Connectedness of Self-affine Sets Associated with 3-digit Sets

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Abstract

Let $A \in M_2(\mathbb{Z})$ be expanding (all its eigenvalues have modulii >1) with characteristic polynomial $f(x) = x^2 + px \pm 3$. Let $D = \{0, v, kAv + lv\} \subset \mathbb{Z}^2$ be a 3-digit set where $v \in \mathbb{Z}^2 \setminus \{0\}$ and $\{v, Av\}$ is linearly indepenedent. It is well-known that there exists a unique compact set T satisfying $T = A^{-1}(T + D) = \left\{ \sum_{i=1}^{\infty} A^{-i} v_i : v_i \in D \right\}$.

We study the connectedness of T and give a complete characterization of T for the two cases (i) $k = 0$ and (ii) $l = 0$, in terms of l and k respectively.

Introduction

- Let $A \in M_2(\mathbb{Z})$ be expanding (all its eigenvalues have modulii >1) with char. poly. $f(x) = x^2 + px + q$.
- Let $\mathscr{D} = \{d_1, d_2, \ldots, d_{|q|}\} \subset \mathbb{Z}^2$. $\mathscr{D}$ is called a $|q|$-*digit set*.
- Let $D = \{0, 1, 2, \ldots, |q|-1\}$ and $\mathscr{D} = Dv$, where $v \in \mathbb{Z}^2 \setminus \{0\}$, $\mathscr{D}$ is called a *consecutive collinear (CC) digit set*.
- $\exists!$ compact set $T = T(A, \mathscr{D})$ satisfying
  $T = A^{-1}(T + \mathscr{D}) = \left\{ \sum_{i=1}^{\infty} A^{-i} v_i : v_i \in \mathscr{D} \right\}$. T is said to be *self-affine*.
- If $intT \neq \emptyset$, then T is called a *self-affine tile*.
- The above definitions and results can be generalized to $\mathbb{R}^n$.

Some known results

- Any T with a 2-digit set is always pathewise connected (Hacon et al).
- *Height Reducing Property* of $f(x)$ (monic and expanding): $\exists g(x)$ (monic) s.t. $h(x) = g(x)f(x) = x^m + c_{m-1}x^{m-1} + \cdots + c_1 x + c_0$ with $|c_0| = |f(0)|$ and $|c_i| < |f(0)| (i \neq 0)$.
- Any planar T with a CC digit set is connected (Kirat and Lau). They conjectured that T (in higher dim.) with a CC digit set is connected. Akiyama and Gjini solved it up to deg. 4.
- Laarakker and Curry considered the connectedness of T generated by A with rational eigenvalues and a *centered canonical* digit set.
- Deng and Lau, and Kirat studied a class of planar self-affine tiles generated by *product digit set*.
- Applying results of Bandt and Wang, Leung and Lau proved that: T with a CC digit set is *disklike* (homeo. to the closed unit disk) iff $2|p| \leq |q+2|$. Akiyama and Loridant re-established this result by parametrizing ∂T.

Our problem

- Let $A \in M_2(\mathbb{Z})$ be expanding with char. poly. $f(x) = x^2 + px \pm 3$.
- Let $\mathscr{D} = \{0, v, kAv + lv\} \subset \mathbb{Z}^2$ be a 3-digit set where $v \in \mathbb{Z}^2 \setminus \{0\}$ and $\{v, Av\}$ is lin. indep.
- $\exists!$ compact set T satisfying $T = A^{-1}(T + \mathscr{D}) = \left\{ \sum_{i=1}^{\infty} A^{-i} v_i : v_i \in \mathscr{D} \right\}$.
- Find conditions (in terms of k, l and $f(x)$) for T to be connected.
- We solved the problem for the two cases: (i) $k = 0 (l \neq 0)$ and (ii) $l = 0 (k \neq 0)$.
- If $f(x) = x^2 - x - 3$ and $\mathscr{D} = \{0, 1, b\}v$, then T is connected if $8/5 \leq b \leq 8/3$ and discon. if $b < (\sqrt{13} - 1)/2$ or $b > (\sqrt{13} + 5)/2$ (Tan).
- 10 eligible char. poly.: $x^2 \pm 3$; $x^2 \pm x + 3$; $x^2 \pm 2x + 3$;$x^2 \pm 3x + 3$;$x^2 \pm x - 3$ (Bandt and Gelbrich)

