Diophantine approximation of the orbit of 1 in beta-transformation dynamical system

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Outline

1. Diophantine approximation of the orbits of 1 under beta-transformations

2. β-transformation and β-expansion

3. Distribution of regular cylinders in parameter space

Diophantine approximation of the orbits of 1 under beta-transformations

Backgrounds

- Poincaré Recurrence Theorem
  Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system (probability space) and $B \subset X$ with positive measure. Then
  $$\mu\{x \in B : T^n x \in B \text{ infinitely often (i.o.)}\} = \mu(B).$$
- Birkhoff ergodic theorem
  Assume that μ is ergodic, then
  $$\mu\{x \in X : T^n x \in B \text{ i.o.}\} = 1.$$
- dynamical Borel-Cantelli Lemma or shrinking target problem
  Let $\{B_n\}_{n\geq 1}$ be a sequence of measurable sets with $\mu(B_n)$ decreasing to 0 as $n \to \infty$. Consider the metric properties of the following set
  $$\{x \in X : T^n x \in B_n \text{ i.o.}\}$$

Backgrounds

- well-approximable set
  Let d be a metric on X consistent with the probability space $(X, \mathcal{B}, \mu)$. Given a sequence of balls $B(y_0, r_n)$ with center $y_0 \in X$ and shrinking radius $\{r_n\}$, the set

$$F(y_0, \{r_n\}) := \{x \in X : d(T^n x, y_0) < r_n \text{ i.o.}\}$$

  is called the well-approximable set.
- inhomogeneous Diophantine approximation
  Let $S_\alpha : x \mapsto x + \alpha$ be the irrational rotation map on the circle with $\alpha \notin \mathbb{Q}$. The classic inhomogeneous Diophantine approximation can be written as

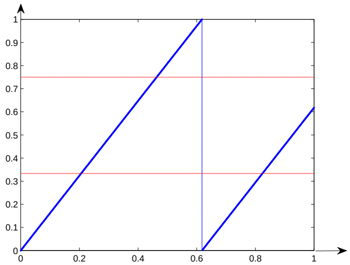
$$\left\{\alpha \in \mathbb{Q}^c : |S_\alpha^n 0 - y_0| < r_n, \text{ i.o. } n \in \mathbb{N}\right\}.$$

beta-transformations

- $\beta > 1$
- β-transformation $T_\beta : [0,1] \to [0,1]$

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

  where $\lfloor \beta x \rfloor$ denotes the integer part of βx.
- Example : $\beta = \frac{1+\sqrt{5}}{2}$



- the orbit of 1 under T_β is crucial (we will see later)

Main problem

- well-approximable set
  Fix a point $x_0 \in [0,1]$ and a given sequence of integers $\{\ell_n\}_{n\geq 1}$.

$$E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \big\{\beta > 1 : |T_\beta^n 1 - x_0| < \beta^{-\ell_n},\ \text{i.o.}\big\}$$

- Question :

$$\dim_{\mathrm{H}} E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = ?$$

- (Persson and Schmeling, 2008)
  When $x_0 = 0$ and $\ell_n = \gamma n (\gamma > 0)$, then

$$\dim_H E(\{\gamma n\}_{n\geq 1}, 0) = \frac{1}{1+\gamma}.$$

Main result

Theorem

Let $x_0 \in [0,1]$ and let $\{\ell_n\}_{n\geq 1}$ be a sequence of integers such that $\ell_n \to \infty$ as $n \to \infty$. Then

$$\dim_H E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \frac{1}{1+\alpha}, \text{ where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

β-transformation and β-expansion

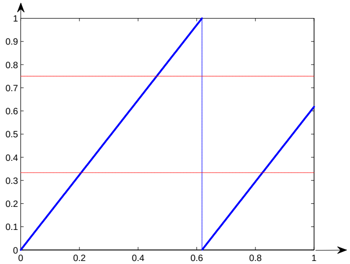
Recall beta-transformations

- $\beta > 1$
- β-transformation $T_\beta : [0,1] \to [0,1]$

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \beta x \rfloor$ denotes the integer part of βx.

- Example : $\beta = \frac{1+\sqrt{5}}{2}$



Invariant measure

- (Rényi 1957)
  When β is not an integer, there exists a unique invariant measure μ_β which is equivalent to the Lebesgue measure.

$$1-\frac{1}{\beta}\leq\frac{d\mu_\beta}{d\mathcal{L}}(x)\leq\frac{1}{1-\frac{1}{\beta}}$$

- Equivalent invariant measure μ_β (Parry 1960 and Gel'fond 1959)

$$\frac{d\mu_\beta}{d\mathcal{L}}(x)=\frac{1}{F(\beta)}\sum_{\substack{n\geq 0\\ x<T_\beta^n(1)}}\frac{1}{\beta^n}$$

where $F(\beta)=\int_0^1\sum_{n\geq 0 x<T_\beta^n(1)}1/\beta^n dx$ is a normalizing factor.

