Fluctuations of recentered maxima of discrete Gaussian Free Fields on a class of recurrent graphs

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Advances on fractals and related topics at CUHK

1 Introduction

Gaussian free field (GFF) in a d-dim box of size $(2N+1)$ (D-bd cond)

$V_N := ([-N,N] \cap \mathbb{Z})^d$, $\partial V_N = V_N \setminus V_{N-1}$

$\{w_n\}_n$: simple RW on V_N killed at $\tau_N := \min\{n \geq 0 : w_n \in \partial V_N\}$

$g_N(x,y) := E^x[\sum_{m=0}^{\tau_N} 1_{\{w_m=y\}}]/\mu_y$ (μ_y: $\sharp$ of bonds that contains y)

$\{X_x^N\}_{x\in V_N}$: zero-mean GFF, i.e. each X_x^N is centered Gaussian,

covariance g_N, $X_x^N = 0$ for $x \in \partial V_N$

Set $X_N^* := \max_{x\in V_N} X_x^N$.

(Q) Is $\{X_N^* - EX_N^*\}_{N\geq 1}$ tight?

$$E[X_N^*] \asymp \begin{cases} \sqrt{N} & \text{for } d=1 \\ \log N & \text{for } d=2 \\ (\log N)^{1/2} & \text{for } d\geq 3 \end{cases}$$

Fluctuations of X_N^*: $\sqrt{N} (\asymp E[X_N^*])$ for $d = 1$

$O(1)$ for $d = 2$ (long time open prob $\Rightarrow$ Bramson-Zeitouni '12)

$O(1)$ for $d \geq 3$ (use transience of SRW and Borell's ineq.)

* Borell's ineq.: $P(X_N^* - EX_N^* > \lambda) < 2\exp(-\frac{1}{2}\lambda^2/\sigma_N^2)$, $\sigma_N^2 = \sup_x E[(X_x^N)^2]$.

So, (A) $\{X_N^* - EX_N^*\}_{N\geq 1}$ is tight iff $d \geq 2$.

Now

(Q) For a general graph, when is $\{X_N^* - EX_N^*\}_{N\geq 1}$ tight?

2 Framework

Weighted graphs

$G = (V(G), E(G))$: con. loc. fnt. graphs

$\mu^G : V(G) \times V(G) \to \mathbb{R}_+$ weight: $\mu_{xy} = \mu_{yx}$, $\mu^G_{xy} > 0 \Leftrightarrow \{x, y\} \in E(G)$

For $B \subset G$ with $B \neq G$ and for $x \neq y \in V(G)$, not both in B, define *resistance* by

$$R_B(x,y)^{-1} := \inf\{\frac{1}{2}\sum_{w,z\in V(G)} (f(w) - f(z))^2 \mu^G_{wz} : f(x) = 1, f(y) = 0, f|_B = \text{constant}\}.$$

(Set $R_B(x,x) = 0$, $R_B(x,y) = 0$ if $x, y \in B$ and, for $x \in V(G) \setminus B$.)

μ^G: meas on G s.t. $\mu^G(A) = \sum_{x\in A} \mu^G_x$, where $\mu^G_x := \sum_{y\in V(G)} \mu^G_{xy}$.

$\{w^G_m\}_{m\geq 0}$: corresp. (discrete time) Markov chain with D-bd on B, i.e.

$$P(w^G_{m+1} = y | w^G_m = x) = \mu^G_{xy}/\mu^G_x, \qquad \forall x, y \in V(G),\ x \notin B,$$

and w^G_m is killed upon hitting B.

GFF on weighted graphs

$\{G^N\}_{N\geq 1}$: seq. of fnt con. graphs s.t. $|G^N| \geq 2$ and $\lim_{N\to\infty} |G^N| = \infty$.

μ^{G^N}: weight, $B^N \subset G^N$ $(B^N \neq G^N)$ boundary: assume $G^N \setminus B^N$ is con.