Main result 1

Theorem

If $\mathscr{D} = \{0, 1, m\}v$ *where* $2 \leq m \in \mathbb{Z}$, *then*
(i) when $m = 2, T$ *is always a connected tile;*
(ii) when $m \geq 4, T$ *is always a disconnected set;*
(iii) when $m = 3$, T *is connected if* $f(x) = x^2 \pm 2x + 3$ *or* $x^2 \pm 3x + 3$ *or* $x^2 \pm x - 3$ *and discon. if* $f(x) = x^2 \pm 3$ *or* $x^2 \pm x + 3$.

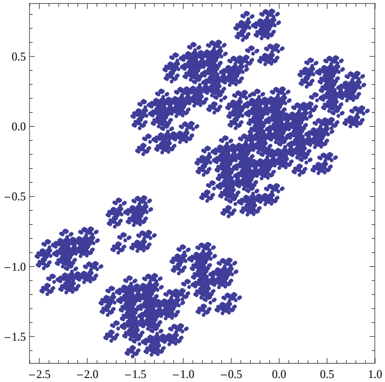
Main result 2

Theorem

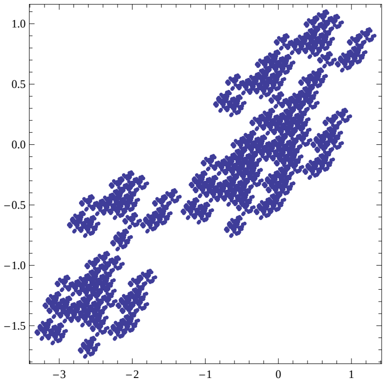
If $\mathscr{D} = \{0,1,b\}v$ where $b > 1$, then
Case 1:$f(x) = x^2 \pm x + 3$ T is discon. if $b \geq 67/25$ or $b \leq 67/42$;
Case 2:$f(x) = x^2 \pm 2x + 3$ T is discon. if $b \geq 37/10$ or $b \leq 37/27$;
Case 3:$f(x) = x^2 \pm 3x + 3$ T is discon. if $b \geq 33/10$ or $b \leq 33/23$;
Case 4:$f(x) = x^2 \pm x - 3$ T is discon. if $b > 19/5$ or $b < 19/14$.

- We conjecture that $\exists c \geq 2$ (dependent on $f(x)$) s.t. T is connected iff $c/(c-1) < b \leq c$.

Some figures ($p \neq 0, m = 4$)

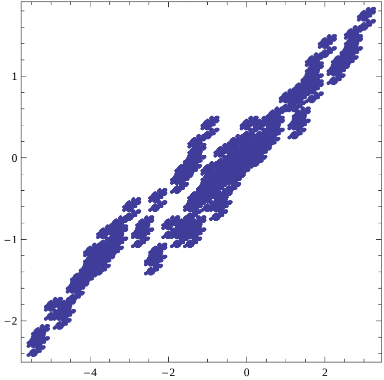


$f(x) = x^2 + x + 3$

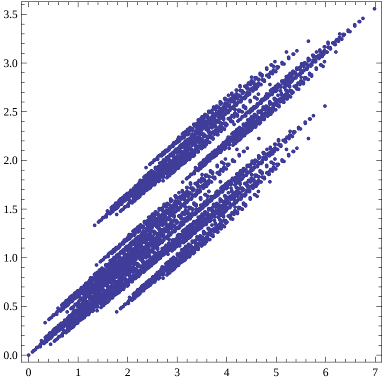


$f(x) = x^2 + 2x + 3$

Some figures ($p \neq 0, m = 4$)

