Bi-Affine Fractal Interpolation Functions and their Box Dimension

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General iterated functions systems (IFSs)

Let $(\mathbb{X}, d)$ be a complete metric space with metric $d = d_{\mathbb{X}}$.

Definition. Let $M \in \mathbb{N}$. If $f_m : \mathbb{X} \to \mathbb{X}$, $m = 1, 2, \ldots, M$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, ..., f_M)$ is called an *iterated function system* (IFS).

Define $\mathcal{F} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by $\mathcal{F}(B) := \bigcup_{f \in \mathcal{F}} f(B), \quad \forall\, B \in 2^{\mathbb{X}}$.

Let $\mathbb{H} = \mathbb{H}(\mathbb{X})$ be the hyperspace of nonempty compact subsets of $\mathbb{X}$ endowed with the Hausdorff metric $d_{\mathbb{H}}$.

Since $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat $\mathcal{F}$ as a mapping $\mathcal{F} : \mathbb{H} \to \mathbb{H}$.

Theorem.

(i) The metric space $(\mathbb{H}, d_{\mathbb{H}})$ is complete.

(ii) If $(\mathbb{X}, d_{\mathbb{X}})$ is compact then $(\mathbb{H}, d_{\mathbb{H}})$ is compact.

(iii) If $(\mathbb{X}, d_{\mathbb{X}})$ is locally compact then $(\mathbb{H}, d_{\mathbb{H}})$ is locally compact.

(iv) If $\mathbb{X}$ is locally compact, or if each $f \in \mathcal{F}$ is uniformly continuous, then $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is continuous.

(v) If $f : \mathbb{X} \to \mathbb{X}$ is a contraction mapping for each $f \in \mathcal{F}$, then $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ is a contraction mapping.

Attractor of an IFS

Definition. A nonempty compact set $A \subset \mathbb{X}$ is said to be an *attractor* of the IFS $\mathcal{F}$ if

(i) $\mathcal{F}(A) = A$ and

(ii) $\exists$ an open set $U \subset \mathbb{X}$ such that $A \subset U$ and $\lim_{k\to\infty} \mathcal{F}^k(B) = A$, $\forall\, B \in \mathbb{H}(U)$, where the limit is taken with respect to the Hausdorff metric.

The largest open set U such that (ii) is true is called the *basin of attraction* (for the attractor A of the IFS $\mathcal{F}$).

[For more details and generalizations, see M. F. Barnsley & A. Vince, The chaos game on a general iterated function system, *Ergod. Th. & Dynam. Syst.* **31** (2011) 1073-1079.]

Fractal interpolants as fixed points of operators

Let $1 < N \in \mathbb{N}$ and let $\{(X_j, Y_j) : j = 0, 1, ..., N\}$ be finite set of points in the Euclidean plane with $X_0 < X_1 < ... < X_N$.

Set $I := [X_0, X_N]$.

Let $\ell_n : I \to [X_{n-1}, X_n]$ be continuous bijections. $(n = 1, 2, ..., N)$

Let $L : I \to I$ be bounded with $L(x) = \ell_n^{-1}(x),$ for $x \in (X_{n-1}, X_n)$.

Let $S : [X_0, X_N] \to \mathbb{R}$ be bounded and piecewise continuous where the only possible discontinuities occur at the points in $\{X_1, X_2, ..., X_{N-1}\}$.

Let $s := \max\{|S(x)| : x \in [X_0, X_N]\}$.

For the complete metric space $(C(I), d_\infty)$, define subspaces

$$C^* := C^*(I) := \{f \in C(I) : f(X_0) = Y_0, f(X_N) = Y_N\},$$
$$C^{**} := C^{**}(I) := \{f \in C(I) : f(X_j) = Y_j, \text{ for } j = 0, 1, ..., N\}.$$

Note that:

- $C^{**} \subset C^* \subset C(I)$ are closed subspaces of $C(I)$.
- $f \in C^{**}$ interpolates the data $\{(X_j, Y_j) \,:\, j = 0, 1, \ldots, N\}$.

Let $b \in C^*$ and $h \in C^{**}$.

