Schrödinger operator with Sturm potentials —Fractal dimensions

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Hong Kong, Dec. 12, 2012

Outline

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   - recent result
2. Cookie-Cutter-like sets
   - Cantor set
   - Cookie-Cutter set and Cookie-Cutter like set
3. Sketch of proof
   - Bounded variation and bounded covariation
   - Deal with different types
   - Homogeneous Moran set

Schrödinger operator with Sturm potential

Schrödinger operator on $l^2(\mathbb{Z})$:

$$(H_{\alpha,V}\,\psi)_n = \psi_{n-1} + \psi_{n+1} + v_n\psi_n,\ \forall n \in \mathbb{Z},\ \forall \psi \in l^2(\mathbb{Z}).$$

- $(v_n)_{n\in\mathbb{Z}}$: potential. Sturm potential:

$$v_n = V\ \chi_{[1-\alpha,1)}(n\alpha + \phi \mod 1), \quad \forall n \in \mathbb{Z},$$

- $\alpha = [0; a_1, a_2, \cdots]$: frequency
- $V > 0$: coupling; $\phi \in [0,1)$: phase (take $\phi = 0$)
- Spectrum
  $\sigma(H_{\alpha,V}) = \{x \in \mathbb{R}\ :\ xI - H_{\alpha,V}$ no bounded inverse $\} := \sigma$.
- 1989, Bellissard and et. al.(BIST), *Commun. Math. Phys.*

  $\forall V > 0$, α irrational, $\mathscr{L}[\sigma] = 0$.

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Fractal dimensions

Let $\alpha = [0; a_1, a_2, \cdots]$,

$$K_* = \liminf_n (a_1 \cdots a_n)^{1/n}, \quad K^* = \limsup_n (a_1 \cdots a_n)^{1/n}$$

- 2004, L., Wen, *Potential Analysis*, $V > 20$,
  - if $K_* < \infty$, then $0 < \dim_H \sigma < 1$
  - if $K_* = \infty$, then $\dim_H \sigma = 1$.
- L., Qu, Wen, preprint, $V > 25$,
  - if $K^* < \infty$, then $0 < \overline{\dim}_B \sigma < 1$
  - if $K^* = \infty$, then $\overline{\dim}_B \sigma = 1$.

Asymptotic property of Fractal dimension

- 2008, Damanik et. al., *CMP*, $\alpha = [0; a_1, a_2, \cdots]$, $a_n \equiv 1$,

$$\lim_{V\to\infty}(\log V)\,\overline{\dim}_B\sigma = -\log(\sqrt{2}-1).$$

- 2007, L., Peyrière, Wen, *Comptes Randus Mathematique*, $\sup_n a_n < \infty$, $V > 20$, s_*, s^* pre-dim,
  - $\dim_H \sigma \ \le\ s_* \ \le\ s^* \ \le \overline{\dim}_B\sigma$,
  - $\lim\limits_{V\to\infty} s_* \log V = -\log f_*(\alpha)$, $\quad \lim\limits_{V\to\infty} s^* \log V = -\log f^*(\alpha)$.
- 2011, Fan, L., Wen, *Ergodic Theory and Dynamical Systems*, $\sup_n a_n < \infty$, then $\dim_H \sigma \ =\ s_* \ \le\ s^* \ = \overline{\dim}_B\sigma$
- L., Qu, Wen, preprint, $V > 25$, no restriction on $\{a_n\}$,

$$\lim_{V\to\infty}(\log V)\dim_H \sigma = -\log f_*(\alpha),$$
$$\lim_{V\to\infty}(\log V)\overline{\dim}_B\sigma = -\log f^*(\alpha).$$

Case of bounded quotient

2011, Fan, L., Wen, *Ergodic Theory and Dynamical Systems*.

Theorem

Let $\alpha = [0; a_1, a_2, \cdots]$, $\sup_n a_n < \infty$, $V > 20$,

$$\dim_H \sigma = s_*, \quad \overline{\dim}_B \sigma = s^*.$$

Theorem

If $\alpha = [0; a_1, a_2, a_3, \cdots]$ *with* $(a_n)_{n\geq 1}$ *ultimate periodic*, $V > 20$

$$s_* = s^*.$$

For $(a_n)_{n\geq 1}$ ultimately periodic, we give an algorithm so that one can estimation s_* in any accuracy.

Case of unbounded quotient

L., Qu, Wen, preprint.

Theorem

Let $\alpha = [0; a_1, a_2, \cdots]$, $V > 25$,

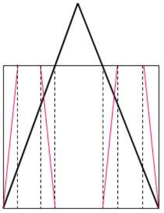
- $\dim_H \sigma = s_*, \quad \overline{\dim}_B \sigma = s^*.$
- $\lim\limits_{V\to\infty} s_* \cdot \log V = -\log f_*(\alpha), \quad \lim\limits_{V\to\infty} s^* \cdot \log V = -\log f^*(\alpha).$
- s_*, s^* *are continuous on* V.