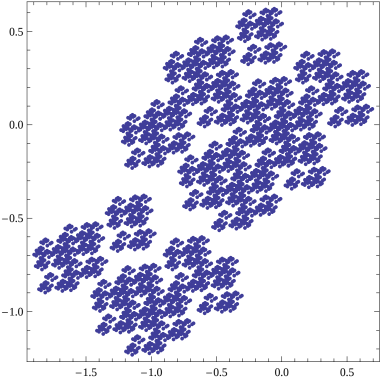


$f(x) = x^2 + 3x + 3$

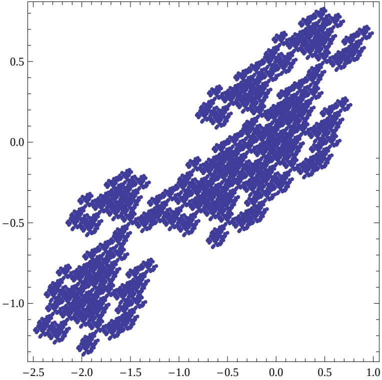


$f(x) = x^2 + x - 3$

Some figures ($p \neq 0, m = 3$)

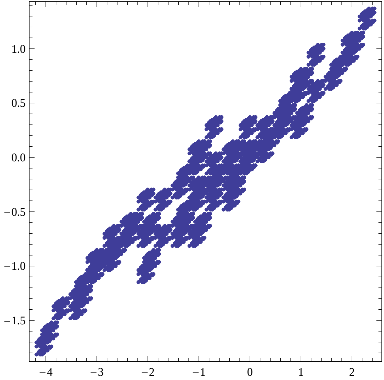


$f(x) = x^2 + x + 3$

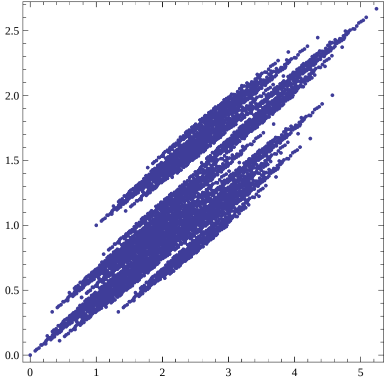


$f(x) = x^2 + 2x + 3$

Some figures ($p \neq 0, m = 3$)



$f(x) = x^2 + 3x + 3$



$f(x) = x^2 + x - 3$

Results on $|detA| > 3$

Let $f(x) = x^2 + px \pm q, p > 0, q \geq 2$ and $\mathscr{D} = Dv$ be a collinear q-digit set s.t. $D = \{0 = \mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_\mathrm{q}\} \subset \mathbb{Z}$ in incr. order with $\mathrm{d}_{\mathrm{i}+1} - \mathrm{d}_\mathrm{i} = 1$ or 2 for all i, $\mathrm{d}_{\mathrm{j}+1} - \mathrm{d}_\mathrm{j} = 1$ for at least one j and $\mathrm{d}_{\mathrm{k}+1} - \mathrm{d}_\mathrm{k} = 2$ for at least one k.

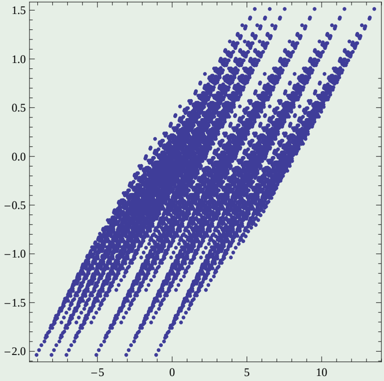
Theorem

(i) Let $f(x) = x^2 + px + q$ with $2p > q + 2$ and $\{0, \pm 1, \pm 2, \ldots, \pm q\} \subset \triangle D$. Then T is connected if $2p - 2 \in \triangle D$ and $2q - p \in \triangle D$.
(ii) Let $f(x) = x^2 + px - q$ with $2p > q - 2$ and $\{0, \pm 1, \pm 2, \ldots, \pm (q-1)\} \subset \triangle D$. Then T is connected if $2p + 1 \in \triangle D$ and $2q - p - 2 \in \triangle D$.

