Mandelbrot's cascade in a Random Environment

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1.Introduction

We present asymptotic properties for generalized Mandelbrot's cascades, formulated by consecutive products of random weights whose distributions depend on a random environment indexed by time, which is supposed to be iid.

We also present limit theorems for a closely related model, called branching random walk on $\mathbb{R}$ with random environment in time, in which the offspring distribution of a particle of generation n and the distributions of the displacements of their children depend on a random environment ξ_n indexed by the time n.

Why Random Environment

In random environment models, the controlling distributions are realizations of a stochastic process, rather then a fixed (deterministic) distribution.

The random environment hypothesis is very natural, because in practice the distributions that we observe are usually realizations of a (measure-valued) stochastic process, rather then being constant.

This explains partially why random environment models attract much attention of many mathematicians and physicians.

2. Description of the model

Mandelbrot's cascade on a Galton-Watson tree. Let

$$(N_u, A_{u1}, A_{u2}, ...)$$

be a famille of independent and identically distributed random variables, indexed by all finite sequences u of positive integers, with values in $\mathbb{N} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \cdots$. By convention, $N = N_\emptyset, A_i = A_{\emptyset i}$. We are interested in the total weights of generation n:

$$Y_n = \sum A_{u_1} A_{u_1 u_2} \cdots A_{u_1 \ldots u_n}, \quad n \geq 1,$$

where the sum is taken over all particles $u = u_1...u_n$ of gen. n of the Galton-Watson tree T associated with (N_u): $\emptyset \in T$; if $u \in T$, then $ui \in T$ iff $1 \leq i \leq N_u$.

$$\{\frac{Y_n}{EY_n} : \ n \geq 1\}$$

forms a martingale, called generalized Mandelbrot's martingale.

Mandelbrot's cascade in a Random Environment

Instead of the assumption of identical distribution, we consider the case where the distributions of

$$(N_u, A_{u1}, A_{u2}, \ldots)$$

depend on an environment $\xi = (\xi_n)$ indexed by the time n: given the environment $\xi = (\xi_n)$, the above vector is of distribution $\mu_n = \mu(\xi_n)$ if $|u| = n$; the random distributions ξ_n are supposed to be iid (as measure-valued random variables). Notice that if $A_u = 1$ for all u, then

$$Y_n = \text{ card } \{u \in T : \ |u| = n\}, n \geq 1,$$

is a branching process in a random environment.

boundary of the branching tree T in RE

Let

$$\partial T = \{u = u_1 u_2 ... : \ u|n := u_1 \cdots u_n \in T \ \forall n \geq 0\}$$

(with $u|0 = \emptyset$) be the boundary of the Galton-Watson tree T, equipped with the ultrametric

$$d(u, v) = e^{-|u \wedge v|},$$

$u \wedge v$ denoting the maximal common sequence of u and v.

We consider the supercritical case where $\partial T \neq \emptyset$ with positive probability.

Quenched and annealed laws

Let (Γ, P_ξ) be the probability space under which the process is defined when the environment ξ is fixed. As usual, P_ξ is called *quenched law*.

The total probability space can be formulated as the product space $(\Theta^{\mathbb{N}} \times \Gamma, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive g, we have

$$\int g dP = \int \int g(\xi, y) dP_\xi(y) d\tau(\xi),$$

where τ is the law of the environment ξ. P is called *annealed law*. P_ξ may be considered to be the conditional probability of P given ξ.

Without loss of generality we suppose that

$$\mathbb{E}_\xi \sum_{i=1}^{N} A_i = 1 \quad \text{a.s.}$$

(otherwise we replace A_{ui} by A_{ui}/m_n, where $m_n = E_\xi \sum_{i=1}^{N} A_{ui}$ with $|u| = n$). Then

$$Y_n = \sum_{|v|=n} X_v, \quad \text{with } X_v = A_{v_1} \cdots A_{v_1 \cdots v_n}, \text{ if } v = v_1 \cdots v_n$$

is a martingale associated with the natural filtration (both under P_ξ and under P), called Mandelbrot's martingale in a random environment. Hence the limit

$$Y = \lim_{n\to\infty} Y_n$$

exists a.s. with $E_\xi Y \leq 1$ a.s.

