Periodic and non-periodic aspects of the heat kernel asymptotics on Sierpiński carpets

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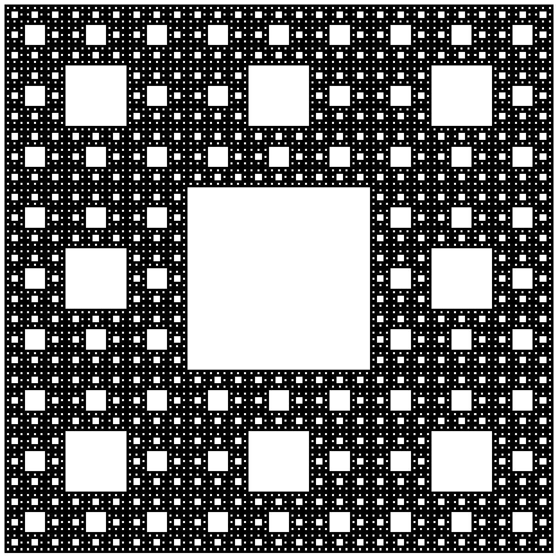
Advances on Fractals and Related Topics

@ Chinese Univ. Hong Kong



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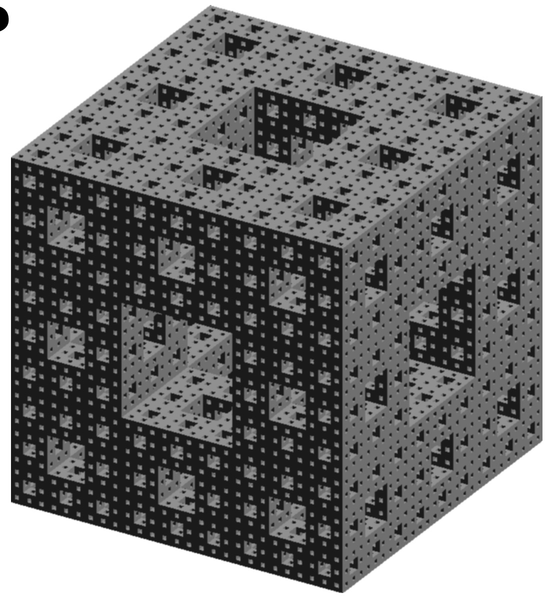

0 Main Question

Given a "Laplacian" Δ, let $p_t(x,y)$ be the heat kernel (transition density of the diffusion):

$$e^{t\Delta}f(x) = \int p_t(x,y)f(y)dy.$$

Question. How does $p_t(x,x)$ behave as $t \downarrow 0$?

cf. M^d: Riem. mfd

$$\Longrightarrow p_t^M(x,x) \overset{t\downarrow 0}{=} (4\pi t)^{-d/2}\Big(1 + \frac{S_M(x)}{6}t + O(t^2)\Big),$$

M cpt $\Rightarrow \mathcal{Z}_M(t) := \sum_n e^{-\lambda_n^M t} = \int_M p_t^M(x,x) \overset{t\downarrow 0}{\sim} \dfrac{\mathrm{vol}_d(M)}{(4\pi t)^{d/2}}$.

