Regularity of the entropy for random walks

François Ledrappier
University of Notre Dame/Université Paris 6

香港中文大學, 2012/12/11

- Free Groups

- Hyperbolic Groups

- Manifolds of negative curvature

- IFS

Free Groups

$\mathbb{F}_d$ free group with generators S

$$S = \{i^{\pm 1}; i = 1, \ldots, d\},$$

$|x|$ the length of the S-name of x and
$\partial\mathbb{F}_d$ the boundary at infinity of $\mathbb{F}_d$.

$\partial\mathbb{F}_d$ can be seen as the set of infinite reduced words in letters from S; the distance ρ extends to $\partial\mathbb{F}_d$, where

$$\rho(x, x) = 0, \rho(x, x') := e^{-x \wedge x'}$$

and, for $x \neq x'$, $x \wedge x'$ is the number of common initial letters in the S-name of x and x'.

F a finite subset of $\mathbb{F}_d$ such that $\cup_n F^n = \mathbb{F}_d$.

$\mathcal{P}(F)$ the set of probability measures p on $\mathbb{F}_d$ such that the support of p is F.

$X_n = \omega_1 \omega_2 \cdots \omega_n$ the right random walk associated with p, where ω_i are i.i.d. random elements of $\mathbb{F}_d$ with distribution p.

$p^{(n)}$ the distribution of X_n.

Define, by subadditivity:

$$\ell_p \ := \ \lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{F}_d} d(x,e)p^{(n)}(x)$$

$$h_p \ := \ \lim_{n\to\infty} -\frac{1}{n} \sum_{x\in\mathbb{F}_d} p^{(n)}(x) \ln p^{(n)}(x).$$

ℓ_p is the *linear drift* of the random walk and h_p is the *entropy* of the random walk ([Avez, 1972]).
h_p/ℓ_p is the Hausdorff dimension of the exit measure p^∞ (L, 2001).

Theorem [L, 2012] *The mappings* $p \mapsto \ell_p$ *and* $p \mapsto h_p$ *are real analytic on* $\mathcal{P}(F)$.

The proof rests on formulas giving ℓ_p and h_p.

There is a unique *stationary* probability measure p^∞ on $\partial\mathbb{F}_d$, i.e. p^∞ satisfies:

$$p^\infty(A) = \sum_{x\in F} p(x) p^\infty(x^{-1}A).$$

Then, by [Kaimanovich & Vershik, 1983] and [Derriennic, 1980],

$$h_p = -\sum_{x\in F}\left(\int_{\partial\mathbb{F}_d} \ln \frac{d(x^{-1})_* p^\infty}{dp^\infty}(\xi)\, dp^\infty(\xi)\right) p(x),$$

$$\ell_p = \sum_{x\in F}\left(\int_{\partial\mathbb{F}_d} B_\xi(x^{-1})\, dp^\infty(\xi)\right) p(x).$$

$\frac{dx_* p^\infty}{dp^\infty}(\xi)$ is the Martin kernel [Derriennic, 1975] :

$$\frac{dx_* p^\infty}{dp^\infty}(\xi) = K_\xi(x) := \lim_{y \to \xi} \frac{G(x^{-1}y)}{G(y)},$$

where $G(z) = \sum_n p^{(n)}(z)$

and B_ξ is the Busemann function:

$$B_\xi(x) = \lim_{y \to \xi} |x^{-1}y| - |y|.$$

We prove the regularity of all the elements of the above formulas.

Let $\mathcal{K}_\alpha$ be the Banach space of Hölder continuous real functions on $\partial\mathbb{F}_d$.

Fact [L, 2001] *For each $p \in \mathcal{P}(F)$, there is $\alpha > 0$ such that the mapping $p \mapsto p^\infty$ is real analytic from a neighbourhood of p into the dual space $\mathcal{K}_\alpha^*$.*

Indeed, for α small enough, the natural Markov operator, which depends analytically on p, preserves $\mathcal{K}_\alpha$ and p^∞ is an eigenvector for an isolated eigenvalue of the dual operator

Since, for a fixed x, $\xi \mapsto B_\xi(x) \in \mathcal{K}_\alpha$, for all α, the regularity of $p \mapsto \ell_p$ follows.