Example

Let A be the companion matrix of $f(x) = x^2 + 5x + 6, v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

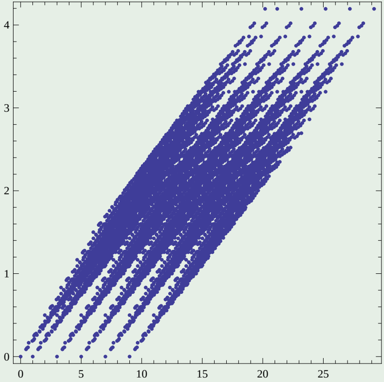
$D = \{0, 1, 2, 4, 6, 8\}$. T is connected.



Example

Let A be the companion matrix of $f(x) = x^2 + 4x - 6, v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$,

$D = \{0, 1, 3, 5, 7, 9\}$. T is connected.



Tool #1

- $\mathscr{E} := \{(d_i, d_j) : (T + d_i) \cap (T + d_j) \neq \emptyset, d_i, d_j \in \mathscr{D}\}$, the set of edges for $\mathscr{D}$.
- d_i and d_j are said to be $\mathscr{E}$-connected if $\exists$ a finite sequence $\{d_{j_1}, \ldots, d_{j_k}\} \subset \mathscr{D}$ s.t. $d_i = d_{j_i}, d_j = d_{j_k}$ and $(d_{j_l}, d_{j_{l+1}}) \in \mathscr{E}, 1 \leq l \leq k-1\}$.
- $(d_i, d_j) \in \mathscr{E}$ iff $d_i - d_j = \sum_{k=1}^{\infty} A^{-i} v_i$, where $v_i \in \triangle\mathscr{D} := \mathscr{D} - \mathscr{D}$.
- T is connected iff any two $d_i, d_j \in \mathscr{D}$ are $\mathscr{E}$-connected.

Tool #2

- Define α_i, β_i by $A^{-i}v = \alpha_i v + \beta_i Av, i = 1, 2, \ldots,$ where $\{v, Av\}$is lin. indep.
- $q\alpha_{i+2} + p\alpha_{i+1} + \alpha_i = 0$ and $q\beta_{i+2} + p\beta_{i+1} + \beta_i = 0$,
  $\alpha_1 = -p/q, \alpha_2 = (p^2 - q)/q^2; \beta_1 = -1/q, \beta_2 = p/q^2$
- α_i, β_ican also be expressed in terms of the roots of $qx^2 + px + 1 = 0$.
- $\tilde{\alpha} := \sum_{i=1}^{\infty} |\alpha_i|, \tilde{\beta} := \sum_{i=1}^{\infty} |\beta_i|$.
- $\tilde{\alpha} \leq \sum_{i=1}^{n-1} |\alpha_i| + \frac{2q^{-(n-1)/2}}{(1-q^{-1/2})(4q-p^2)^{1/2}}$,
  $\tilde{\beta} \leq \sum_{i=1}^{n-1} |\beta_i| + \frac{2q^{-n/2}}{(1-q^{-1/2})(4q-p^2)^{1/2}}$