$\{w_m^N\}_{m\geq 0}$: corresp. MC with D-bd cond. on B^N, $\tau^N := \min\{m \geq 0 : w_m^{G^N} \in B^N\}$,

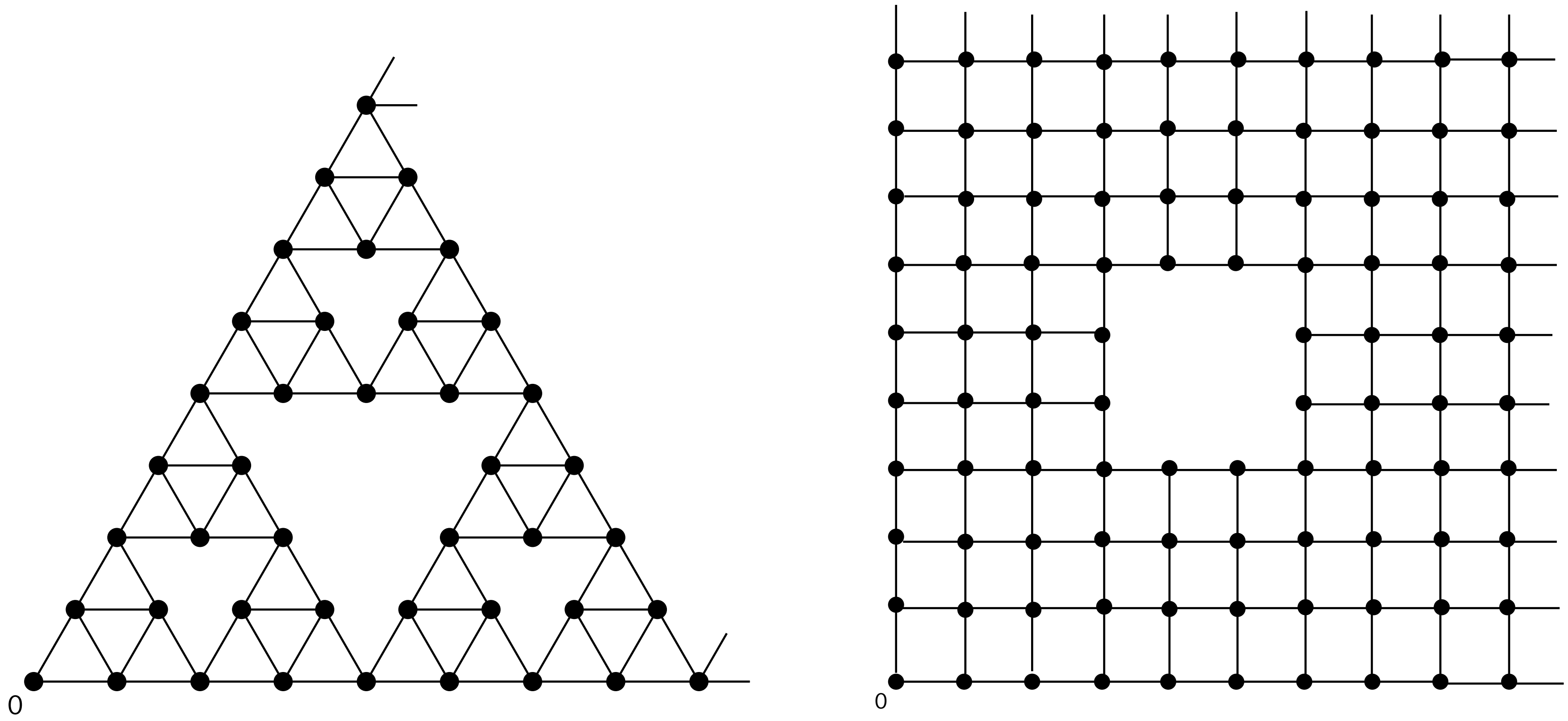
$g_N(x,y) := (\mu_y^N)^{-1} E_{G^N}^x[\sum_{m=0}^{\tau^N} 1_{\{w_m^{G^N}=y\}}]$ for $x, y \in V(G^N) \setminus B^N$,

$\{X_z^N\}_{z\in V(G^N)}$: GFF on G^N with D-bd on B^N, i.e.

zero-mean Gaussian field with covariance $g_N(\cdot,\cdot)$, $X_z^N \equiv 0$ for $z \in B^N$.