Define a Read-Bajraktarević operator $T : C(I) \to C(I)$ by

$$T(g) = h + S \cdot (g \circ L - b \circ L).$$

Theorem. The mapping $T : C(I) \to C(I)$ obeys

$$d_\infty(Tg, Th) \le s\, d_\infty(g, h),$$

$\forall g, h \in C(I)$. In particular, if $s < 1$ then T is a contraction and thus possesses a unique fixed point $f \in C^{**}$.

Note that $Tg = H + S \cdot g \circ L$ where $H = h - S \cdot b \circ L$.

A fractal interpolation function f is uniquely defined by these three functions: H, S, and L.

$$f = \lim_{k \to \infty} T^k(f_0), \qquad f_0 \in C^*.$$

The rate of convergence of $\{T^k f_0 \,:\, k \in \mathbb{N}\}$ is governed by

$$\left\| f - T^k(f_0) \right\|_\infty \le s^k \left\| f - f_0 \right\|_\infty.$$

The metric space $(I \times \mathbb{R}, d_q)$

The following metric is a generalization of the "taxi cab metric."

Theorem. Let $\alpha, \beta > 0$ and $q : I \to \mathbb{R}$. Define a mapping $d_q : (I \times \mathbb{R}) \times (I \times \mathbb{R}) \to [0, \infty)$ by

$$d_q((x_1, y_1), (x_2, y_2)) = \alpha |x_1 - x_2| + \beta |(y_1 - q(x_1)) - (y_2 - q(x_2))|,$$

$\forall (x_1, y_1), (x_2, y_2) \in I \times \mathbb{R}$. Then d_q is a metric on $I \times \mathbb{R}$. If q is continuous then $(I \times \mathbb{R}, d_q)$ is a complete metric space.

Fractal interpolants as attractors of IFSs

Define $w_n : I \times \mathbb{R} \to I \times \mathbb{R}$ by

$$w_n(x,y) = (\ell_n(x), h(\ell_n(x)) + S(l_n(x))(y - b(x)))$$

Define an IFS by $\mathcal{W} = (I \times \mathbb{R}; w_1, w_2, ..., w_N)$.

Let $B \geq 0$ and let $\mathbb{X} = \{(x,y) : x \in I, |y - f(x)| \leq B\}$.

Theorem. Let $s < 1$ and let $f \in C^{**}$ be the fixed point of T. Let $\exists\, \lambda_\ell < 1$ so that $|\ell_n(x_1) - \ell_n(x_2)| \leq \lambda_\ell \,|x_1 - x_2|\ \forall x_1, x_2 \in I$, $\forall n$. Let $\exists\, \lambda_S > 0$ so that $|S(x_1) - S(x_2)| \leq \lambda_S \,|x_1 - x_2|\ \forall x_1, x_2 \in I$. Then the IFS $(\mathbb{X}; w_1, w_2, ..., w_N)$ is contractive with respect to the metric d_f with $\alpha = 1$ and $0 < \beta < \ (1 - \lambda_\ell)\,/\lambda_S B \lambda_\ell$. In particular, under these conditions, the IFS $\mathcal{W}$ has a unique attractor $A = \text{graph}\,(f)$.

$$\text{graph}\,(T(g)) = \mathcal{W}\,(\text{graph}\,(g))\,,\ \text{for all } g \in C(I).$$

We have not provided a metric with respect to which $\mathcal{W}$ is contractive!

Bi-affine fractal interpolation

Let

$$\ell_n(x) := X_{n-1} + \left(\frac{X_n - X_{n-1}}{X_N - X_0}\right)(x - X_0),$$

$$S(x) = s_n(\ell_n^{-1}(x)), \quad \text{for } x \in [X_{n-1}, X_n],\ \ n = 1, \ldots, N,$$

$$s_n(x) = s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_n - X_{n-1}}\right)(x - X_{n-1}),$$

with $\{s_j : j = 0, 1, 2, ..., N\} \subset (-1, 1)$.

Then S is continuous and $|S(x)| \leq \max\{|s_j| : j = 0, 1, ..., N\} =: s < 1$.

