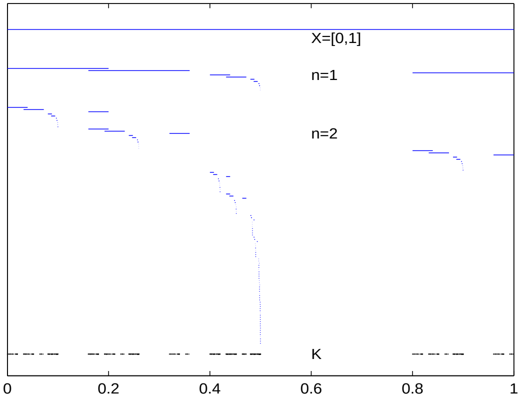


Figure: First two iterations of the set $X = [0, 1]$ under the IIFS, with $r = 1/5$ and $t = 1/2$. The limit set $K$ is also shown.

# Topological pressure

Let $\mathcal{S}_n = \mathcal{S}_n(I) := \{S_{\mathbf{i}} : \mathbf{i} \in I^n\}$.

Definition

Upper and lower topological pressure functions:

$$\underline{P}(s) := \varliminf_{n\to\infty} \frac{1}{n} \ln \sum_{\phi\in\mathcal{S}_n} R_\phi^s, \qquad \overline{P}(s) := \varlimsup_{n\to\infty} \frac{1}{n} \ln \sum_{\phi\in\mathcal{S}_n} R_\phi^s.$$

*If $\overline{P}(s) = \underline{P}(s)$, we denote the common value by $P(s)$ and call $P$ the* topological pressure function. *Define*

$$\mathrm{dom} P = \{s \in \mathbb{R} : P(s) < \infty\} \qquad (\text{Domain of } P).$$

# Topological pressure properties

- BDP $\Rightarrow$ $\overline{P}_V, \underline{P}_V$ are independent of $V$.
- Assume BDP and WSC. Then $[d, \infty) \subseteq \mathrm{dom} P$, the limit defining $P$ exists, $P$ is strictly decreasing, convex on $\mathrm{dom} P$ and continuous on $(\mathrm{dom} P)^\circ$.

# Dimension result for FIFS under BDP and WSC

## Theorem

*(Lau-X.Wang-N., 2009) Assume that a FIFS satisfies BDP and WSC. Then*

(a) $\alpha := \dim_{\mathrm{H}}(F) = \dim_{\mathrm{P}}(F) = \dim_{\mathrm{B}}(F)$;

(b) $0 < \mathcal{H}^{\alpha}(F) \leq \mathcal{P}^{\alpha}(F) < \infty$.

# Dimension formula

**Theorem**

*(Q. Deng-N., 2011) Assume that a FIFS satisfies BDP and WSC. Then* $\dim_{\mathrm{H}}(K)$ *is the unique zero of* $P$.

This result extends those by Y.Wang-N., 2001 and Lau-N. 2007 for similitudes satisfying FTC.

# Finite weak separation condition

Another natural extension of WSC to IIFSs. Let

$$\mathcal{F} = \mathcal{F}(I) := \{J \subset I : J \text{ is finite}\}$$

be the collection of all finite subsets of $I$.

**Definition**

Finite weak separation condition (FWSC): $\forall\, J \in \mathcal{F}(I)$, *the FIFS* $\{S_j\}_{j \in J}$ *satisfies WSC.*

# FWSC is strictly weaker than WSC

IIFS satisfying FWSC but not WSC.