Tool #3

- $L := \{\gamma v + \delta Av : \gamma, \delta \in \mathbb{Z}\}$ the *lattice* generated by $\{v, Av\}$.
- For $l \in L \setminus \{0\}$, $T + l$ is called a neighbour of T if $T \cap (T + l) \neq \emptyset$.
- Let $\mathscr{D} = Dv$ be a collinear digit set. $T + l$ is a nb. of T iff $l = \sum_{i=1}^{\infty} b_i A^{-i} v \in T - T$, where $b_i \in \triangle D$.
- If $T + l$ is a nb. of T, where $l = \gamma v + \delta Av = \sum_{i=1}^{\infty} b_i A^{-i} v$, then $|\gamma| \leq max_i |b_i| \tilde{\alpha}, |\delta| \leq max_i |b_i| \tilde{\beta}$.
  Moreover, $T + l'$ is also a nb. of T, where $l' = Al - b_1 v = \gamma' v + \delta' Av = -(q\delta + b_1)v + (\gamma - p\delta)Av$.
- Let $T_1 = T(A, \mathscr{D})$ and $T_2 = T(-A, \mathscr{D})$. Then $T_1 + l$ is a nb. of T_1 iff $T_2 + l$ is a nb. of T_2.
- If the char. poly. of A is $x^2 + px + q$ and that of B is $x^2 - px + q$. Then $T(A, \mathscr{D})$ is connected iff $T(B, \mathscr{D})$ is connected.

Proof of main result 1 ($p = 0$)

When $p = 0$

- Consider the case $f(x) = x^2 - q(q \geq 2)$ only.
- $f(A) = 0 \Rightarrow A^{-2} = q^{-1}I \Rightarrow A^{-2}v = q^{-1}v$
- Let $y = \sum_{i=1}^{\infty} a_i A^{-i} v \in T$, where $a_i \in D = \{0 = \mathrm{d}_1, \mathrm{d}_2, \ldots, \mathrm{d}_\mathrm{q}\} \subset \mathbb{Z}$. Then $y = (\sum_{i=1}^{\infty} a_{2i-1} q^{-i})Av + (\sum_{i=1}^{\infty} a_{2i} q^{-i})v$.
- T connected $\Rightarrow \{\sum_{i=1}^{\infty} a_{2i-1} q^{-i} : a_{2i-1} \in D\}$ and $\{\sum_{i=1}^{\infty} a_{2i} q^{-i} : a_{2i} \in D\}$ are intervals$\Rightarrow$ $D = \{0, 1, 2, ..., q-1\}a$ for some $a > 0$.
- $D = \{0, 1, 2, ..., q-1\}a \Rightarrow T$ connected (HRP)

Proof of main result 1 ($p \neq 0, m \geq 4$)

- Consider $p > 0$ only.
- Prove that $T \cap (T + mv) = \emptyset$ and $(T + v) \cap (T + mv) = \emptyset$. Hence T is disconnected.

Proof of main result 1 ($p \neq 0, m = 3$)

- Consider $p > 0$ only.
- Prove that $T \cap (T + 2v) = \emptyset$ and $(T + v) \cap (T + 3v) = \emptyset$ for $f(x) = x^2 \pm 2x + 3$ or $x^2 \pm 3x + 3$ or $x^2 \pm x - 3$. Hence T is disconnected.
- For the other cases, show that $T \cap (T + v) \neq \emptyset$ and $(T + v) \cap (T + 3v) \neq \emptyset$ (equivalently, $T \cap (T + 2v) \neq \emptyset$).
- Example: $f(x) = x^2 - x - 3$.
  $0 = f(A) = A^2 - A - 3I \Rightarrow v = -2A^{-1}v + 3A^{-2}v \in T - T \Rightarrow$
  $T \cap (T + v) \neq \emptyset$.
  $0 = f(A)(2A - I) = (2A - 3I)(A^2 - I) - 3A \Rightarrow 2A - 3I = 3\sum_{i=1}^{\infty} A^{-2i-1} \Rightarrow$
  $2v = 3A^{-1}v + 3\sum_{i=1}^{\infty} A^{-2i}v \in T - T \Rightarrow (T + v) \cap (T + 3v) \neq \emptyset$.