Let

$$b(x) = Y_0 + \left(\frac{Y_N - Y_0}{X_N - X_0}\right)(x - X_0)$$

and let

$$h(x) = Y_{n-1} + \left(\frac{Y_n - Y_{n-1}}{X_n - X_{n-1}}\right)(x - X_{n-1}).$$

Bi-affine fractal interpolants

T has a unique fixed point f satisfying the set of functional equations

$$f(\ell_n(x)) - h(\ell_n(x)) = [s_{n-1} + (s_n - s_{n-1})x][f(x) - b(x)],\ x \in I.$$

f is called a *bi-affine fractal interpolant.*

Define an IFS $\mathcal{W}$ by

$$\begin{aligned} w_n(x,y) = (\ell_n(x), Y_{n-1} &+ \left(\frac{Y_n - Y_{n-1}}{X_N - X_0}\right)(x - X_0) \\ &+ \left[s_{n-1} + \left(\frac{s_n - s_{n-1}}{X_N - X_0}\right)(x - X_0)\right]\left[y - Y_0 - \left(\frac{Y_N - Y_0}{X_N - X_0}\right)(x - X_0)\right]. \end{aligned}$$

Note:

$$w_n(X_N, y) = (X_n, Y_n + s_n(y - Y_N)) \text{ and } w_{n+1}(X_0, y) = (X_n, Y_n + s_n(y - Y_0)).$$

Example of a bilinear interpolant

The images of any (possibly degenerate) parallelogram with vertices at $(X_0, Y_0 \pm H)$ and $(X_N, Y_N \pm H)$, for $H \in \mathbb{R}$ under the IFS $\mathcal{W}$ fit together neatly.

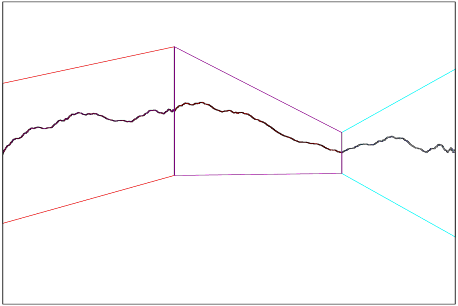

Figure : A bilinear fractal interpolant.

Box dimension of bi-affine interpolants

Box-counting or box dimension of a bounded set $M \subset \mathbb{R}^n$:

$$\dim_B M := \lim_{\varepsilon \to 0+} \frac{\log \mathcal{N}_\varepsilon(M)}{\log \varepsilon^{-1}}, \qquad (*)$$

where $\mathcal{N}_\varepsilon(M)$ is the minumum number of square boxes, with sides parallel to the axes, whose union contains M.

"$\dim_B M = D$" $\Longleftrightarrow$ the limit in (*) exists and equals D.

Theorem. *Let $\mathcal{W}$ denote the bi-affine IFS defined above, and let $\Gamma(f)$ denote its attractor. Let $a_n = 1/N$ for $n = 1, 2, ..., N$, and let $\sum_{n=1}^{N} \frac{s_{n-1}+s_n}{2} > 1$. If $\Gamma(f)$ is not a straight line segment then*

$$\dim_B \Gamma(f) = 1 + \frac{\log \left(\displaystyle\sum_{n=1}^{N} \frac{s_{n-1} + s_n}{2} \right)}{\log N};$$

otherwise $\dim_B \Gamma(f) = 1$.

Idea of Proof

Arguments based on approach in Hardin & M. (1985) and Barnley, Elton, Hardin, M. (1989)

Denote by $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ the image of $\Gamma(f)$ under the maps $w_{\sigma_1\cdots\sigma_r} := w_{\sigma_1}\circ\cdots\circ w_{\sigma_r}$ over the subinterval $\ell_{\sigma_1\cdots\sigma_r}(I)$.

Then one can show there that exist constants $0 < \underline{c} \le \overline{c}$ such that

$$\underline{c}\,\lambda_{\sigma_1}\cdots\lambda_{\sigma_r}\,N^{|\sigma|} \le \mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|) \le \overline{c}\,\lambda_{\sigma_1}\cdots\lambda_{\sigma_r}\,N^{|\sigma|},$$

Here, $\mathcal{N}_{\sigma_1\cdots\sigma_r}(|\sigma|)$ = minimum number of $N^{-|\sigma|}\times N^{-|\sigma|}$-squares needed to cover $w_{\sigma_1\cdots\sigma_r}(\Gamma(f))$ and $\lambda_i := \frac{s_{i-1}+s_i}{2}$.

Nonlinearity (xy-term) rather tricky; delicate estimates are needed.

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