Uniformity of measures with Fourier frames

Chun-Kit Lai, McMaster University

Dec 2012

Background and Motivations

Background and Motivations

Background and Motivations

Definition

Let μ be a compactly supported probability measure on $\mathbb{R}^d$, $\{e^{2\pi i\langle\lambda,\cdot\rangle}\}_{\lambda\in\Lambda}$ is called a Fourier frame of μ if for all $f \in L^2(\mu)$,

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} \left|\int f(x)e^{2\pi i\langle\lambda,x\rangle}d\mu(x)\right|^2 \leq B\|f\|^2.$$

If such Fourier frame exists, then μ is called an F-spectral measure and Λ is called an F-spectrum of μ.

1. Fourier frame generalizes orthonormal basis, and it is "overcomplete" i.e. every f can be expanded using linear combination of the frame, but it is not unique.
2. If it is unique, then it is called an *(exponential) Riesz basis*.

Background and Motivations

If a measure μ exists an exponential orthonormal basis, μ is called a *spectral measure*. The frequency set is called a *spectrum*.

Conjecture (Fuglede)

$\Omega \subset \mathbb{R}^d$ is a spectral set if and only if Ω is a translational tile.

He showed that

1. Any fundamental domain given by a discrete lattice are spectral sets with its dual lattice as its spectrum.
2. Triangles and circles on $\mathbb{R}^2$ are not spectral.
3. $[0,1] \cup [2,3]$ is not a fundamental domain, but it is still spectral (clearly it is a tile).

However, until 2004, Tao [T] gave a counterexample in $\mathbb{R}^d$, $d \geq 5$. The examples was modified later so that the conjecture are false in both directions on $\mathbb{R}^d$, $d \geq 3$.

Background and Motivations

Other fractal probability measures were also found to have Fourier frame.

[Jorgensen and Pedersen, 1998]
Let μ_4 be the Cantor measures supported on Cantor sets of $1/4$ contractions.

$$\mu_4(E) = \frac{1}{2}\mu_4(4E) + \frac{1}{2}\mu_4(4E-2).$$

For such measure, It was found that

$$\Lambda = \{0,1\} \oplus 4\{0,1\} \oplus 4^2\{0,1\} \oplus ...$$

is an orthonormal basis spectrum of μ_4.

The same also works for μ_{2n}. More spectral self-similar measures was found by Łaba and Wang based on certain algebraic conditions.

Background and Motivations

However, for the μ_3, the Cantor measures supported on Cantor sets of 1/3 contractions.

$$\mu_3(E) = 1/2\mu_3(3E) + 1/2\mu_3(3E-2).$$

For such measure, there are at most 2 mutually orthogonal exponentials. Hence, there is no complete orthogonal exponentials in $L^2(d\mu_3)$. The same for μ_{2n+1}.

Qu: Is μ_3 F-spectral?

More generally, we ask
Qu: which self-similar/self-affine measures are F-spectral?

Background and Motivations

In the following, we will decompose a measure μ as its Lebesgue decomposition.

$$\mu = \mu_d + \mu_s + \mu_c.$$

μ_d: discrete part
μ_s: singular (w.r.t. Lebesgue) part
μ_c: absolutely continuous

Theorem (He, Lai and Lau, 2011)

Let μ be an F-spectral measure on $\mathbb{R}^d$. Then it must be one of the three pure types: discrete (and finite), singularly continuous or absolutely continuous.

Theories and Main results

For discrete measures,

Theorem (He, Lai and Lau, 2011)

(1) Every discrete measure admits some exponential Riesz basis.
(2) Suppose
(i)μ be a spectral measure on $\mathbb{R}^1$ with Zero set of $\widehat{\mu}$ are integers.
(ii)η be a discrete measure of atoms in $\mathbb{Z}$.
Then $\eta * \mu$ admits an exponential Riesz basis (some are not spectral measure).