**Example**

*Let* $X = [0,1]$ *and*

$$S_{k,i} := \frac{x}{2^k} + \frac{i}{2^k}, \qquad i = 0, 1, \ldots, 2^k - 1, \quad k \in \mathbb{N}.$$

*That is, for each* $k$, $S_{k,i}[0,1]$, $i = 0, 1, \ldots, 2^k - 1$, *is the union of all nonoverlapping dyadic intervals in* $[0,1]$ *with length* $1/2^k$. *Then* $K = [0,1]$ *and the IIFS satisfies FWSC but not WSC.*

# Topological pressure star

## Definition

*For each $J \in \mathcal{F}$, let $P_J$ be the topological pressure function for the FIFS $\{S_i\}_{i\in J}$, i.e.,*

$$P_J(s) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{\sigma\in\mathcal{S}_n(J)} R_\sigma^s.$$

*Define*

$$P^*(s) := \sup_{J\in\mathcal{F}} P_J(s).$$

# Auxiliary topological pressure

**Definition**

*For any $b \in (0,1)$, define*

$$\underline{Q}(s) := \varliminf_{b \to 0^+} \frac{1}{-\ln b} \ln \sum_{\tau \in \mathcal{A}_b} R_\tau^s, \qquad \overline{Q}(s) := \varlimsup_{b \to 0^+} \frac{1}{-\ln b} \ln \sum_{\tau \in \mathcal{A}_b} R_\tau^s,$$

*and let $Q(s)$ denote the common value if $\underline{Q}(s) = \overline{Q}(s)$.*

# "Zeros" of topological pressures

For each $J \in \mathcal{F}$, denote the limit set of the FIFS $\{S_i\}_{i \in J}$ by $K_J$. Define

$$\begin{aligned} \alpha_J &:= \dim_{\mathrm{H}}(K_J), & \hat{\alpha} &:= \sup\{\alpha_J : J \in \mathcal{F}\}, \\ \xi &:= \inf\{s \geq 0 : P(s) < 0\}, & \xi^* &:= \inf\{s \geq 0 : P^*(s) < 0\}, \\ \underline{\zeta} &:= \inf\{s \geq 0 : \underline{Q}(s) < 0\}, & \overline{\zeta} &:= \inf\{s \geq 0 : \overline{Q}(s) < 0\}. \end{aligned}$$

# Main results

## Theorem

*(N-Tong) Assume BDP and WSC.*

(a) *If $K$ is a pre-WSC set, then*

$$\dim_{\mathrm{H}}(K) = \underline{\zeta} = \overline{\zeta} = \widehat{\alpha} = \xi^* \leq \xi.$$

(b) *If a WSC set $D$ satisfies CC, then $\overline{D}$ is a WSC set. In particular, $K$ is a pre-WSC set and thus the conclusion of part (a) holds.*

# Outline of Proof

- Combining Lau-N-X. Wang (2009) and Q. Deng-N(2011), we have the following key lemma:

**Lemma**

*Assume BDP and WSC hold and $K$ is a pre-WSC set. Then for any $J \in \mathcal{F}$ and any $b \in (0,1)$,*

$$\sum_{\tau \in \mathcal{A}_b} R_\tau^{\alpha_J} \le c_1^{\alpha_J} \gamma.$$

- This lemma allows us to obtain the lower bound: $\underline{\zeta} \le \overline{\zeta} \le \dim_{\mathrm{H}}(K)$.
- The upper bound can be obtained more easily by using covers provided by the definition of various topological pressures.

# Growth dimension

*Growth dimension* (Zerner, 1996) of a FIFS is

$$\lim_{b\to 0^+} \frac{\ln \#\mathcal{A}_b}{-\ln b}.$$

For IIFS, since $\#A_b = \infty,\ \forall b$, we extend the definition to IIFSs as follows.

### Definition

*For $J \in \mathcal{F} = \mathcal{F}(I)$, let $d_G^J$ be the growth dimension of the finite IFS $\{S_j\}_{j\in J}$. Define the* growth dimension *of $\{S_i\}_{i\in I}$ as*

$$d_G = \sup_{J\in\mathcal{F}} d_G^J.$$

## Result concerning growth dimension

### Corollary

*Assume BDP holds.*

(a) $d_G \leq \dim_{\mathrm{H}}(K)$.