Proof of main result 2

- Consider $b \geq 2$ only. If $1 < b < 2$, then $b/(b-1) > 2$. Replace $\mathscr{D}$ by $\mathscr{D}' = \{0, 1, b/(b-1)\}v$.
- Example: $f(x) = x^2 + x + 3$
- T is connected $\Rightarrow (b-y)v = \sum_{i=1}^{\infty} b_i A^{-i} v$ holds for $y = 0$ or 1, where $b_i \in \triangle D = \{0, \pm 1, \pm(b-1), \pm b\}$
  $\Rightarrow T + (b-y)v$ is a nb. of $T \Rightarrow T + A(b-y)v - b_1 v$ is a nb. of $T \Rightarrow T + A^2(b-y)v - b_1 Av - b_2 v$ is a nb.
  $\Rightarrow T - (3(b-y) + b_2)v - (b - y + b_1)Av$ is a nb.$\Rightarrow |3(b-y) + b_2| < \tilde{\alpha} b$ and $|3(b-y) + b_2| \geq 3(b-1) - b = 2b - 3 \Rightarrow b < 67/25$. On the other hand, $b/(b-1) < 67/25 \Rightarrow b > 67/42$.

Proof of results on $|detA| > 3$

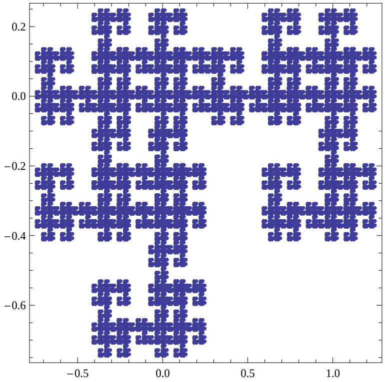
- Find two expressions for v: $v = \sum_{i=1}^{\infty} b_i A^{-i} v$ (*) and $v = \sum_{i=1}^{\infty} b'_i A^{-i} v$ (**).
- Adding them to give an expression for $2v$.

Non-collinear 3-digit sets

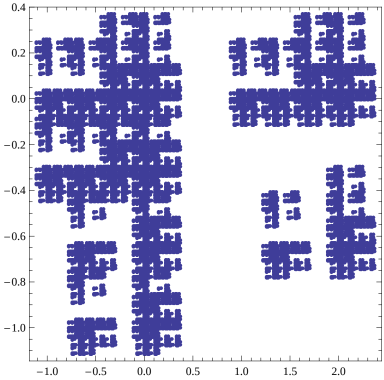
Theorem

Let A be a 2×2 integral expanding matrix with $|detA| = 3$, and let $\mathscr{D} = \{0, v, kAv\}$ be a digit set where $k \in \mathbb{Z} \backslash \{0\}$ and $v \in \mathbb{R}^2$ s.t. $\{v, Av\}$ is linearly independent. Then $T(A, \mathscr{D})$ is connected iff $k = \pm 1$.