Theories and Main results

Theories and Main results

Theories and Main results

Absolutely continuous measures

Theorem

Let μ be a compactly supported absolutely continuous probability measure on $\mathbb{R}^d$ with $d\mu = \varphi(x)dx$. Then μ is an F-spectral measure if and only if there exists $0 < m, M < \infty$ such that $m \leq \varphi(x) \leq M$ a.e. on suppμ.

There are three proofs to this theorem.

1. Comparing Beurling densities with its subset of the support
2. Convolution inequality and Beurling density (with Gabardo)
3. Translational absolute continuity (with Dutkay)

Theories and Results

Windowed exponentials:
$\bigcup_{j=1}^{q} \mathcal{E}(g_j, \Lambda_j) = \bigcup_{j=1}^{q} \{g_j(x)e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda_j\}$

Theorem (Gabardo and Lai)

Let $\mu = \varphi(x)dx$ be an absolutely continuous measures with support $\Omega = \{\varphi \neq 0\}$ and let g_j, $j = 1, 2\cdots, q$ be a finite set of functions in $L^2(\varphi dx)$. Then there exists Λ_j such that $\bigcup_{j=1}^{q} \mathcal{E}(g_j, \Lambda_j)$ form a frame in $L^2(\varphi dx)$ if and only if we can find $0 < m \leq M < \infty$ such that

$$\frac{m}{\sqrt{\varphi}} \leq \max_{\{j: g_j \in L^{\infty}(\varphi dx)\}} |g_j| \leq \frac{M}{\sqrt{\varphi}}$$

almost everywhere on Ω.

Translational Absolute Continuity

2. Translational Absolute Continuity.
Let F be such that $\mu(F) > 0$. Denote $\omega(\cdot) = \mu((\cdot + a) \cap (F + a))$ with $a \in \mathbb{R}^d$. We have the following theorem.

Theorem

Let μ be a finite Borel measure on $\mathbb{R}^d$ and suppose there exists a Fourier frame for μ, with frame bounds $A, B > 0$. Assume $\omega \ll \mu$. Then

$$\frac{B}{A} \geq \left\| \frac{d\omega}{d\mu} \right\|_{\infty}.$$

Theories and Main results

2. Translational Absolute Continuity. Let $h = d\omega/d\mu$, then

$$\int f\, d\omega = \int_{F+a} f(x-a)\, d\mu(x) = \int f(x)h(x)d\mu(x)$$

Let $M = \|h\|_\infty$. By restricting to a subset, we may assume it is finite. Let

$$E = \{x \in F : M - \epsilon \leq h \leq M\},\ f_1 := \frac{1}{\sqrt{\mu(E)}}\chi_E$$

$$\|f_1(\cdot - a)\|^2_{L^2(\mu)} = \int |f_1(x-a)|^2\, d\mu(x) = \int |f_1(x)|^2 h(x)\, d\mu(x),$$

$$(f_1(\cdot - a)\, d\mu)\widehat{\ }(\xi) = e^{-2\pi i t\cdot a}\widehat{f_1 h\, d\mu}(\xi)$$

Theories and Main results

2. Translational Absolute Continuity.

Denote $\nu = \sum_{\lambda\in\Lambda}\delta_\lambda$, Compare using the frame inequality

$$
\begin{aligned}
AM^2 &\leq \int |\widehat{Mf_1 d\mu}(\xi)|^2 d\nu(\xi) \\
&\leq \left| \|\widehat{Mf_1 d\mu}\|^2_{L^2(\nu)} - \|f_1\widehat{(\cdot - a)d\mu}\|^2_{L^2(\nu)} \right| \\
&\quad + \|f_1\widehat{(\cdot - a)d\mu}\|^2_{L^2(\nu)} \\
&\leq C\epsilon + B\int |f_1(\cdot - a)|^2 \leq C\epsilon + BM.
\end{aligned}
$$

Theories and Main results

Theorem (Dutkay and Lai)

Let $\mu = g\,dx$ be an absolutely continuous measure on $\mathbb{R}^d$. If μ has a Fourier frame bounds $A, B > 0$ then on the support of μ

$$\frac{B}{A} \geq \frac{\sup(g)}{\inf(g)}.$$

Theories and Main results

Theorem (Dutkay and Lai)