From the proof in [Derriennic, 1975], follows that there is $\alpha > 0$ such that, for all fixed x, $\xi \mapsto \ln K_\xi(x)$ belongs to $\mathcal{K}_\alpha$. The regularity of $p \mapsto h_p$ follows from

Proposition [L, 2010] *For each $p \in \mathcal{P}(F)$, each $x \in \mathbb{F}_d$, there is $\alpha > 0$ such that the mapping $p \mapsto \ln K_\xi(x)$ is real analytic from a neighbourhood of p into the space $\mathcal{K}_\alpha$.*

Derriennic used the Birkhoff Contraction Theorem for linear maps that preserve cones. The proof of the Proposition uses a recent complex extension of Birkhoff Theorem due to H.H. Rugh (2010).

Previous works:

- D. Ruelle, Analyticity properties of the characteristic exponents of random matrix products (1979)
- Y. Peres, Domains of analytic continuation for the top Lyapunov exponent (1992)
- A. Erschler & V.A. Kaimanovich, Continuity of entropy for random walks on hyperbolic groups.
- G. Han & B. Marcus, Analyticity of entropy rate of hidden Markov chains (2006).
- G. Han, B. Marcus & Y. Peres, A note on a complex Hilbert metric with application to domain of analyticity for entropy rate of hidden Markov processes (2011).

Extension 1: Hyperbolic groups

A group G is called hyperbolic if geodesic triangles in the Cayley graph are thin.

Consider now G a finitely generated hyperbolic group. As before, we note

S a symmetric generator,
d the associated word distance,
F a finite subset of G such that $\cup_n F^n = G$,
$\mathcal{P}(F)$ the set of probability measures p on G such that the support of p is F
and $X_n = \omega_1 \omega_2 \cdots \omega_n$ the right random walk associated with p.

Define again the linear drift and the entropy:

$$\ell_p := \lim_{n\to\infty} \frac{1}{n} \sum_{g\in G} d(g,e) p^{(n)}(g)$$

$$h_p := \lim_{n\to\infty} -\frac{1}{n} \sum_{g\in G} p^{(n)}(g) \ln p^{(n)}(g)$$

Theorem [L, 2012] *With the above notations, if G is a finitely generated hyperbolic group, the mappings $p \mapsto \ell_p$ and $p \mapsto h_p$ are Lipschitz continuous on $\mathcal{P}(F)$.*

Boundaries of G:

Geometric boundary $\partial_G G$: geodesic rays, up to bounded Hausdorff distance away from one another.

Martin boundary $(\partial_M G, p)$: compactification by the functions $x \mapsto \frac{G(x^{-1}y)}{G(y)}$, as $y \to \infty$.

Busemann boundary $\partial_B G$: compactification by the functions $x \mapsto d(x,y) - d(e,y)$, as $y \to \infty$.

[Ancona, 1990] For a finitely supported random walk on a hyperbolic group, the Martin boundary and the geometric boundary coincide and there is a unique stationary measure p^∞ on this boundary.

[Izumi, Neshveyev & Okayasu, 2008]
Moreover, $\ln K_\xi \in \mathcal{K}_\alpha$ for some $\alpha > 0$.

[Coornaert & Papadopoulos, 2001]
The Busemann boundary has a nice Markov structure.

BUT...

The Busemann boundary and the geometric boundary do not necessarily coincide (see the discussion in [Webster & Winchester, 2005]).

There might be several stationary measures on the Busemann boundary.

The geometric boundary doesn't necessary have a nice Markov structure.

There are still formulas for h_p and ℓ_p:

$$h_p = -\sum_{x\in F}\left(\int_{\partial_G G}\ln K_\xi(x^{-1})\,dp^\infty(\xi)\right)p(x),$$
$$\ell_p = \sup_m\left\{\sum_{x\in F}\left(\int_{\partial_B G}B_\xi(x^{-1})\,dm\right)p(x)\right\},$$

where the sup in the second formula is over the stationary probability measures on $\partial_B G$; see [Kaimanovich (2000)] for the entropy, [Karlsson & L (2007)] for the linear drift.

Assume **(BA)**: The Busemann boundary coincide with the geometry boundary. Then,

Proposition *Under (BA), for each* $p \in \mathcal{P}(F)$*, there is* $\alpha > 0$ *such that the mapping* $p \mapsto p^\infty$ *is real analytic from a neighbourhood of* p *into the dual space* $\mathcal{K}_\alpha^*$.

