Projection and slicing theorems in Heisenberg groups

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10.12.2012

papers

- Z. Balogh, E. Durand Cartagena, K. Fässler, P. Mattila, J. Tyson: The effect of projections on dimension in the Heisenberg group, to appear in Revista Math. Iberoamericana
- Z. Balogh, K. Fässler, P. Mattila, J. Tyson: Projection and slicing theorems in Heisenberg groups, Advances in Math. 231 (2012), pp. 569-604

Heisenberg group $\mathbb{H}^n$

Heisenberg group $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$ equipped with a non-abelian group structure, with a left invariant metric and with natural dilations.

The first Heisenberg group $\mathbb{H}^1$

- $\mathbb{H} = \mathbb{C} \times \mathbb{R}$, $p = (w, s), q = (z, t) \in \mathbb{H}$
- $p \cdot q = (w + z, s + t + 2Im(w\bar{z}))$
- $||p|| = (|z|^4 + t^2)^{1/4}$
- $d(p, q) = ||p^{-1} \cdot q|| = (|w - z|^4 + |s - t - 2Im(w\bar{z})|^2)^{1/4}$
- $\delta_r(p) = (rz, r^2 t)$
- $d(\delta_r(p), \delta_r(q)) = rd(p, q)$
- $d(p \cdot q_1, p \cdot q_2) = d(q_1, q_2)$
- $\dim_H \mathbb{H} = 4$, Heisenberg Hausdorff dimension

Projections in $\mathbb{H}^1$

- $V_\theta = \{te_\theta : t \in \mathbb{R}\}, e_\theta = (\cos\theta, \sin\theta, 0), 0 \leq \theta < \pi$, horizontal line in $\mathbb{H}^1$
- $W_\theta = V_\theta^\perp$ vertical plane in $\mathbb{H}^1$
- $\mathbb{H}^1 = W_\theta \cdot V_\theta$, that is, for $p \in \mathbb{H}^1$, $p = Q_\theta(p) \cdot P_\theta(p), \; P_\theta(p) \in V_\theta, Q_\theta(p) \in W_\theta$
- $P_\theta : \mathbb{H}^1 \to V_\theta, Q_\theta : \mathbb{H}^1 \to W_\theta, 0 \leq \theta < \pi$, are the group projections

Projections in $\mathbb{H}^1$

- $p = (z, t) = (x + iy, t) \in \mathbb{H}^1$
- $P_\theta(p) = ((x \cos\theta + y \sin\theta)e_\theta, t)$;
  P_θ is the standard linear projection
- $Q_\theta(p) =$
  $((y \cos\theta - x \sin\theta)e_\theta^\perp, t - 2(\cos\theta)xy + \sin(2\theta)(x^2 - y^2))$;
  Q_θ is a non-linear projection

Marstrand's projection theorem

If $A \subset \mathbb{R}^2$ is a Borel set, then ($\dim_E$ is the Euclidean Hausdorff dimension) for almost all $\theta \in [0, \pi)$,

$$\dim_E P_\theta(A) = \dim_E A \text{ for almost all } \theta \in (0,\pi) \text{ if } \dim_E A \leq 1,$$

$$\mathcal{H}^1(P_\theta(A) > 0 \text{ for almost all } \theta \in (0,\pi) \text{ if } \dim_E A > 1.$$

Kaufman's proof for the first part:
Let $0 < s < \dim_E A$. Then there is a non-trivial Borel measure μ on A such that $I_s(\mu) = \iint |x-y|^{-s} d\mu x d\mu y < \infty$. Let $P_\theta \mu$ be the push-forward under P_θ: $P_\theta\mu(B) = \mu(P_\theta^{-1}(B)$. Then

$$\int_0^\pi I_s(P_\theta\mu) d\theta = \iiint |P_\theta(x-y)|^{-s} d\mu x d\mu y d\theta$$
$$\approx \int_0^\pi |\theta|^{-s} d\theta I_s(\mu) < \infty.$$