Lemma 2.1

$$E[(X_x^N - X_y^N)^2] = R_{B^N}(x,y).$$



Examples: 2-dimensional Sierpinski gasket graph and carpet graph

3 Main theorem

$h : \mathbb{N} \to \mathbb{N}$: strict incr. with $h(0) = 0$ s.t. $0 < \exists \beta_1 \leq \exists \beta_2 < \infty$ and $C > 0$

$$C^{-1} \left(\frac{R}{r} \right)^{\beta_1} \leq \frac{h(R)}{h(r)} \leq C \left(\frac{R}{r} \right)^{\beta_2} \quad 0 < \forall r \leq \forall R < \infty. \tag{1}$$

Assumption 3.1 $\exists \alpha > 0$ *and* $c_1, c_2, c_3 > 0$ *s.t. the following hold* $\forall N$ *large:*

(i) $R_{B^N}(x, y) \leq c_1 h(d_{G^N}(x, y))$, $\forall x, y \in G^N$.

(ii) $\max_{x \in G^N} R_{B^N}(x, B^N) \geq c_2 \max_{x \in G^N} h(d_{G^N}(x, B^N))$, $\forall x \in G^N$.

(iii) $\mathcal{N}_{G^N}(\delta d^N_{max}) \leq c_3 \delta^{-\alpha}$, $\forall \delta \in (0, 1]$ *where* $d^N_{max} := \max_{x \in G^N} d_{G^N}(x, B^N)$ *and*

$\mathcal{N}_{G^N}(\varepsilon)$*: minimal* $\sharp$ *of* d_{G^N}*-balls of radius* ε *needed to cover* G^N*.*

Furthermore, $d^N_{max} \to \infty$ *as* $N \to \infty$*.*

Let $X_N^* := \max_{z \in V(G^N)} X_z^N$, $\tilde{X}_N := X_N^* / \bar{\sigma}_N$, where $\bar{\sigma}_N = (\max_{z \in G^N} E[(X_z^N)^2])^{1/2}$.

[Note: $\bar{\sigma}_N^2 = \max_{x \in G^N} R_{B^N}(x, B^N)$.]

Theorem 3.2 *Under Assumption 3.1,* $\exists A, B, A' > 0$ *and* $\exists g : (0, \infty) \to (0, 1)$ *s.t.*

$$P(\tilde{X}_N < A) > B, \quad P(\tilde{X}_N > c) \geq g(c) \ \ \forall c > 0, \quad E(\tilde{X}_N) \leq A' \quad \forall N \ \textit{large}. \tag{2}$$

In particular, $\{X_N^* - EX_N^*\}_{N \geq 1}$ is NOT tight (since they fluctuate with order $\bar{\sigma}_N$).

Normalized local time at B^N at cover time

$\bar{G}^N$: shorting, i.e. making B^N to a point $\{b\}$. Modified weight function

$$\mu_{xy}^{\bar{G}^N} = \begin{cases} \mu_{xy}^{\bar{G}^N}, & x \in V(\bar{G}^N),\ y \in V(\bar{G}^N) \setminus \{b\}, \\ \sum_{z \in B^N} \mu_{xz}^{G^N}, & x \in V(\bar{G}^N) \setminus \{b\},\ y = b. \end{cases}$$

$\{\bar{w}_t^N\}_{t \geq 0}$: (non-killed) cont. time MC with jump rates $\mu^{\bar{G}^N}$ (holding time $\exp(1/\mu^{\bar{G}^N})$).