Key techniques

- Cookie-Cutter-like structure
- trace formula
- Homogeneous Moran set

Cantor set

- Let $I = [0,1]$, $f : I \to \mathbb{R}$, $f(x) = \begin{cases} 3x, & 0 \le x \le \frac{1}{2} \\ 3(1-x), & \frac{1}{2} < x \le 1 \end{cases}$.



Then $E = \{x \in I : \forall n \ge 0,\ f^n(x) \in I\} =$ Cantor set, and

$$\dim_H E = \dim_P E = \overline{\dim}_B E = \frac{\log 2}{\log 3} = \sup_{\mu: f-inv} \frac{h_\mu(f)}{\int \log |Df| d\mu}.$$

- Cookie-Cutter: f non-linear.
- Cookie-Cutter-like: change f^n to $f_n \circ f_{n-1} \circ \cdots \circ f_1$

Definition for Cookie-Cutter set

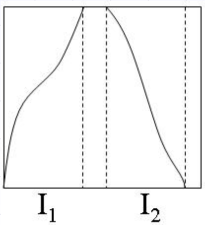
Let $I = [0, 1]$, $I_1, I_2 \subset I$, and $f : I_1 \cup I_2 \to I$ satisfy:

(i) $f|_{I_1}, f|_{I_2}$ is an $1-1$mapping to I.

(ii) f is $c^{1+\gamma}$ Hölder: $|Df(x) - Df(y)| \leq c|x - y|^{\gamma}$.

(iii) f is Expansive, $1 < b \leq |Df(x)| \leq B < \infty$.

- $E = \{x \in I : \forall n \geq 0,\ f^n(x) \in I\}$ Cookie-Cutter set of f.



Definition of Cookie-Cutter like set

Given $\{(f_k, \bigsqcup_{j=1}^{q_k} I_j^k, c_k, \gamma_k, b_k, B_k)\}_{k\geq 1}$ satisfy:

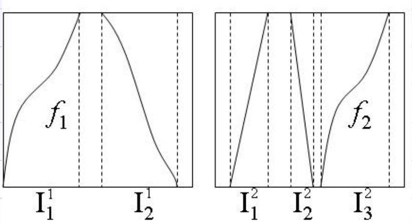
(i') $f_k|_{I_j^k}$ is an $1-1$ mapping to I.

(ii') f_k is $c^{1+\gamma_k}$ Hölder

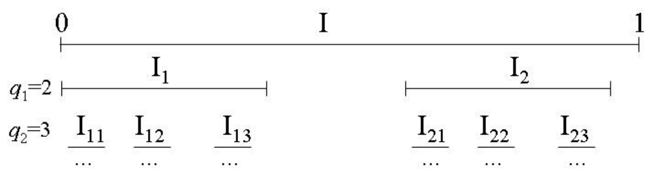
(iii') f_k is Expansive.

- Cookie-Cutter-like set (CC-like set)

$$E = \{x \in I : f_k \circ \cdots \circ f_1(x) \in I, \forall k \geq 0\}.$$



Symbol system and pre-dimension



- Let $\Omega_n = \prod_{k=1}^{n}\{1, \cdots, q_k\}$, $\quad F_n = f_n \circ \cdots \circ f_1$, $\quad \forall \omega \in \Omega_n$,

  F_n is monotone on I_ω, $\qquad F_n(I_\omega) = I$.

- $\forall n > 0$, $\{I_\omega\}_{\omega \in \Omega_n}$ is a covering of E.
- $\forall k \geq 1$, let s_k satisfies $(\exists.1.)$ $\sum_{\omega\in\Omega_k} |I_\omega|^{s_k} = 1$, and

$$s_* = \liminf_k s_k, \qquad s^* = \limsup_k s_k.$$

Ma, Rao, Wen, Sci. China A, 2001

Let E be CC-like set for $\{(f_k, \bigsqcup_{j=1}^{q_k} I_j^k, c_k, \gamma_k, b_k, B_k)\}_{k\geq 1}$.

Theorem

$\dim_H E = s_*$, $\dim_P E = \overline{\dim}_B E = s^*$.

Theorem

s_*, s^* depend continuously on $\{(f_k, \bigsqcup_{j=1}^{q_k} I_j^k, c_k, \gamma_k, b_k, B_k)\}_{k\geq 1}$.

- $\sigma(H_{\alpha,V})$ has a kind of CC-like structure (multi-type).
- Let $\alpha = [0; a_1, a_2, \cdots]$, a_k partly determines f_k.
- $(a_k)_{k\geq 1}$ bounded implies bounded expansive.

key properties [MRW01]

Recall $F_n = f_n \circ \cdots \circ f_1$, $\forall \omega \in \Omega_n$, $F_n(I_\omega) = I$.