Mandelbrot's measure in a random environment

For each finite sequence u we define Y_u with the weighted tree T^u beginning with u just as we defined Y with the weighted tree T beginning with $\emptyset$ (so that $Y_\emptyset = Y$). It is clear that for each finit sequence u,

$$X_u Y_u = \sum_{i=1}^{N_u} X_{ui} Y_{ui}$$

($X_\emptyset = 1$). Therefore by Kolmogorov's consistency theorem there is a unique measure $\mu = \mu_\omega$ on ∂T such that for all $u \in T$,

$$\mu([u]) = P_u Z_u, \quad \text{where } [u] = \{v \in \partial T : \; u < v\}$$

with mass $\mu(\partial T) = Z$. Notice that when $Z \neq 0$,

$$\frac{\mu([u])}{Z} = \lim_{k\to\infty} \frac{\sum_{v>u,|v|=k} P_v}{\sum_{|v|=k} P_v},$$

describing the proportion of the weights of the descendants of u over the total weights of all individuals (in gen. k).

Problems that we consider

Following Mandelbrot (1972), Kahane- Peyrière (1976) and others, we consider:

1) Non degeneration of Y;

2) Existence of moments and weighted moments of Y;
3) Hausdorff dim of μ and its multifractal spectrum

3. Main results on Mandelbrot's cascades in RE

Non-degeneration of Y. For $x \in \mathbb{R}$, write

$$\rho(x) = \mathbb{E} \sum_{i=1}^{N} A_i^x. \quad (1)$$

Theorem 0 (Biggins - Kyprianou (2004) ; Kuhlbusch (2004))

Assume that

$$\rho'(1) := \mathbb{E} \sum_{i=1}^{N} A_i \ln A_i$$

is well-defined with value in $[-\infty, \infty)$. Then the following assertions are equivalent:

(a) $\rho'(1) < 0$ and $\mathbb{E} Y_1 \ln^+ Y_1 < \infty$;

(b) $\mathbb{E} Y = 1$;

(c) $\mathbb{P}(Y = 0) < 1$.

Moments

Theorem 1 (Liang and Liu (2012)

For $\alpha > 1$, the following assertions are equivalent:

(a) $\mathbb{E} Y_1^\alpha < \infty$ and $\rho(\alpha) < 1$;

(b) $\mathbb{E} Y^\alpha < \infty$.

Recall:

$$Y_1 = \sum_{i=1}^{N} A_i,$$

$$\rho(\alpha) = \mathbb{E} \sum_{i=1}^{N} A_i^\alpha.$$

Comments on the Moments

For the deterministic case:

(a) When N is constant or bounded: Kahane and Peyrière (1976), Durrett and Liggett (1983); direct estimation using

$$Y = A_1 Y_1 + \ldots + A_N Y_N.$$

(b) When $A_i \leq 1$: Bingham and Doney (1975), using Tauberian theorems and the functional equation for $\phi(t) = \mathbb{E}e^{-tY}$:

$$\phi(t) = \mathbb{E} \prod_{i=1}^{N} \phi(A_i t).$$

(c) In the general case: Liu (2000), using the Peyrière measure to transform the above distributional equation to

$$Z = AZ + B \quad \text{in law},$$

where (A, B) is indep. of Z, $\mathbb{P}(Z \in dx) = x\mathbb{P}(Y \in dx)$.

In the random environment case: We failed to prove the result using these methods; new ideas are needed.

More Comments on the Moments in RE

For branching process in a random environment:

(a) Afanasyev (2001) gave a sufficient condition (which is not necessary) with several pages of calculation

(b) Guivarc'h and Liu (2001) gave the criterion.

Weighted Moments of order $\alpha > 1$

The preceding theorem suggests that if $\rho(\alpha) < 1$, then Y_1 and Y would have similar tail behavior. We shall ensure this by establishing comparison theorems for weighted moments of Y_1 and Y.

Let $\ell : [0, \infty) \mapsto [0, \infty)$ be a measurable function slowly varying at ∞ in the sense that

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1 \ \forall \lambda > 0.$$

Theorem 2 (Liang and Liu (2012))

For $\alpha \in \mathrm{Int}\{a > 1 : \rho(\alpha) < 1\}$, the following assertions are equivalent:

(a) $\mathbb{E} Y_1^{\alpha} \ell(Y_1) < \infty$;

(b) $\mathbb{E} Y^{\alpha} \ell(Y) < \infty$.