Define $\quad L_t^{x,N} := \frac{1}{\mu_x^{\bar{G}^N}} \int_0^t \mathbf{1}_{\{\bar{w}_s^N = x\}} ds$ ((weight normalized) local time at x),

$\quad \tau_{\mathrm{COV}}^N := \inf\{t > 0 : L_t^{x,N} > 0, \forall x \in \bar{G}^N\}$ (Cover time).

$L^N := \sqrt{L^b_{\tau^N_{\mathrm{COV}}}}$: the square-root of the normalized local time at B^N at cover time,

"L^N should behave similarly to $|X_N^*|$."

Gener. Ray-Knight Isom. thm: $\{L^x_{\tau_t} + \frac{1}{2} X_x^2 : x \in G\} \stackrel{\text{law}}{=} \{\frac{1}{2}(X_x + \sqrt{2t})^2 : x \in G\}$, $\tau_t = \inf\{s : L_s^b > t\}$.

Proposition 3.3 *Assumption 3.1* $\Rightarrow$ *Theorem 3.2 hold with* $L^N / \bar{\sigma}_N$ *replacing* $\tilde{X}_N$.

4 Examples

2-dim Sierpinski gasket and carpet graphs: G

Detailed heat kernel estimates (Jones '96, Barlow-Bass '99):

$$p_k(x,y) \le c_1 k^{-d_f/d_w} \exp\left(-c_2\left(\frac{d(x,y)^{d_w}}{k}\right)^{1/(d_w-1)}\right) \quad \forall x,y\in G, k>0,$$

$$p_k(x,y)+p_{k+1}(x,y) \ge c_3 k^{-d_f/d_w} \exp\left(-c_4\left(\frac{d(x,y)^{d_w}}{k}\right)^{1/(d_w-1)}\right) \quad \forall k>d(x,y).$$

Here d_f: volume growth, $d_w > 2$: walk dimension (note $d_w > d_f$).

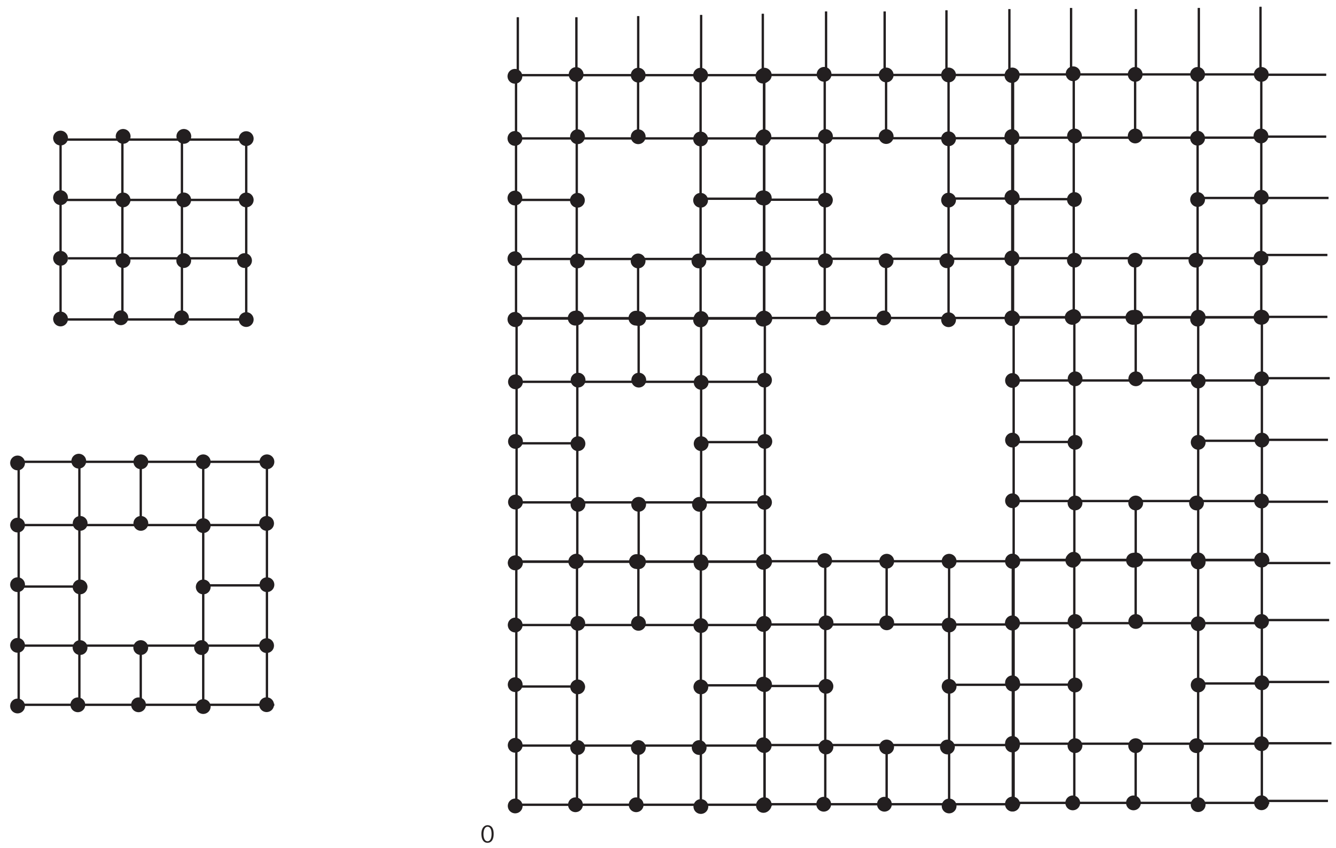
$\Rightarrow$ $$c_5 R^{d_f} \le \mu(B(x,R)) \le c_6 R^{d_f}, \quad \forall x\in G, R\ge 1,$$