Let $\mu = g\,dx$ be an absolutely continuous measure on $\mathbb{R}^d$. If μ has a Fourier frame bounds $A, B > 0$ then on the support of μ

$$\frac{B}{A} \geq \frac{\sup(g)}{\inf(g)}.$$

Sketch of Proof. Restricting on the subset $\{N^{-1} \leq g \leq N\}$, we may assume upper and lower bound M, m. Now, consider

$$C = \{x : m \leq g(x) \leq m + \epsilon\}, \ D = \{x : M - \epsilon \leq g(x) \leq M\}$$

Take a set F of positive Lebesgue measure such that $F \subset C$ and $F + a \subset D$ (it is possible by considering $\chi_C * \chi_D$ and take Fourier transform).

Theories and Main results

Consider $\omega(\cdot) = \mu((\cdot + a) \cap (F + a))$, then

$$\left\|\frac{d\omega)}{d\mu}\right\|_\infty = \left\|\frac{g(x+a)}{g(x)}|_E\right\|_\infty \geq \frac{M-\epsilon}{m+\epsilon}.$$

Hence, $\frac{B}{A} \geq \frac{M-\epsilon}{m+\epsilon}$.

Theories and Results

Corollary

Suppose $\mu = \varphi\, dx$ admits a tight frame ($A = B$), then φ is a constant multiple of a characteristic function.

Note that

1. Laba and Wang (2006) showed that this corollart is true when the support is an interval. Dutkay, Han and Jorgensen (2009) showed it is true for finite union of intervals.
2. The proof is just simply note that $B = A$ implies $\inf \varphi \geq \sup \varphi$.

Theories and Main results

(IV) **Self-affine measures**
Let R be a real $d \times d$ expanding matrix, i.e., all its eigenvalues λ have absolute value $|\lambda| > 1$. Let $\mathcal{B}$ be a finite subset of $\mathbb{R}^d$ and let $(p_b)_{b\in\mathcal{B}}$ be a set of positive probability weight, $p_b > 0$ and $\sum_{b\in\mathcal{B}} p_b = 1$. We define the *affine iterated function system*(IFS)

$$\tau_b(x) := R^{-1}(x+b), \quad (x \in X, b \in \mathcal{B}).$$

There is a unique Borel probability measure $\mu = \mu_{\mathcal{B}}$ on $\mathbb{R}^s$ called the *invariant measure*, such that

$$\mu(E) = \sum_{b\in\mathcal{B}} p_b \mu(\tau_b^{-1}(E)), \quad \text{for all Borel sets } E. \tag{1}$$

In addition, the measure μ is supported on the attractor X.

Theories and Main results

If for all $b, b' \in \mathcal{B}$, $b \neq b'$ we have $\mu_{\mathcal{B}}(\tau_b(X) \cap \tau_{b'}(X)) = 0$ then we say that the affine IFS has *measure disjoint condition*.
Checking also the condition of translational absolute continuity theorem we have

Theorem

Let $(\tau_b)_{b\in\mathcal{B}}, (p_b)_{b\in\mathcal{B}}$ be an affine iterated function system with measure-disjoint condition. Suppose the invariant measure $\mu_{\mathcal{B}}$ is an F-spectral measure. Then all the probabilities p_b, $b \in \mathcal{B}$ must be equal.

Theories and Main results

Proof. Pick two elements $b \neq c$ in $\mathcal{B}$. For any $n \in \mathbb{N}$, let $F = \tau_b^n(X)$ and $F + a = \tau_c^n(X)$ for some a. This is possible since they are affine maps.

