Hausdorff dimension of metric spaces and Lipschitz maps onto cubes

Tamás Keleti

Eötvös Loránd University, Budapest

AFRT2012, Hong Kong, December 10, 2012

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joint work with

András Máthé (Warwick) and Ondřej Zindulka (Prague)

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Main question

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$$f : X \to [0,1]^k \textit{ Lipschitz and onto} \Longrightarrow \mathcal{H}^k(X) \geq \mathcal{H}^k(f(X)) = \mathcal{H}^k([0,1]^k) > 0,$$

where $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure. Therefore

Obvious necessary condition:

$$\mathcal{H}^k(X) > 0$$

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Is this sufficient? No!

Vitushkin, Ivanov and Melnikov (1963)

There exists a compact set $K \subset \mathbb{R}^2$ with $\mathcal{H}^1(K) > 0$ that cannot be mapped onto a segment by a Lipschitz map.

A special case

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The following long standing conjecture states exactly this:

Conjecture of Laczkovich (1991)

Every compact subset of $\mathbb{R}^k$ with positive Lebesgue measure can be mapped onto a k-dimensional cube by a Lipschitz map.

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For $k \geq 3$ it is open.

Our main result

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The slightly stronger condition that $\dim_H(X) > k$ is already sufficient:

Main Theorem

If X is a compact metric space with $\dim_H(X) > k$ then X can be mapped onto a k-dimensional cube by a Lipschitz map.

More general metric spaces

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In fact we proved this result for a larger class of metric spaces:

Theorem - for more general metric spaces

If X is an analytic (for example Borel) subset of a complete separable metric space and $\dim_H(X) > k$ then X can be mapped onto a k-dimensional cube by a Lipschitz map.

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But some assumption is needed for the metric space.

Theorem - negative result

There exist separable metric spaces with arbitrarily large Hausdorff dimension that cannot be mapped onto a segment by a uniformly continuous function.

About the proof of the negative result

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It is well known that the following hypothesis is independent of the standard ZFC axioms.

cov $\mathcal{M} < \mathfrak{c}$

The real line can be covered by less than continuum many sets of first category.

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To prove the negative result we give different constructions depending on the validity of this hypothesis:

- If cov $\mathcal{M} < \mathfrak{c}$ holds then our example is a separable metric space of cardinality less than continuum.
- If cov $\mathcal{M} < \mathfrak{c}$ is false then we can give an example in $\mathbb{R}^n$.

Motivation - Transfinite Hausdorff Dimension

Definition (Urbanski)

$$\mathrm{tHD}(X) = \sup\{\mathrm{ind}\, f(Y) : Y \subset X,\ f : Y \to Z \text{ Lipschitz, } Z \text{ a metric space}\},$$

where ind denotes the transfinite small inductive topological dimension.

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Combining a theorem of Urbanski and our general theorem we get the following.

Theorem

Let A be an analytic subset of a complete separable metric space.

- If $\dim_{\mathrm{H}} A$ is finite but not an integer then $\mathrm{tHD}(A) = \lfloor \dim_{\mathrm{H}} A \rfloor$,
- if $\dim_{\mathrm{H}} A$ is an integer then $\mathrm{tHD}(A)$ is $\dim_{\mathrm{H}} A$ or $\dim_{\mathrm{H}} A - 1$, and
- if $\dim_{\mathrm{H}} A = \infty$ then $\mathrm{tHD}(A) \geq \omega_0$.

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Using our negative result we get that for general separable metric spaces nothing can be said:

Theorem

There exist separable metric spaces with zero transfinite Hausdorff dimension and arbitrarily large Hausdorff dimension.

Monotone metric spaces are nice

Definition (Zindulka)

A metric space (X, d) is *monotone* if there exists a linear order $<$ and a C s.t.

$$(*) \qquad \operatorname{diam}([a, b]) \leq C \cdot d(a, b) \qquad (\forall a, b \in X),$$

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Theorem - s-Hölder map for compact monotone metric spaces

(X, d) compact, monotone and $\mathcal{H}^s(X) > 0 \Longrightarrow \exists g : X \to [0, 1]$ s-Hölder onto

Sketch of the proof:

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Let $g(x) = \mu((-\infty, x))$, where $(-\infty, x) = \{y \in X : y < x\}$.

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Then g is s-Hölder since $g(b) - g(a) = \mu([a, b)) \leq \operatorname{diam}([a, b))^s \leq (C \cdot d(a, b))^s$.

Peano curves can be useful!

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Proof.

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Let $h : [0,1] \to [0,1]^k$ be a $1/k$-Hölder Peano curve.

Then $f = h \circ g : X \to [0,1]^k$ Lipschitz onto.

Ultrametric spaces are also useful!

Definition

An *ultrametric space* is a metric space with the stronger triangle inequality

$$d(x,y) \leq \max(d(x,z), d(y,z))$$

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Every compact ultrametric space is monotone.

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Second good news:

Theorem (Mendel-Naor, 2012)

For any (X, d) compact metric space and $\varepsilon > 0$ there exists a closed subset $Y \subset X$ s.t. $\dim_H Y \geq (1 - \varepsilon) \dim_H X$ and (Y, d) is and (Y, d) is bi-Lipschitz equivalent to an ultrametric space.

Now we can put together the proof of the main result

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A different argument in $\mathbb{R}^n$ using self-similar sets

Special case of the Main Theorem

If $X \subset \mathbb{R}^n$ is a compact with $\dim_H(X) > k$ then X can be mapped onto a k-dimensional cube by a Lipschitz map.

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Special case of the Main Theorem

If $X \subset \mathbb{R}^n$ is a compact with $\dim_H(X) > k$ then X can be mapped onto a k-dimensional cube by a Lipschitz map.

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Let $S \subset \mathbb{R}^n$ be a self-similar set with the SSC with large enough Hausdorff dimension.

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Sketch of the proof.

Let $S \subset \mathbb{R}^n$ be a self-similar set with the SSC with large enough Hausdorff dimension.

By a theorem of Mattila: $\exists$ isometry ϕ s.t. $\dim_H(X \cap \phi(S)) > k$.

Every self-similar set with SSC is bi-Lipshitz equivalent to an ultrametric space, so it is monotone, then so is $X \cap \phi(S)$.

$\implies$ $X \cap \phi(S)$ can be mapped onto $[0,1]^k$ by a Lipschitz map

$\implies$ X can be mapped onto $[0,1]^k$ by a Lipschitz map


THE END