$$R(x,y) \le c_7 d(x,y)^{d_w-d_f}, \quad R(x,B^c(x,R)) \ge c_8 R^{d_w-d_f}, \quad \forall x,y\in G, \ \forall R\ge 1.$$

$\{G^N, B^N\}_{N\ge 0}$ G^N: N-level box containing 0, $B^N = L^N V_0$: L^{-1} contraction rate

$\Rightarrow$ Assumption 3.1 holds with $h(s)=s^{d_w-d_f}, \alpha = d_f$ so $\{X_N^* - EX_N^*\}_{N\ge 1}$ is NOT tight!

Homogeneous random Sierpinski carpet graph: G



Example of 2-dimensional homogeneous random Sierpinski carpet graph

$I := \{1, \cdots, \ell\}$. For each $k \in I$, $\{\psi_i^k\}_{i=1}^{K_k}$: family of L_k-similitudes that construct SC

For $\xi = (k_1, \cdots, k_n, \cdots) \in I^\infty$, $n \in \mathbb{N}$, write $\xi|_N := (k_1, \cdots, k_N) \in I^N$, and let

$$V(G^N_{\xi|_N}) := \bigcup_{\substack{i_j \in \{1, \cdots, K_{k_j}\}, \\ 1 \le j \le N}} L_{k_1} \cdots L_{k_N} \psi^{k_N}_{i_N} \circ \ldots \circ \psi^{k_1}_{i_1}(V_0), \quad G_\xi := \bigcup_{N=1}^{\infty} V(G^N_{\xi|_N}).$$

$B_n := L_{k_1} \cdots L_{k_n}$ (space scale), $M_n := K_{k_1} \cdots K_{k_n}$ (mass scale), $T_n := R_n M_n$ (time scale)

$$d_f(n) := \frac{\log M_n}{\log B_n}, \qquad d_w(n) := \frac{\log T_n}{\log B_n}.$$

Let $V_d(x, r)$ be $\sharp$ of vertices in $B(x, r)$ w.r.t. graph distance. Then

$$c_1 r^{d_f(n)} \le V_d(x, r) \le c_2 r^{d_f(n)} \quad \text{if} \quad B_n \le r < B_{n+1}, \; x \in G_\xi.$$