$f(x) = x^2 + 3, v = (1,0)^t$

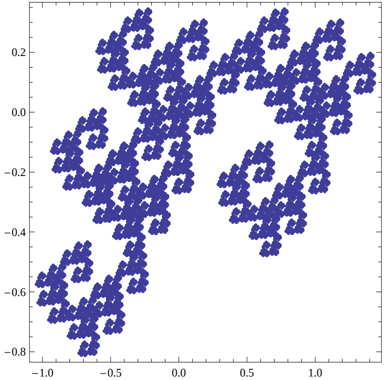


$k = 1$, T connected

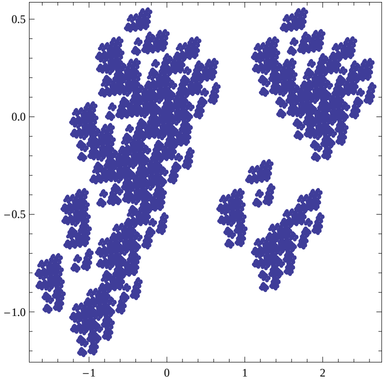


$k = 2$, T disconnected

$f(x) = x^2 + x + 3, v = (1,0)^t$

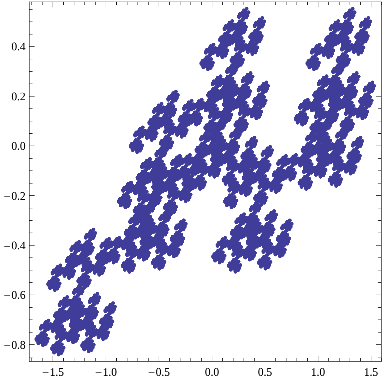


$k = 1,\ T$ connected

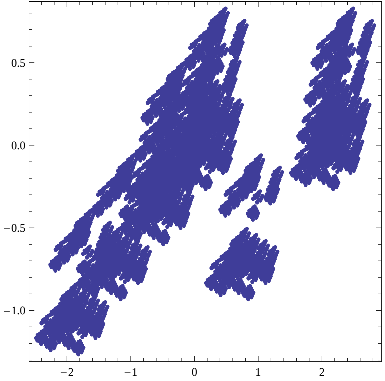


$k = 2,\ T$ disconnected

$f(x) = x^2 + 2x + 3, v = (1,0)^t$

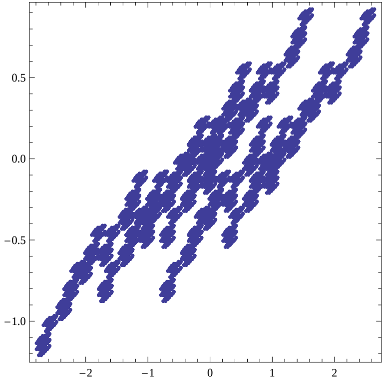


$k = 1,\ T$ connected

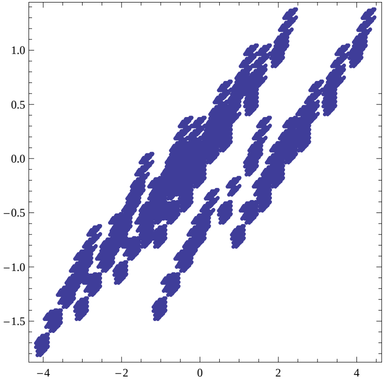


$k = 2,\ T$ disconnected

$f(x) = x^2 + 3x + 3, v = (1,0)^t$

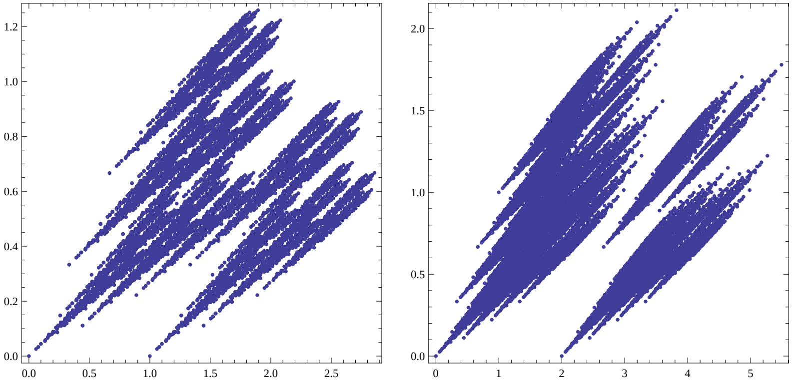


$k = 1,\ T$ connected



$k = 2,\ T$ disconnected

$f(x) = x^2 + x - 3, v = (1,0)^t$



$k = 1$, T connected

$k = 2$, T disconnected

Proof of Theorem (non-collinear 3-digit set)(i)

Proof.