- **Bounded variation**. $\exists \xi \geq 1$, $\forall n \geq 1$, $\omega \in \Omega_n$, $x, y \in I_\omega$,

$$\xi^{-1} \leq \frac{|DF_n(x)|}{|DF_n(y)|} < \xi, \qquad |I_\omega| \sim |DF_n(x)|^{-1}.$$

- **Bounded covariation**. $\forall m > k \geq 1$, $\omega_1, \omega_2 \in \Omega_k$, $\tau \in \Omega_{k+1,m}$,

$$\xi^{-2}\frac{|I_{\omega_2 * \tau}|}{|I_{\omega_2}|} \leq \frac{|I_{\omega_1 * \tau}|}{|I_{\omega_1}|} \leq \xi^2 \frac{|I_{\omega_2 * \tau}|}{|I_{\omega_2}|}.$$

- **Existence of Gibbs-like measure**. Given $\beta > 0$, there exist $\eta > 0$ and a probability measure μ_β supported by E such that for any $n \geq 1$ and $\omega_0 \in \Omega_n$, we have

$$\eta^{-1}\frac{|I_{\omega_0}|^\beta}{\sum_{\omega \in \Omega_n} |I_\omega|^\beta} \leq \mu_\beta(I_{\omega_0}) \leq \eta \frac{|I_{\omega_0}|^\beta}{\sum_{\omega \in \Omega_n} |I_\omega|^\beta}.$$

Method to proof bounded variation for spectrum

- Let $I_{n+1} \subset I_n \subset I_{n-1}$ be interval of order $n+1$, n and $n-1$,
  - F_i is monotone on I_i,
  - $F_i(I_i) = [-2, 2], \qquad i = n+1, n, n-1.$
- In stead of $F_n = f_n \circ \cdots \circ f_1$ in CC-like case, we have

$$F_{n+1} = z(F_n, F_{n-1}) S_p(F_n) - F_{n-1} S_{p-1}(F_n), \qquad *$$

where

  - $z(x, y)$ is a solution of the equation $x^2 + y^2 + z^2 - xyz = V^2$,
  - $S_p(\cdot)$ chebishev polynomial,
  - p determined by a_n and type of $I_{n+1}, \ I_n, \ I_{n-1}$.
- From $(*)$, for any $x, y \in I_{n+1}$, we can estimate by iteration

$$\frac{DF_{n+1}(x)}{DF_n(x)}, \qquad \frac{DF_{n+1}(x)}{DF_n(x)} - \frac{DF_{n+1}(y)}{DF_n(y)}.$$

Case of $\{a_n\}$ unbounded

- Illustrate in simple case of $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_1$,

$$\begin{aligned}
\ln \frac{|DF_n(x)|}{|DF_n(y)|} &= \ln |DF_n(x)| - \ln |DF_n(y)| \\
&\le \sum_{i=1}^{n} \left| \ln |Df_i(F_{i-1}(x))| - \ln |Df_i(F_{i-1}(y))| \right| \\
&\le \sum_{i=1}^{n} |Df_i(F_{i-1}(x)) - Df_i(F_{i-1}(y))| \\
&\le \sum_{i=1}^{n} |F_{i-1}(x) - F_{i-1}(y)|^{\gamma} < \ln \xi
\end{aligned}$$

- For any $b > a > 1$,
  replace $\ln b - \ln a < b - a$
  by $\ln b - \ln a < a^{-1}(b - a)$.

Deal with different types for unbounded $\{a_n\}$

For $i = 1, 2, 3,\ m \geq k > 1$, define

$$b_{m,s}^{(k,i)} = \text{Sum}\left\{ |J|^s \ : \begin{array}{l} J \text{ is an order } m \text{ interval,} \\ \text{its order-}k\text{-father is of type } i \end{array} \right\}.$$

We have to estimate

- ratio between $b_{m,s}^{(m,i)}$, $i = 1, 2, 3$.
- ratio between $b_{m,s}^{(k,i)}$, $i = 1, 2, 3,\ m \gg k$.

Fractal dimensions

- It is direct to prove that

$$\dim_H \sigma \ \leq \ s_* \ \leq \ s^* \ \leq \overline{\dim}_B \sigma.$$

- We only need to prove

$$\dim_H \sigma \ \geq \ s_*, \quad \overline{\dim}_B \sigma \leq s^*.$$

- For $\{a_k\}$ unbounded, they are more difficult.
- Our idea come from Feng, Wen, Wu's (Sci. China, 1997) study on Homogeneous Moran set.

Homogeneous Moran sets

- $\mathscr{M}(\{n_k\},\{c_k\})$ a class of Homogeneous Moran sets ($n_k \geq 2$) any $E \in \mathscr{M}(\{n_k\},\{c_k\})$ has a Homogeneous Moran structure:

- Classical Cantor set is in $\mathscr{M}(\{n_k\},\{c_k\})$ with

$$n_k \equiv 2, \quad c_k \equiv \frac{1}{3}.$$

- multi-type, non-linear, throw ε-interval away($\varepsilon \to 0$)

Thank you !