Isodiametric Problem w.r.t. Hausdorff Measure

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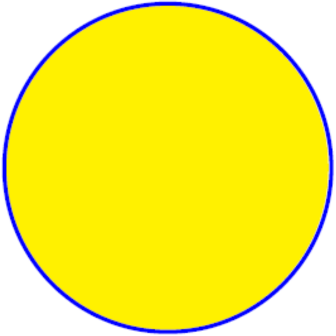
- **Isodiametric problem** in $\mathbb{R}^n$ asks for convex domains $C \subset \mathbb{R}^n$ *of diameter* $|C| = 1$ *that have maximum volume.*
- (Bieberbach, 1915) proved that for any compact domain $\Omega \subset \mathbb{R}^n$

$$\mathrm{Volume}(\Omega) \leq \mathrm{Volume}(\underline{\text{Ball of diameter 1}}) \left(\frac{|\Omega|}{2}\right)^n$$

and that *equality holds* if and only if Ω is a ball.
- Therefore, the ball $B \subset \mathbb{R}^n$ of diameter one is the *unique solution* of classical isodiametric problem in $\mathbb{R}^n$.
- This is "isodiametric problem with respect to Lebesgue measure". How about replacing *Lebesgue measure* by *Hausdorff measure restricted to a self-similar set with OSC* ?



- For a self-similar set $E \subset \mathbb{R}^n$ with OSC, it is known that
$$\sup\left\{\frac{\mathcal{H}^s(X \cap E)}{|X|^s} : |X| > 0\right\} = 1.$$
- Isodiametric Problem on E then asks for compact convex domain $\Omega \subset \mathbb{R}^n$ with
$$\frac{\mathcal{H}^s(\Omega \cap E)}{|\Omega|^s} = 1.$$
- We call such a domain Ω an *extremal set*.

AIM to find extremal sets Ω for specific E and to study structure of Ω.

1. "Shape" of Ω
2. "Relative location" of Ω in E
3. Diameter $|\Omega|$ of Ω

- When Hausdorff dimension s of E is smaller than or equal to 1, there are many examples of E for which an extremal set Ω is found.

$\cdots\cdots\cdots$

- When Hausdorff dimension s of E is strictly greater than 1, every known example of E s.t. *an extremal set Ω has been found* satisfies the following 2 properties:

  (1) $s \in \mathbb{Z}$;

  (2) $\mathcal{H}^s|_E$ and Lebesgue measure on E differ by a constant.

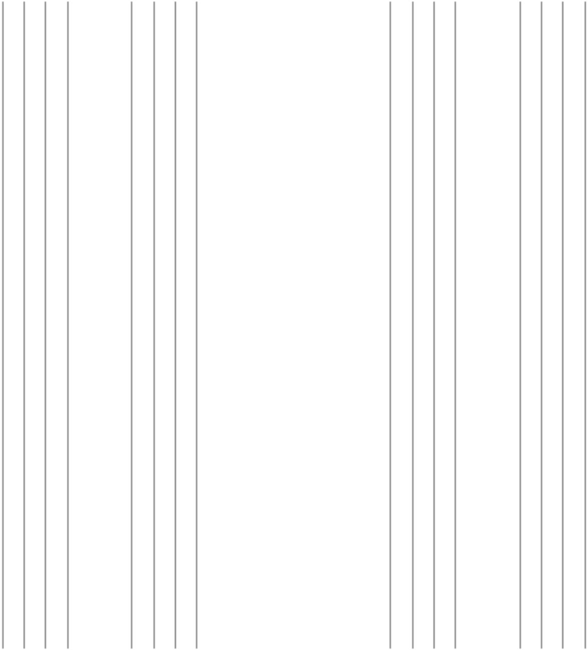
- We will consider concrete self-similar sets E of dimension s in $(1, \infty) \setminus \mathbb{Z}$ and try to find extremal sets Ω.
- Current Talk is about a family of self-similar fractals E on $\mathbb{R}^2$. We can determine the shape and location of extremal sets Ω.

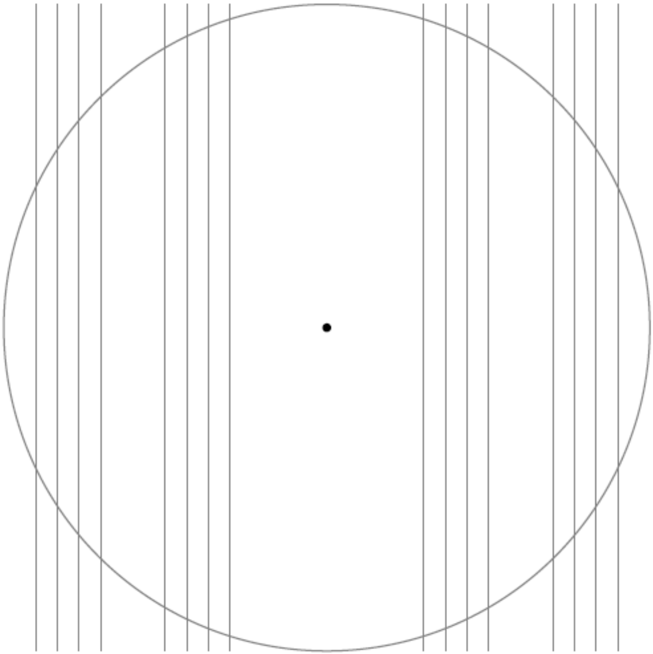
- Let $F_\lambda = E_\lambda \times \mathbb{R}$ for $0 < \lambda < \frac{1}{2}$, where E_λ denotes the middle $(1-2\lambda)$ Cantor set.
- By Marstrand's formula the set F_λ has Hausdorff dimension $s = 1 - \frac{\ln 2}{\ln \lambda} \in (1, 2)$.
- **Observations**:

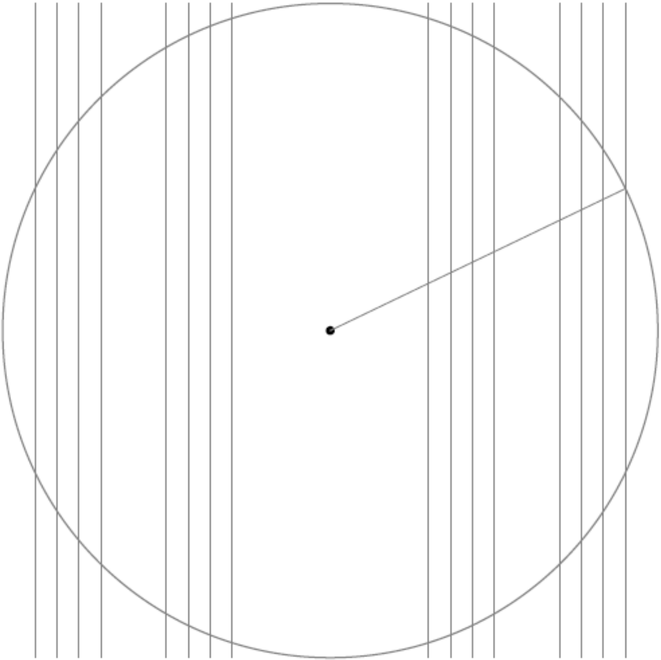
1. $\mathcal{H}^s(X \cap F_\lambda) \leq |X|^s$ for any compact set X.
2. For $0 < \lambda < \frac{1}{2}$, there is an extremal set Ω for IP on F_λ.

Theorem

- *If* $\lambda \leq \frac{1}{5}$, Ω *is a copy of some extremal set* Ω_λ *with* $[0,1] \subset \mathrm{proj}_1(\Omega_\lambda)$ *such that* $\mathcal{M} \circ \mathcal{S}\,(\Omega_\lambda) \cap F_\lambda = \mathrm{Disk} \cap F_\lambda$.
- $\mathcal{H}^s(X \cap F_\lambda) = \mathcal{H}^s(E_\lambda \times [0,1])\mathcal{H}^{s-1} \times \mathcal{H}^1(X)$ *for compact* $X \subset \mathbb{R}^2$.







Theorem

If $\lambda \leq \frac{1}{5}$ *diameter* t_λ *of* Ω_λ *(*$> \frac{2}{\sqrt{3}}$*) is determined by*

$$2\int_0^\lambda \frac{t dF_\lambda(x)}{\sqrt{t^2 - (1-2x)^2}} = \frac{sf(\lambda, t)}{t} \tag{1}$$

where D_t *is a disk of diameter* $t \geq \frac{2}{\sqrt{3}}$ *centered on the line* $x = \frac{1}{2}$*,*
$F_\lambda(x) = \mathcal{H}^{s-1}(E_\lambda \cap [0, x])$*,* $f(\lambda, t) = \mathcal{H}^{s-1} \times \mathcal{H}^1(D_t \cap F_\lambda) =$

$$4\int_0^\lambda \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x\right)^2} dF_\lambda(x) = 2\int_0^\lambda \sqrt{t^2 - (1-2x)^2} dF_\lambda(x)$$

and

$$\varphi_t(\lambda, t) = \frac{f_t(\lambda, t)}{t^s} - \frac{sf(\lambda, t)}{t^{s+1}} = t^{-s}\left[2\int_0^\lambda \frac{t dF_\lambda(x)}{\sqrt{t^2-(1-2x)^2}} - \frac{sf(\lambda, t)}{t}\right]$$

$$\begin{cases} \text{lower bound}: & f_L(t,\lambda,n) = \frac{4}{2^n} \sum_{x \in A_n} \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x\right)^2} \\ \text{upper bound}: & f_U(t,\lambda,n) = \frac{4}{2^n} \sum_{x \in A_n} \sqrt{\frac{t^2}{4} - \left(\frac{1}{2} - x - \lambda^n\right)^2} \end{cases}$$

Key fact: $|f_U - f_L| \leq 3\lambda^n$. Considering the case $n = 4$, we have

Value of λ	Upper Bound of $\varphi(\lambda, t_\lambda)$	Lower Bound of $\varphi(\lambda, t_\lambda)$	Interval containing $\mathcal{H}^s(E_\lambda \times [0,1])$
$\frac{1}{5}$	0.702626	0.701483	$(1.423232, 1.425551)$
$\frac{1}{6}$	0.706784	0.706297	$(1.414859, 1.415835)$
$\frac{1}{7}$	0.711554	0.711314	$(1.405375, 1.405849)$
$\frac{1}{8}$	0.716226	0.716096	$(1.396207, 1.396461)$
$\frac{1}{9}$	0.720599	0.720522	$(1.387734, 1.387825)$
$\frac{1}{10}$	0.724629	0.724581	$(1.3800165, 1.38010795)$

Thank you !