$f(x) = x^2 + x + 3$

(i) $k = 1$, i.e., $\mathscr{D} = \{0, v, Av\}$ and $\triangle\mathscr{D} = \{0, \pm v, \pm(Av - v), \pm Av\}$.

$0 = f(A)(A - I) \Rightarrow v = \sum_{i=1}^{\infty} A^{-3i}(-2Av + 2v) =$

$\sum_{i=1}^{\infty} A^{-3i}(A^{-2}(-v) + A^{-3}(v - Av) + A^{-4}(Av)) \Rightarrow T \cap (T + v) \neq \emptyset$

Moreover,

$Av = \sum_{i=1}^{\infty} A^{-3i}(A^{-1}(-v) + A^{-2}(v - Av) + A^{-3}(Av)) \Rightarrow T \cap (T + Av) \neq \emptyset$.

Hence T is connected. $\square$

Proof of Theorem (non-collinear 3-digit set)(ii)

Proof.

(cont'd) (ii) $k = -1$, i.e., $\mathscr{D} = \{0, v, -Av\}$ and
$\triangle\mathscr{D} = \{0, \pm v, \pm Av, \pm(Av + v)\}$.
$0 = f(A) \Rightarrow I = (-A - 2I)(A^2 + I)^{-1}$
$\Rightarrow v = -A^{-1}v - 2A^{-2}v + A^{-3}v + 2A^{-4}v - A^{-5}v - 2A^{-6}v + A^{-7}v + 2A^{-8}v - \cdots = A^{-2}(-Av - v) + A^{-3}(-Av) + A^{-4}(Av + v) + A^{-5}(Av) + A^{-6}(-Av - v) + A^{-7}(-Av) + A^{-8}(Av + v) + A^{-9}(Av) + \cdots \in T - T$
$\Rightarrow T \cap (T + v) \neq \emptyset$.
Multiply the above by A,
$Av = A^{-1}(-Av - v) + A^{-2}(-Av) + A^{-3}(Av + v) + A^{-4}(Av) + A^{-5}(-Av - v) + A^{-6}(-Av) + A^{-7}(Av + v) + A^{-8}(Av) + \cdots \in T - T$
$\Rightarrow T \cap (T + Av) \neq \emptyset$.
Hence T is connected.

□

Proof of Theorem (non-collinear 3-digit set)(iii)

Proof.

(cont'd) (iii) $|k| > 1$, i.e., $\mathscr{D} = \{0, v.kv\}$ and $\triangle\mathscr{D} = \{0, \pm v, \pm(k-1)v, kv\}$.
A pt. in $T - T$ can be written as
$X = \sum_{i=1}^{\infty} A^{-i}(k_i Av + l_i v), k_i Av + l_i \in \triangle\mathscr{D}$. Using $A^{-i}v = \alpha_i v + \beta_i Av$, X can be rewritten as
$X = (k_1 + \sum_{i=1}^{\infty}(k_{i+1} + l_i)\alpha_i)\, v + (\sum_{i=1}^{\infty}(k_{i+1} + l_i)\beta_i)\, Av := \mathscr{L} v + \mathscr{K} Av$.
$|l_i + k_{i+1}| \leq 1 + |k|$ and
$\tilde{\beta} < 0.63 \Rightarrow |\mathscr{K}| < 0.63(1 + |k|) < |k| \Rightarrow T \cap (T + kAv) = \emptyset$ and
$(T + v) \cap (T + kAv) = \emptyset \Rightarrow T$ diconnected. $\square$

Further questions

- How about $\mathscr{D} = \{0, v, kAv + lv\}(k \neq 0, l \neq 0)$?
- How about $\mathscr{D}$ with more than three elements?

References

K.S. Leung and J.J. Luo, *Connectedness of planar self-affine sets associated with non-consecutive collinear digits*, J. Math. Anal. Appl. 395 (2012) 208-217.
http://arxiv.org/pdf/1208.3759v1.pdf

End

Thank you