Define $\tau : [1, \infty) \to [1, \infty)$ (time scale) and $h : [1, \infty) \to [1, \infty)$ (resistance scale) as

$$\tau(s) = s^{d_w(n)}, \; h(s) = s^{d_w(n) - d_f(n)} \quad \text{if} \quad T_n \le s < T_{n+1}, \quad \tau(0) = h(0) = 0.$$

Detailed heat kernel estimates (Cont. version Hambly-Kusuoka-K-Zhou '00):

$$
\begin{aligned}
p_k(x,y) &\leq \frac{c_3}{V_d(x,\tau^{-1}(k))}\exp(-c_4(\frac{\tau(d(x,y))}{k})^{1/(\beta_1-1)}),\\
p_k(x,y)+p_{k+1}(x,y) &\geq \frac{c_5}{V_d(x,\tau^{-1}(k))} \quad \text{for } \ k \geq c_6\tau(d(x,y))
\end{aligned}
$$

for $k \in \mathbb{N}$, $x, y \in G_\xi$. **If** the following limits exist and the inequality holds

$$
d_f := \lim_{n\to\infty} d_f(n), \quad d_w := \lim_{n\to\infty} d_w(n), \quad d_w > d_f, \tag{3}
$$

then (by Barlow-Coulhon-K '05), we have

$$
c_7\frac{\tau(d(x,y))}{V_d(x,d(x,y))} \leq R(x,y) \leq c_8\frac{\tau(d(x,y))}{V_d(x,d(x,y))}, \quad \forall x,y \in G_\xi.
$$

$\{G^N_{\xi|_N}, B^{\xi|_N}\}_{N\geq 0}$ where $B^{\xi|_N} = B_N V_0$

$\Rightarrow$ Assumption 3.1 holds so $\{X^*_N - EX^*_N\}_{N\geq 1}$ is NOT tight!

<<Checking (3): introduce randomness>>

$(I^{\mathbb{N}}, \mathcal{F}, \mathbb{P})$: Borel prob. space $\mathbb{P}$ is stationary and ergodic (w.r.t. the shift)

$\Rightarrow$ by the sub-additive ergodic thm, $\exists$ the first two limits in (3) $\mathbb{P}$-a.e. ξ.

Further, if $d_w^i > d_f^i$ for all $i \in I$ (Hdff and walk dim for $G_{\mathbf{i}}$, $\mathbf{i} = (i, i, i, \cdots)$),

then the third inequality in (3) $\mathbb{P}$-a.e. ξ. also holds for G_ξ for $\mathbb{P}$-a.e. ξ.

[Special case] $d = 3, \ell = 2$, $\mathbb{P}$: Bernoulli with $\mathbb{P}(\xi_1 = 1) = p$, $\mathbb{P}(\xi_1 = 2) = 1 - p$.

Take carpets in such a way that $d_w^1 > d_f^1$ and $d_w^2 < d_f^2$.

$\Rightarrow$ $\exists p_* \in (0, 1)$ s.t. (3) holds $\mathbb{P}$-a.e. for all $p < p_*$.

(Open problem) Is $\{X_N^* - EX_N^*\}_{N \geq 1}$ tight when $p = p_*$??

–Related question: Can one construct 'deterministic' carpet with $d_w = d_f$?

Remark:

$\mathbb{Z}^2$ case:

Bolthausen-Deuschel-Zeitouni ('11): tightness holds along $\exists$ subseq.

(Soft arguments, applicable for fractals as well.)

In the paper, they also showed that

$$(*)\quad EX^*_{2N} \le EX^*_N + C, \quad \forall N = 2^n \quad \Rightarrow \quad \text{tightness for full seq.}$$

Bramson-Zeitouni ('12) proved

$$EX^*_{2^n} = (2\sqrt{2/\pi}\log 2)\, n - (\frac{3}{4}\sqrt{2/\pi})\log n + O(1),$$

which certainly implies $(*)$.

Method, NOT robust.