β-expansion

- digit set

$$\mathcal{A} = \begin{cases} \{0, 1, \dots, \beta - 1\} & \text{when } \beta \text{ is an integer} \\ \{0, 1, \dots, \lfloor \beta \rfloor\} & \text{otherwise.} \end{cases}$$

- digit function

$$\varepsilon_1(\cdot, \beta) : [0, 1] \to \mathcal{A} \text{ as } x \mapsto \lfloor \beta x \rfloor$$

- $\varepsilon_n(x, \beta) := \varepsilon_1(T_\beta^{n-1} x, \beta)$
- β-expansion (Rényi, 1957)

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \frac{\varepsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots$$

- notation :

$$\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \varepsilon_2(x, \beta), \dots, \varepsilon_n(x, \beta), \dots)$$

admissible sequence

- admissible sequence/word
  $\Sigma_\beta = \{\omega \in \mathcal{A}^{\mathbb{N}} : \ \exists\, x \in [0,1) \text{ such that } \varepsilon(x,\beta) = \omega\}$

  $\Sigma_\beta^n = \{\omega \in \mathcal{A}^n : \ \exists\, x \in [0,1) \text{ such that } \varepsilon_i(x,\beta) = \omega_i \text{ for all } i = 1, \cdots, n\}$

- β is an integer

$$\Sigma_\beta = \mathcal{A}^{\mathbb{N}} \ \text{(except countable points)}$$

- Example : $\beta_0 = \frac{\sqrt{5}+1}{2}$

$$\Sigma_{\beta_0} = \{\omega \in \{0,1\}^{\mathbb{N}} : \text{ the word } 11 \text{ dosen't appear in } \omega\}$$

- number of admissible words of length n

$$\beta^n \le \sharp\Sigma_\beta^n \le \frac{\beta^{n+1}}{\beta - 1}$$

admissible sequence

- the infinite expansion of the number 1

$$\varepsilon^*(1,\beta) = \begin{cases} \varepsilon(1,\beta) & \text{if there are infinite many } \varepsilon_n(1,\beta) \neq 0 \text{ in } \varepsilon(1,\beta) \\ \big(\varepsilon_1(1,\beta), \cdots, (\varepsilon_n(1,\beta)-1)\big)^\infty & \text{otherwise, where } \varepsilon_n(1,\beta) \text{ is the last non-zero element in } \varepsilon(1,\beta). \end{cases}$$

Theorem (Parry, 1960)

Let $\beta > 1$ be a real number and $\varepsilon^*(1,\beta)$ the infinite expansion of the number 1. Then $\omega \in \Sigma_\beta$ if and only if

$$\sigma^k(\omega) \prec \varepsilon^*(1,\beta) \quad \textit{for all} \quad k \geq 0,$$

where $\prec$ means the lexicographical order.

self-admissible sequence

Corollary (Parry, 1960)

w is the β-expansion of 1 for some $\beta \Longleftrightarrow \sigma^k(w) \preceq w$ for all $k \geq 0$

- self-admissible sequence

$$\sigma^k(w) \preceq w \quad \text{for all} \quad k \geq 0$$

distribution of full cylinders

- cylinder of order n ($(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \in \Sigma_\beta^n$)

$$I_n(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = \{x \in [0,1) : \varepsilon_k(x) = \varepsilon_k, 1 \le k \le n\}$$

- full cylinder

$$\left|I_n(w_1, \cdots, w_n)\right| = \beta^{-n}$$

Theorem

Every $n+1$ consecutive cylinders of order n contains a full cylinder.

- The quantities $n+1$ can be improved, for example, if S_β satisfies the specification property, then $n+1$ can be optimally improved to a constant just depends on β and independent of n. But for the other β's, we still do not the optimal estimate for this quantity.

Corollary

Let $\beta > 1$. For any $y \in [0,1]$ and an integer $\ell \in \mathbb{N}$, the ball $B(y, \beta^{-\ell})$ can be covered by at most $4(\ell+1)$ cylinders of order ℓ in the β-expansion.