(b) *If, in addition,* $\{S_i\}_{i \in I}$ *WSC holds and* $K$ *is a pre-WSC set, then* $d_G = \dim_{\mathrm{H}}(K)$.

# Example on computing dimension

**Example**

*Let* $X = [0,1]$, $0 < r < (2-\sqrt{2})/2 \approx 0.292893\dots$,
$r(2-r)/(1-r) < t < 1-r$.

$$S_1(x) = rx + (1-r), \qquad S_{2k}(x) = r^k x + t(1-r^{k-1}),$$
$$S_{2k+1}(x) = r^k x + t(1-r^{k-1}) + r^k(1-r), \qquad k \geq 1.$$

*Then OSC fails, but BDP and WSC hold with* $D = X$.

$$\dim_{\mathrm{H}}(K) = \ln(2+\ln 2)/(-\ln r).$$

*In particular, for* $r = 1/5$, *and* $t = 1/2$, $\alpha = 0.762966\dots$.

# Figure

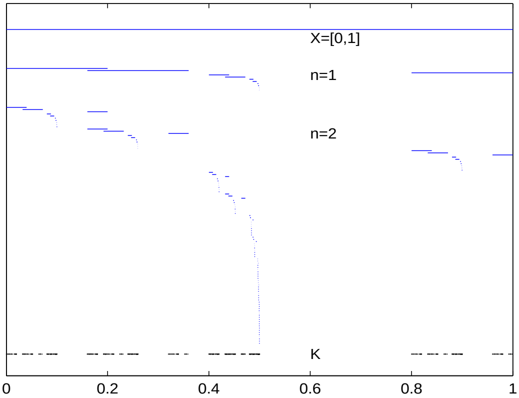


Figure: The first two iterations of the set $X = [0, 1]$, with $r = 1/5$ and $t = 1/2$. The limit set $K$ is also shown.

# Example of a conformal IIFS with WSC

## Example

*Let* $X = [0,1]$*,* $r < 13/16$*,* $23/(32(1-r)) < t < 13/16$ *and define*

$$S_1(x) = \frac{x^2}{8} + \frac{x}{16} + \frac{13}{16}, \quad S_2(x) = \frac{x}{2}, \quad S_3(x) = \frac{x^2}{4} + \frac{x}{16} + \frac{13}{32},$$
$$S_{2k}(x) = r^{k-1}S_2(x) + t(1-r^{k-1}),$$
$$S_{2k+1}(x) = r^{k-1}S_3(x) + t(1-r^{k-1}),$$

*for* $k \geq 2$*. Then OSC fails, but BDP holds and WSC holds with* $D = X$*.*

# Figure

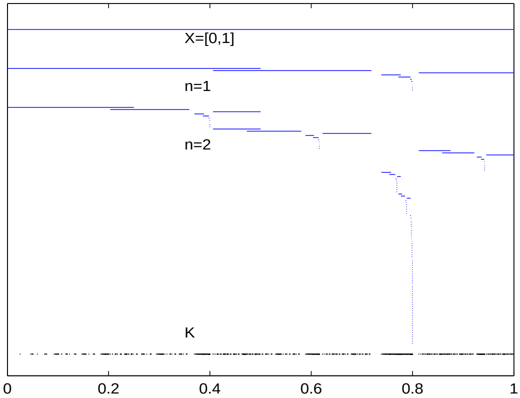


Figure: First two iterations of the set $X = [0,1]$, with $r = 1/13$ and $t = 4/5$.

# Problems for further study

1. Can the condition that $K$ is a pre-WSC set in the main theorem be removed?
2. Is the inequality $\dim_{\mathrm{H}}(K) \leq \xi$ in the main theorem an equality? If not, under what conditions does equality hold?
3. How to find $\dim_{\mathrm{H}}(\overline{K})$?
4. Hausdorff and packing measures of $K$.
5. Self-conformal measures and multifractal decomposition.

Thank you!