Horizontal projection theorem

Theorem

Let $A \subset \mathbb{H}^1$ be a Borel set. Then for almost all $\theta \in [0, \pi)$,

$$\dim_H P_\theta(A) \geq \dim_H A - 2 \text{ if } \dim_H A \leq 3,$$

$$\mathcal{H}^1(P_\theta(A)) > 0 \text{ if } \dim_H A > 3.$$

This is sharp: consider
$A = \{(x, 0, t) : x \in C, t \in [0, 1]\}, C \subset \mathbb{R}$. Then
$\dim_H A = \dim_E C + 2$ and

$$\dim_H P_\theta(A) = \dim_E P_\theta(A) = \dim_E P_\theta(C) = \dim_E C$$

for all but one θ.

Vertical projection theorem

Theorem

Let $A \subset \mathbb{H}^1$ be a Borel set. If $\dim_H A \leq 1$, then for almost all $\theta \in [0, \pi)$,

$$\dim_H A \leq \dim_H Q_\theta(A) \leq 2 \dim_H A.$$

For A with $\dim_H A \leq 1$ this is sharp:
if $A \subset t$-axis, $\dim_H Q_\theta(A) = \dim_H A$ for all θ,
if $A \subset x$-axis, $\dim_H Q_\theta(A) = 2 \dim_H A$ for all but one θ.

Vertical projection theorem

$$p = (z, t), q = (\zeta, \tau) \in \mathbb{H}^1, \varphi_1 = \arg(z - \zeta), \varphi_2 = \arg(z + \zeta)$$

$$d(p, q)^4 = |z - \zeta|^4 + (t - \tau + |z^2 - \zeta^2| \sin(\varphi_1 - \varphi_2))^2$$

$$d(Q_\theta(p), Q_\theta(q))^4$$
$$= |z - \zeta|^4 \sin^4(\varphi_1 - \theta) + (t - \tau - |z^2 - \zeta^2| \sin(\varphi_2 + \varphi_1 - 2\theta))^2$$

To get for $0 < s < 1$, $\int_0^\pi d(Q_\theta(p), Q_\theta(q))^{-s} d\theta \lesssim d(p, q)^{-s}$, one needs for $a \in \mathbb{R}$,

$$\int_0^\pi \frac{d\theta}{|a + \sin\theta|^{s/2}} \lesssim 1$$

Vertical projection theorem

If $\dim_H A > 1$, we have some estimates which quite likely are not sharp.
For example, we don't know if $\dim_H A > 3$ implies $\mathcal{H}^2(Q_\theta(A)) > 0$ for almost all $\theta \in [0, \pi)$.
A related Euclidean question: does $\dim_E A > 2$ imply $\mathcal{H}^2(Q_\theta(A)) > 0$ for almost all $\theta \in [0, \pi)$?

Higher dimensions

- $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R},\ p = (w, s), q = (z, t) \in \mathbb{H}^n$
- $p \cdot q = (w + z, s + t + \omega(w, z)),$
- $\omega(w, z) = 2Im(w \cdot z) = \sum_{j=1}^{n}(v_j x_j - u_j y_j),$
  $w = (u_j + iv_j), z = (x_j + iy_j)$
- $||p|| = (|z|^4 + t^2)^{1/4}$
- $d(p, q) = ||p^{-1} \cdot q|| = (|w - z|^4 + |s - t - \omega(w, z)|^2)^{1/4}$
- $\delta_r(p) = (rz, r^2 t)$
- $d(\delta_r(p), \delta_r(q)) = rd(p, q)$
- $d(p \cdot q_1, p \cdot q_2) = d(q_1, q_2)$
- $\dim_H \mathbb{H}^n = 2n + 2$