It was indeed observed in [Bjorklund, 2010] that, under (BA), p^∞ is an eigenvector for an isolated eigenvalue of the natural dual Markov operator in the suitable $\mathcal{K}_\alpha$.

Corollary *Under (BA), the mapping* $p \mapsto \ell_p$ *is real analytic on* $\mathcal{P}(F)$.

Question *Under (BA), the mapping* $p \mapsto h_p$ *is* C^∞ *on* $\mathcal{P}(F)$.

The other result in the case of hyperbolic groups is for symmetric probability measures. If F is a symmetric set, denote $\mathcal{P}_\sigma(F)$ the set of probability measures with support F and such that $p(x) = p(x^{-1})$.

Theorem [Mathieu (2012)] *With the above notations, if G is a finitely generated hyperbolic group, the mappings $p \mapsto \ell_p$ and $p \mapsto h_p$ are C^1 on $\mathcal{P}_\sigma(F)$.*

Moreover, Mathieu has an expression for the derivative.

Extension 2: Manifolds of negative curvature

M a compact closed manifold
$\widetilde{M}$ the universal cover
$\mathcal{P}(M)$ the set of C^∞ metrics of negative curvature of M, endowed with the C^2 topology.

For $g \in \mathcal{P}(M)$, $\tilde{g}$ the lifted metric on $\widetilde{M}$, $\mathbb{P}_g$ the family of probabilities on $C(\mathbb{R}_+, \widetilde{M})$ that describe the Brownian motion associated to the metric $\tilde{g}$, $p(t,x,y)$ the heat kernel of $\tilde{g}$; $p(t,x,y)dy$ is the distribution of the Brownian particle $\omega(t)$ under $\mathbb{P}_g^x$.

We set, for $g \in \mathcal{P}(M)$,

$$\ell_g := \lim_{t\to\infty} \frac{1}{t} \int_{\widetilde{M}} d(y,x) p(t,x,y) dy$$

$$h_g := \lim_{t\to\infty} -\frac{1}{t} \int_{\widetilde{M}} p(t,x,y) \ln p(t,x,y) dy.$$

By compactness, the limits do not depend on the origin point x;

ℓ_g is the *linear drift* ([Guivarc'h, 1980]) of the Brownian motion on $(\widetilde{M}, \widetilde{g})$

and h_g is the *stochastic entropy* of (M, g) ([Kaimanovich,1986]).

Theorem [L & Shu, 2013]
Let φ be a C^3 function on M and consider the curve $\lambda \mapsto g(\lambda) = e^{\lambda\varphi}g$ of metrics conformal to a metric $g \in \mathcal{P}(M)$. Then, the mappings $\lambda \mapsto \ell_{g(\lambda)}$ and $\lambda \mapsto h_{g(\lambda)}$ are differentiable at $\lambda = 0$.

Observe that the metric $g(\lambda)$ has negative curvature for λ close to 0. The proof extends the techniques of the hyperbolic group case ([L, 2012] and [Mathieu, 2012]). In particular, from the formula for the derivative, we obtain:

Theorem [L & Shu, 2013]
Assume M is a locally symmetric space and consider C^3 curves $\lambda \mapsto g(\lambda) = e^{\varphi_\lambda} g$ of conformal metrics with total area 1 on M. Then, the stochastic entropy $\lambda \mapsto h_{g(\lambda)}$ has a critical point at 0 for all such curves.

In dimension 2, the stochastic entropy depends only on the volume.

The above theorem is meaningful only in higher dimensions. The converse is an open problem.

Extension 3: IFS. The above suggests the following questions about the familiar ICBM

Set, for $0 < p, \lambda < 1$, $\mu_{p,\lambda}$ for the distribution of $\sum_{i=1}^{\infty} \varepsilon_i \lambda^i$, where

$\{\varepsilon_i\}_{i \in \mathbb{N}}$ are i.i.d. $(\{-1, +1\}, (p, 1-p))$.

By [Feng & Hu, 2009], $\mu_{p,\lambda}$ is exact dimensional with dimension $\delta(p, \lambda)$. What is the regularity of $p \mapsto \delta(p, \lambda)$?
In particular for λ^{-1} Pisot?