Comments on Weighted Moments of order $\alpha > 1$

In the deterministic case:

(a) For GW process: Bingham and Doney (1974): α not an integer; additional condition needed otherwise
Alsmeyer and Rösler (2004): α not a power of 2.

(b) For Mandelbrot's martingale: Alsmeyer and Kuhlbusch (2010): α not a power of 2.

Mais tool of the approach: Burkholder-Davis-Gundy inequality (convex inequality for martingales).

Weighted Moments of order 1

The situation for order 1 is different. Let $\ell : [0, \infty) \mapsto [0, \infty)$ be slowly varying at ∞, and concave on $[a_0, \infty)$ for some $a_0 \geq 0$. Set

$$\hat{\ell}(x) = \begin{cases} \int_1^x \frac{\ell(t)}{t} dt & \text{if } x > 1; \\ 0 & \text{if } x \leq 1. \end{cases}$$

Example: if $\ell(x) = (\ln x)^a$, then $\hat{\ell}(x) = (\ln x)^{a+1}/(a+1), x > 1$.

Theorem 3 (Liang and Huang (2012))

Assume that there exists some $\delta > 0$ such that $\rho(1+\delta) < \infty$. If $\mathbb{E} Y_1 \hat{\ell}(Y_1) < \infty$, then $\mathbb{E} Y \ell(Y) < \infty$.

The converse also holds in special cases. The argument leads to a new proof for the non-degeneration of Y.

Hölder exponent and Hausdorff dimension of μ

Theorem 4 (Liang and Liu (2012))

Assume $EY_1(\log^+ Y_1)^2 < \infty$ and $\rho'(1) := \mathbb{E}\sum_{i=1}^{N} A_i \ln A_i < 0$. Then for $\mathbb{P}$-almost all ω and μ_ω-almost all $u \in \partial T$,

$$\lim_{n\to\infty} \frac{\log \mu_\omega([u|n])}{n} = \rho'(1).$$

Consequently,

$$\dim \mu_\omega = -\rho'(1) \quad \text{a.s.}$$

Two critical values t_- and t_+

Let

$$\Lambda(t) = \mathbb{E} \log m_0(t), \quad \text{with } m_0(t) = \mathbb{E}_\xi \sum_{i=1}^{N} A_i^t,$$

be well defined for all $t \in \mathbb{R}$. Set

$$\lambda(t) = t\Lambda'(t) - \Lambda(t).$$

Then $\lambda'(t) = t\Lambda''(t)$, $\lambda(t)$ decreases on $\mathbb{R}_-$, increases on $\mathbb{R}_+$, and attains its minimum at 0 with $\min_t \lambda(t) = \rho(0) = -\Lambda(0) < 0$.
Let

$$t_- = \inf\{t \in \mathbb{R} : \ \lambda(t) \leq 0\},$$

$$t_+ = \sup\{t \in \mathbb{R} : \ \lambda(t) \leq 0\}.$$

Then $-\infty \leq t_- < 0 < t_+ \leq \infty$, and for $t \in \mathbb{R}$,

$$\lambda(t) \begin{cases} = 0 & \text{if} \quad t = t_- \text{ or } t_+, \\ < 0 & \text{if} \quad t_- < t < t_+, \\ > 0 & \text{if} \quad t < t_- \text{ or } t > t_+ \end{cases}$$

Legendre transform of Λ

Let

$$\Lambda^*(x) = \sup_{t \in \mathbb{R}} \{xt - \Lambda(t)\}$$

be the Legendre transform of Λ. Then

$$\Lambda^*(x) = \begin{cases} \lambda(t) & \textit{if} \quad x = \Lambda'(t) \text{ for some } t \in \mathbb{R}, \\ +\infty & \textit{if} \quad x < \Lambda'(-\infty) \text{ or } x > \Lambda'(+\infty), \end{cases}$$

and

$$\min_x \Lambda^*(x) = \Lambda^*(\Lambda'(0)) = -\Lambda(0) = -E \log m_0(0) < 0.$$

Multifractal spectrum of μ_ω

For $x \in \mathbb{R}$, define

$$E(x) = \{u \in \partial T : \lim_{n\to\infty} \frac{\log \mu_\omega([u|n])}{n} = x\}$$

Theorem 5 (Liang and Liu (2012))

Under simple moment conditions, we have a.s.