Projections in $\mathbb{H}^n$

- $G_h(n,m) = \{V \in G(2n,m) : \omega(w,z) = 0 \ \forall w,z \in V\}$, $0 < m \leq n$, isotropic subspaces
- unitary group $U(n) \subset O(2n)$ acts transitively on $G_h(n,m)$; $g \in U(n) : \omega(g(w),g(z)) = \omega(w,z) \ \forall w,z \in \mathbb{C}^n$
- $\mathbb{H}^n = V^\perp \cdot V$, $V^\perp \subset \mathbb{R}^{2n+1}$, $V \in G_h(n,m)$, $p = Q_V(p) \cdot P_V(p)$, $P_V(p) \in V$, $Q_V(p) \in W$, for $p \in \mathbb{H}^n$
- $P_V : \mathbb{H}^n \to V$ is the standard linear projection
- $Q_V(z,t) = (P_{V^\perp}(z), t - \omega((P_{V^\perp}(z), P_V(z)))$ is a non-linear projection, $Q_V : \mathbb{H}^n \to V^\perp$

Horizontal projection theorem in $\mathbb{H}^n$

Theorem

Let $A \subset \mathbb{H}^n$ be a Borel set. If $\dim_H A \leq m + 2$, then

$$\dim P_V(A) \geq \dim_H A - 2$$

for $\mu_{n,m}$ almost all $V \in G_h(n,m)$. Furthermore, if $\dim_H A > m + 2$, then

$$\mathcal{H}^m(P_V(A)) > 0 \text{ for } \mu_{n,m} \text{ almost } V \in G_h(n,m)$$

.

This is again sharp.
Above $\mu_{n,m}$ is the unique $U(n)$-invariant Borel probability measure on $G_h(n,m)$.

Vertical projection theorem in $\mathbb{H}^n$

Theorem

Let $A \subset \mathbb{H}^n$ be a Borel subset with $\dim_H A \leq 1$. Then for $\mu_{n,m}$ almost $V \in G_h(n,m)$,

$$\dim_H A \leq \dim_H Q_V A \leq 2 \dim_H A.$$

This is again sharp when $\dim_H A \leq 1$. Some, probably rather weak, partial results are known when $\dim_H A > 1$.

Vertical projection theorems in $\mathbb{H}^n$

$$d_H(p,q) = \sqrt[4]{|z-\zeta|^4 + (t-\tau-2\omega(\zeta,z))^2}$$

$$d_H(Q_V(p),Q_V(q))^4 = |P_{V^\perp}(z-\zeta)|^4 + (t-\tau-2\omega(P_{V^\perp}(z),P_V(z)) + 2\omega(P_{V^\perp}(\zeta),P_V(\zeta)) - 2\omega(P_{V^\perp}(\zeta),P_{V^\perp}(z)))^2.$$

The key estimate in the proof is

$$\int_{G_h(n,m)} |a-2\omega(v,P_V(w))|^{-s/2} d\mu_{n,m}V \lesssim 1$$

for all $0 < s < 1, a \in \mathbb{R}$ and $v,w \in S^{2n-1}$.
This estimate is false for $s \geq 1$.

Slicing theorems in $\mathbb{H}^n$

Theorem

Let $A \subset \mathbb{H}^n$ be a Borel set with $\dim_H A > m + 2$. Then for $\mu_{n,m}$ almost $V \in G_h(n, m)$,

$$\mathcal{H}^m(\{v \in V : \dim_H(A \cap (V^\perp \cdot v)) = \dim_H A - m\}) > 0.$$

The assumption $\dim_H A > m + 2$ is necessary.

Slicing theorems in $\mathbb{H}^n$

Theorem

Let $A \subset \mathbb{H}^n$ be a Borel set with $0 < \mathcal{H}_H^s(A) < \infty$ for some $s > m + 2$. Then for $\mathcal{H}_H^s$ almost all $p \in A$ we have

$$\dim_H(A \cap (V^{\perp} \cdot p)) = s - m \text{ for } \mu_{n,m} \text{ almost all } V \in G_h(n, m).$$

Thank you

Thank you Ka-Sing, De-Jun and all others