For any $E \subset F$, by the measure-disjoint condition,

$$\mu(E) = p_b^n \mu(\tau_b^{-n}(E)).$$

On the other hand,

$$\omega(E) = p_c^n \mu(\tau_b^{-n}(E)).$$

Hence $d\omega/d\mu = p_c^n/p_b^n$. Hence,

$$\frac{p_c^n}{p_b^n} \leq \frac{B}{A}.$$

Theories and Main results

If the affine iterated function system does not satisfy the no overlap condition, we still have some conclusion on dimension 1. Assume the IFS with functions $\tau_i(x) = \lambda x + b_i$, for $0 < \lambda < 1$, $i = 1, .., N$ and

$$B = \{0 = b_1 < ... < b_N = 1 - \lambda\}.$$

In this case, the self-similar set X_B is a subset $[0, 1]$. The self-similar measure with weight p_i is the unique Borel probability measure satisfying

$$\mu = \sum_{i=1}^{N} p_i \mu \circ \tau^{-1}.$$

Theories and Main results

Theorem

Suppose μ defined is absolutely continuous with respect to $\mathcal{H}^{\alpha}|_X$ and $0 < \mathcal{H}^{\alpha}(X) < \infty$. If μ has a frame measure, then $p_1 = p_N$. If $\alpha = 1$ (i.e. $\mu \ll \mathcal{L}|_X$), then $p_j \le \lambda$ for all j and $p_1 = p_N = \lambda$.

In particular, if the measure is of equal weight, i.e. $p_i = \frac{1}{N}$, then the μ must be a measure supported on a self-similar tile.
However, it's hard to analyze the p_i for $i \neq 1, N$, since we need to tackle overlaps.

Theories and Main results

If we assume more strongly that μ is spectral. We have a solved a special case of Łaba-Wang conjecture of the spectral measure.

Theorem

Suppose μ be the self-similar measure that is absolutely continuous with respect to the Lebesgue measure and suppose μ admits a tight frame. Then

(i) $p_1 = \cdots = p_N = \lambda$.

(ii) $\lambda = \frac{1}{N}$.

(iii) *There exists $\alpha > 0$ such that $\mathcal{D}' := \alpha\mathcal{D} \subset \mathbb{Z}$ and $\mathcal{D}'$ tiles $\mathbb{Z}$.*

In particular, μ is the normalized Lebesgue measure of a self-similar tile.

Further Problems

Further Problems

Further Problems

All the above are proved by based on the assumption of translational absolute continuity. We say that a finite Borel measure μ is *translationally absolutely continuous* if for all Borel sets F with $\mu(F) > 0$ and for all $a \in \mathbb{R}^d$ such that $F, F + a \subset \text{supp}\mu$, $\omega \ll \mu$. (recall $\omega(\cdot) = \mu(\cdot + a \cap F + a)$)

Conjecture

If μ is an $F-$spectral measure. Then μ must be translationally absolutely continuous and it has only one local dimension.

Recall

$$\dim_{loc}\mu(x) := \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}.$$

Further Problems

However, there do exist examples for which such translational absolute continuity fails. The following suggests that singular measures supported essentially on positive Lebesgue measurable sets give such examples.

Example

Let μ be a measure whose support is exactly $[0,1]$. Suppose μ is singular with respect to the Lebesgue measure on $[0,1]$, then there exists $F, F+a \subset [0,1]$ such that ω is singular with respect to μ.

Sketch of Proof. Pick $\mu(E) > 0$ and $\mathcal{L}(E) = 0$.
Consider

$$\int_I \mu(E+x)dx > 0.$$

Further Problems

If we know the measure μ is "uniform" on the support, the above arguments cannot work, there should be some other criterion for F-spectrality. This is again the most interesting question:

(Q1). For the case μ_3, are there any Fourier frame?

(Q2). Find a singular measure with a Fourier frame but which is not absolutely continuous with respect to a spectral measure nor a convolution of spectral measures with some discrete measures.

(Q3) Find a self-similar measure admitting a Fourier frame of the type described in Q2.

Further Problems

There are possibilities of sets we may try.

(1) unbounded sets of finite Lebesgue measure

(2) the surface measure supported on some Riemannian manifold sitting on $\mathbb{R}^d$

(3) 3/8 Bernoulli convolution

(4) Salem construction of Cantor sets $\widehat{\mu}(\xi) = O(|\xi|^{-\beta/2})$.

(5) Riesz product

Thank You !!