(a) If $x < \Lambda'(t_-)$ or $x > \Lambda'(t_+)$, then $E(x) = \emptyset$;

(b) If $x = \Lambda'(t)$ for some $t \in \mathbb{R}, t_- \leq t \leq t_+$, then $E(x) \neq \emptyset$, and

$$\dim E(x) = -\Lambda^*(x) = -\lambda(t).$$

For deterministic case: Holley and Waymire (1992), Molchan (1996), Barral (1997,2000).

4. Branching Random Walk in a Random Env.

The Mandelbrot cascade in a random environment is closely related to the Branching Random Walk with a random environment in time defined as follows:

$$S_\emptyset = 0, \; S_{u_1 \ldots u_n} = \log A_{u_1} + \cdots + \log A_{u_1 \cdots u_n},$$

where S_u denotes the position of $u \in T$ (the i-th child ui of u has displacement $\log A_{ui}$). Let

$$Z_n = \sum_{|u|=n} \delta_{S_u}$$

be the counting measure of particles of gen. n, so that for $A \subset \mathbb{R}$,

$$Z_n(A) = \text{number of particles of gen. n located in} A.$$

Convergence of the free energy

The laplace transform of Z_n is

$$\tilde{Z}_n(t) := \int e^{tx} dZ_n(x) = \sum_{|u|=n} e^{tS_u}.$$

It is also called the partition function. Notice that $\{\tilde{Z}_n(t)/E_\xi \tilde{Z}_n(t)\}$ is a Mandelbrot martingale in random environment.

Theorem 6 (Huang and Liu (2012))

We have a.s.

$$\lim_{n\to\infty} \frac{\log \tilde{Z}_n(t)}{n} = \tilde{\Lambda}(t) := \begin{cases} \Lambda(t) & \text{if} \quad t \in (t_-, t_+) \\ t\Lambda'(t_+) & \text{if} \quad t \geq t_+ \\ t\Lambda'(t_-) & \text{if} \quad t \leq t_- \end{cases}$$

Deterministic case: B. Chauvin and A. Rouault (1997), J. Franchi (1993).

Large Deviation Principle

Let $\tilde{\Lambda}^*(x) = \sup_t\{tx - \tilde{\Lambda}(t)\}$ be the Legendre transform of $\tilde{\Lambda}$. By the preceding theorem and Gärtner- Ellis' theorem, we obtain:

Theorem 7 (Huang and Liu 2012)

A.s. the sequence of finite measures $A \mapsto Z_n(nA)$ satisfies a large deviation principle with rate function $\tilde{\Lambda}^*$: for each measurable subset A of $\mathbb{R}$,

$$\begin{aligned} -\inf_{x\in A^o} \tilde{\Lambda}^*(x) &\leq \liminf_{n\to\infty} \frac{1}{n} \log Z_n(nA) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log Z_n(nA) \leq -\inf_{x\in \bar{A}} \tilde{\Lambda}^*(x), \end{aligned}$$

where A^o denotes the interior of A, and $\bar{A}$ its closure.

For deterministic branching random walk: see Biggins (1977).

Leftmost and rightmost particles

The two critical values t_- and t_+ are related to the positions of leftmost and rightmost particles defined by

$$L_n = \min_{|u|=n} S_u, \;\; R_n = \max_{|u|=n} S_u.$$

Theorem 8 (Huang and Liu 2012)

It is a.s. that

$$\lim_n \frac{L_n}{n} = \Lambda'(t_-),$$

$$\lim_n \frac{R_n}{n} = \Lambda'(t_+).$$

For deterministic branching random walk: see Biggins (1977).

Multifractal spectrum for the BRW

For $x \in \mathbb{R}$, define

$$E(x) = \{u \in \partial T : \lim_n \frac{S_{u|n}}{n} = x\}$$

Theorem 9 (Liang and Liu (2012))

Under simple moment conditions, we have a.s.

(a) If $x < \Lambda'(t_-)$ or $x > \Lambda'(t_+)$, then $E(x) = \emptyset$;

(b) If $x = \Lambda'(t)$ for some $t \in \mathbb{R}, t_- \leq t \leq t_+$, then $E(x) \neq \emptyset$, and

$$\dim E(x) = -\Lambda^*(x) = -\lambda(t).$$

For deterministic environment case and in $\mathbb{R}^d$: Attia and Barral (2012).

Thank you !

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