An introduction to minimal sets and the classification of singularities

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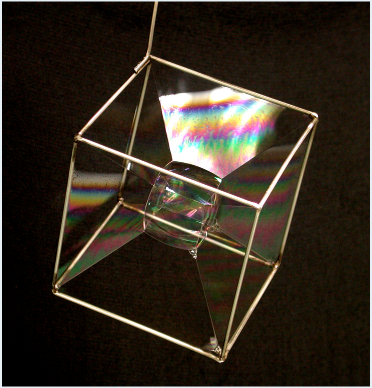
Back ground—Plateau's problem

Aim : try to understand regularity and existence of physical objects that have certain minimizing properties, such as soap films (minimizing "surface area" while spanning a given boundary).

Soap films



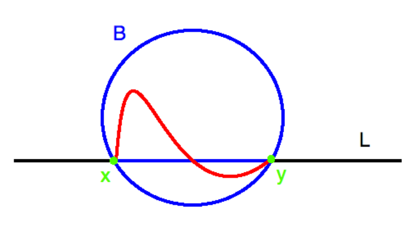
(Almgren-)minimal sets : definition

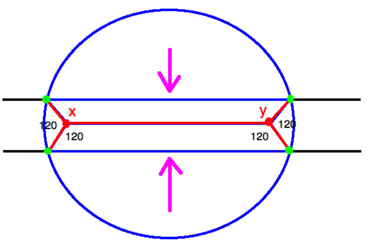
Definition

A closed set $E \subset \mathbb{R}^n$ is said to be a d-dimensional (Almgren-)minimal set in $\mathbb{R}^n$ if for every compact ball B, and every Lipschitz deformation f in B, (i.e. $f : \mathbb{R}^n \to \mathbb{R}^n, f|_{B^c} = id, f(B) \subset B$)

$$H^d(E \cap B) \leq H^d(f(E) \cap B).$$

Example (Lipschitz deformations in a ball)





Main features of Almgren minimal sets

Main differences between minimal sets and other models for Plateau's problem :

- The definition does not come from direct calculus of variation, i.e. no equational description (compared to minimal surfaces and theories with the same idea)
- Algebraic multiplicity is not preserved (unlike most of the other models)
- no orientation (unlike currents)

Therefore

Almost no pure algebraic or dual arguments (e.g. calibration) can apply directly ; slicing does not work directly (we do not know whether the product of a minimal set with $\mathbb{R}$ is minimal or not). Especially for sets of dimension ≥ 1 and codimension ≥ 2, existing methods and ideas are hard to apply.

What is known :

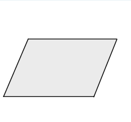
- rectifiable—tangent planes a.e.
- singular points : blow-up limits ("tangent objects") of minimal sets are minimal cones.
- 2-dim minimal sets in $\mathbb{R}^n$: Locally Bi-Hölder or C^1-equivalent to a minimal cone (Guy David 2010, Jean Taylor 1976 for $n = 3$)
- $d-$dim minimal sets for $d \geq 3$? Not known. (2-dim minimal cones are in fact kind of 1-dim minimal set on the sphere)

Goal : list of minimal cones (=classification of singularities). The lists of 1 and 2 dimensional minimal cones in $\mathbb{R}^3$ are known for over a century. But for any higher dimension and codimensions, far from clear.
Procedure : Guess which cones might be minimal, and try to see whether they are or not. Our guess depends generally on intuition.

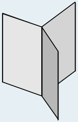
Minimal cones of codimension 1

Up to now we mainly use separation condition and paired calibration methods to prove minimality.

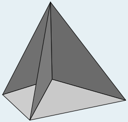
- 2-dimensional minimal cones in $\mathbb{R}^3$: the list is known for over a century



a plane

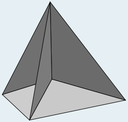


$\mathbb{Y}$



$\mathbb{T}$

- Higher dimensions : Cone over the $n-2$ dimensional skeleton of a regular simplex in $\mathbb{R}^n$ (G.Lawlor & F. Morgan 1994), cone over the $n-2$ dimensional skeleton of a cube in $\mathbb{R}^n$ for $n \geq 4$ (Ken. Brakke 1991)



simplex

cube

competitor

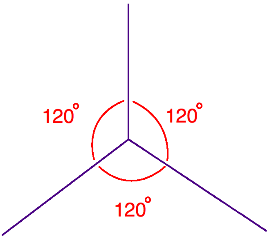
How to find minimal cones ? Unions or products of known minimal cones

The almost orthogonal union of minimal cones

- The union $P_1 \cup P_2$ of two almost orthogonal planes is minimal in $\mathbb{R}^4$. (L. 2010)
- Corollary : There exists a continuous one (or two ?)-parameter family of 2-dimensional minimal cones in $\mathbb{R}^4$.
- Further : Generalization to almost orthogonal unions of several d-planes ($d \geq 2$), as well as almost orthogonal unions of a plane and a 2-dimensional $\mathbb{Y}$ set in $\mathbb{R}^5$. (L. 2010)

$Y \times Y$

The product $Y \times Y$ of two 1-dimensional Y sets is a 2-dimensional minimal cone in $\mathbb{R}^4$. In addition, unlike the unions of planes, this set is an unstable minimal cone, i.e. none of its conic perturbations is minimal. (L. 2012)



Remark

It is in general not known whether the product of two minimal sets should be minimal. Even not known for $E \times \mathbb{R}$ provided that E was minimal. True for E varifying some topological condition on their complementary (L. 2011).

Question :

Are all 2-dim global minimal sets cones ?

Idea

1) The density function of a minimal set E at a point $x \in E$

$$\theta(x,r) = r^{-d}H^d(E \cap B(x,r)) \text{ is monotone.}$$

2) A minimal set of constant density at a point is a minimal cone at this point.
$\Rightarrow$If $\lim_{r\to 0}\theta(x,r) = \lim_{r\to\infty}\theta(x,r)$ for some point x then E is a cone over x.
3) Every global minimal set looks like a minimal cone C at infinity (i.e., its blow-in limits are minimal cones).
So we have to discuss the problem for each minimal cone.

- Planes and $\mathbb{Y}$ sets : yes : topological arguments ;
- Unions of almost orthogonal planes in $\mathbb{R}^4$: Yes (L. 2012) ;
- $\mathbb{T}$ sets : Unknown : potential counter example proposed by David, Hardt, Morgan, etc. Eliminated, and a necessary topological condition for potential counter example is given (L. 2011) ;
- $\mathbb{Y} \times \mathbb{Y}$ in $\mathbb{R}^4$: Unknown.

Thank you !