Distribution of regular cylinders in parameter space

cylinders in parameter space

- Recall :
  a word $w = (\varepsilon_1, \cdots, \varepsilon_n)$ is called self-admissible if $\sigma^i w \preceq w$ for all $1 \leq i < n$, that is,

$$\sigma^i(\varepsilon_1, \cdots, \varepsilon_n) \preceq \varepsilon_1, \cdots, \varepsilon_n.$$

Definition

Let $(\varepsilon_1, \cdots, \varepsilon_n)$ be self-admissible. A cylinder in the parameter space is defined as

$$I_n^P(\varepsilon_1, \cdots, \varepsilon_n) = \Big\{\beta > 1 : \varepsilon_1(1, \beta) = \varepsilon_1, \cdots, \varepsilon_n(1, \beta) = \varepsilon_n\Big\},$$

i.e., the collection of β for which the β-expansion of 1 begins with $\varepsilon_1, \cdots, \varepsilon_n$.

cylinders in parameter space

- (Schmeling, 1997)
  The cylinder $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is a half-open interval $[\beta_0, \beta_1)$. The left endpoint β_0 is given as the only solution in $(1, \infty)$ to the equation

  $$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n}.$$

  The right endpoint β_1 is given as the limit of the solutions $\{\beta_N\}_{N\geq 1}$ in $(1, \infty)$ to the equations

  $$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} + \frac{\varepsilon_{n+1}}{\beta^{n+1}} + \cdots + \frac{\varepsilon_N}{\beta^N},$$

  where $(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_N)$ is the maximal self-admissible word beginning with $\varepsilon_1, \cdots, \varepsilon_n$ in the lexicographical order. Moreover,

  $$\left|I_n^P(\varepsilon_1, \ldots, \varepsilon_n)\right| \leq \beta_1^{-n}.$$

- Remark : If the left endpoint of $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is 1, then the cylinder will be an open interval. For example, $I_2^P(1,0) = (1, \frac{1+\sqrt{5}}{2})$.

maximal self-admissible sequence

Definition

Let $w = (\varepsilon_1, \cdots, \varepsilon_n)$ be a word of length n. The recurrence time $\tau(w)$ of w is defined as

$$\tau(w) := \inf \left\{ k \geq 1 : \sigma^k(\varepsilon_1, \cdots, \varepsilon_n) = \varepsilon_1, \cdots, \varepsilon_{n-k} \right\}.$$

If such an integer k does not exist, then $\tau(w)$ is defined to be n and w is said to be of full recurrence time.

Theorem

Then the periodic sequence

$$(\varepsilon_1, \cdots, \varepsilon_k)^\infty$$

is the maximal self-admissible sequence beginning with $\varepsilon_1, \cdots, \varepsilon_n$.

lengths of cylinders in parameter space

Theorem

Let $w = (\varepsilon_1, \cdots, \varepsilon_n)$ be self-admissible with $\tau(w) = k$. Let β_0 and β_1 be the left and right endpoints of $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$. Then we have

$$\left|I_n^P(\varepsilon_1, \cdots, \varepsilon_n)\right| \geq \begin{cases} C\beta_1^{-n}, & \text{when } k{=}n\,; \\ C\left(\frac{\varepsilon_{t+1}}{\beta_1^{n+1}} + \cdots + \frac{\varepsilon_k + 1}{\beta_1^{(\ell+1)k}}\right), & \text{otherwise.} \end{cases}$$

where $C := (\beta_0 - 1)^2$ is a constant depending on β_0 ; the integers t and ℓ are given as $\ell k < n \leq (\ell + 1)k$ and $t = n - \ell k$.

- regular cylinder
  When $(\varepsilon_1, \cdots, \varepsilon_n)$ is of full recurrence time, the length

$$C\beta_1^{-n} \leq |I_n^P(\varepsilon_1, \cdots, \varepsilon_n)| \leq \beta_1^{-n},$$

in this case, $I_n^P(\varepsilon_1, \cdots, \varepsilon_n)$ is called regular cylinder.

distribution of regular cylinders in parameter space

Proposition

Let w_1, w_2 be two self-admissible words of length n. Assume that $w_2 \prec w_1$ and w_2 is next to w_1 in the lexicographic order. If $\tau(w_1) < n$, then

$$\tau(w_2) > \tau(w_1).$$

- Denote by C_n^P the collection of cylinders of order n in parameter space.

Corollary

Among any n consecutive cylinders in C_n^P, there is at least one with full recurrence time, hence with regular length.

- This corollary was established for the first time by Persson and Schmeling (2008).

Recall main result

Theorem

Let $x_0 \in [0,1]$ and let $\{\ell_n\}_{n\geq 1}$ be a sequence of integers such that $\ell_n \to \infty$ as $n \to \infty$. Then

$$\dim_H E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \frac{1}{1+\alpha}, \text{ where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

- The generality of $\{\ell_n\}_{n\geq 1}$ arises no extra difficulty compared with special $\{\ell_n\}_{n\geq 1}$.
- The difficulty comes from that $x_0 \neq 0$ has no uniform β-expansion for different β.
- When $x_0 \neq 1$, the set $E(\{\ell\}_{n\geq 1}, x_0)$ can be regarded as a type of shrinking target problem. While $x_0 = 1$, it becomes a type of recurrence properties.
- The notion of the recurrence time of a word in symbolic space is introduced to characterize the lengths and the distribution of cylinders in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$.

Thanks for your attention !