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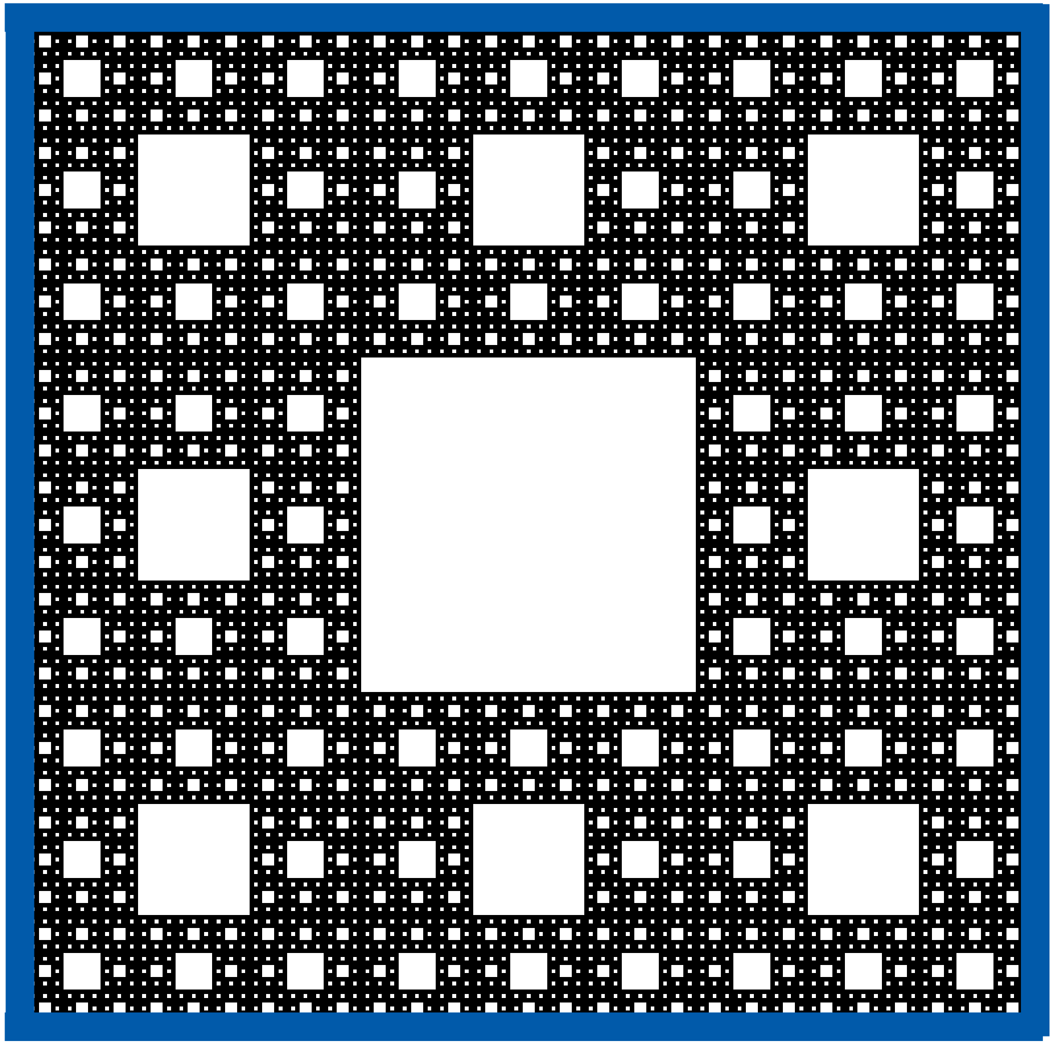
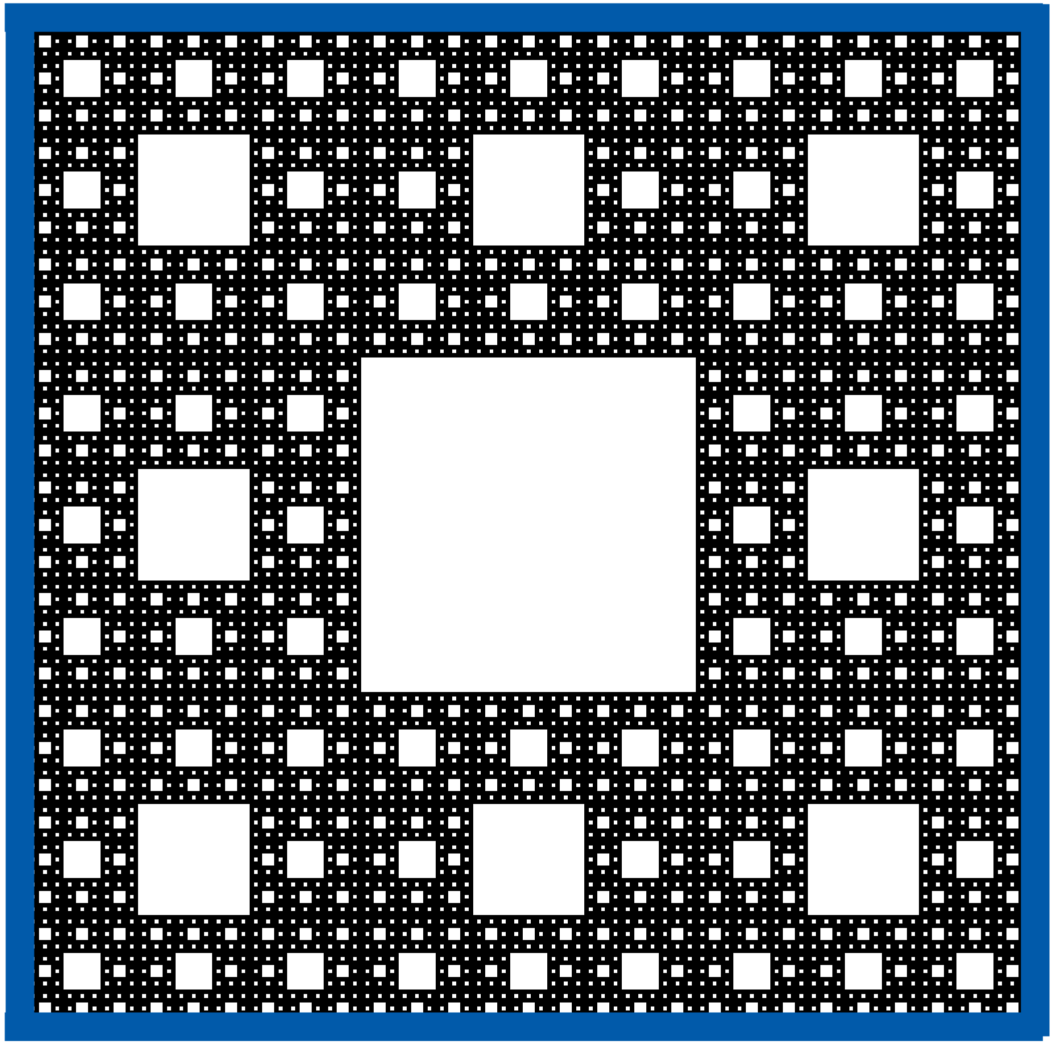
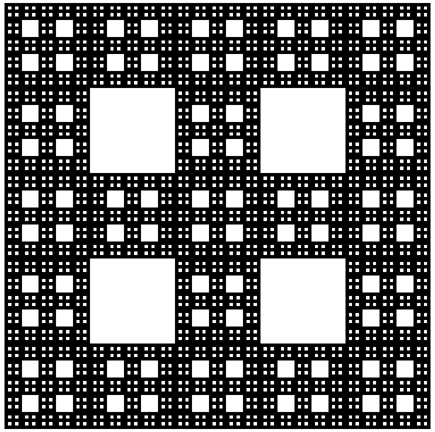
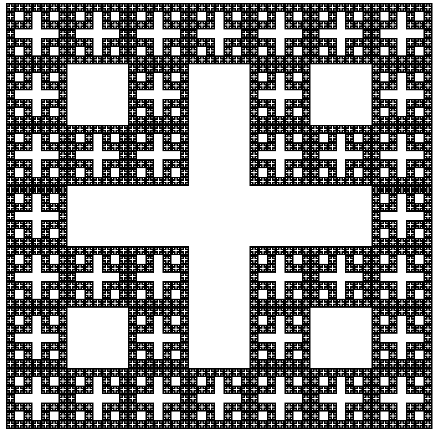


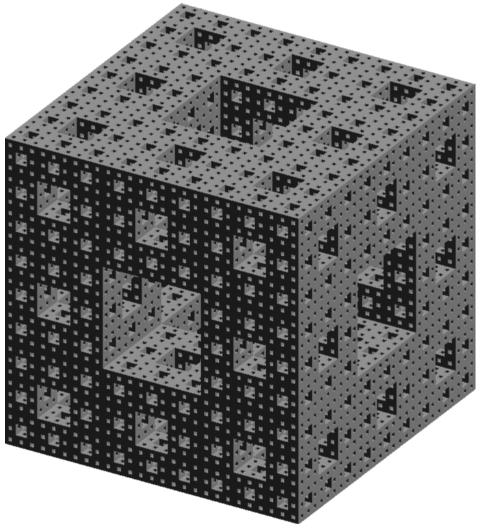
the Sierpiński carpet

$$\partial(\mathrm{SC}) = \partial_{\mathbb{R}^2}[0,1]^2!$$









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generalized SCs

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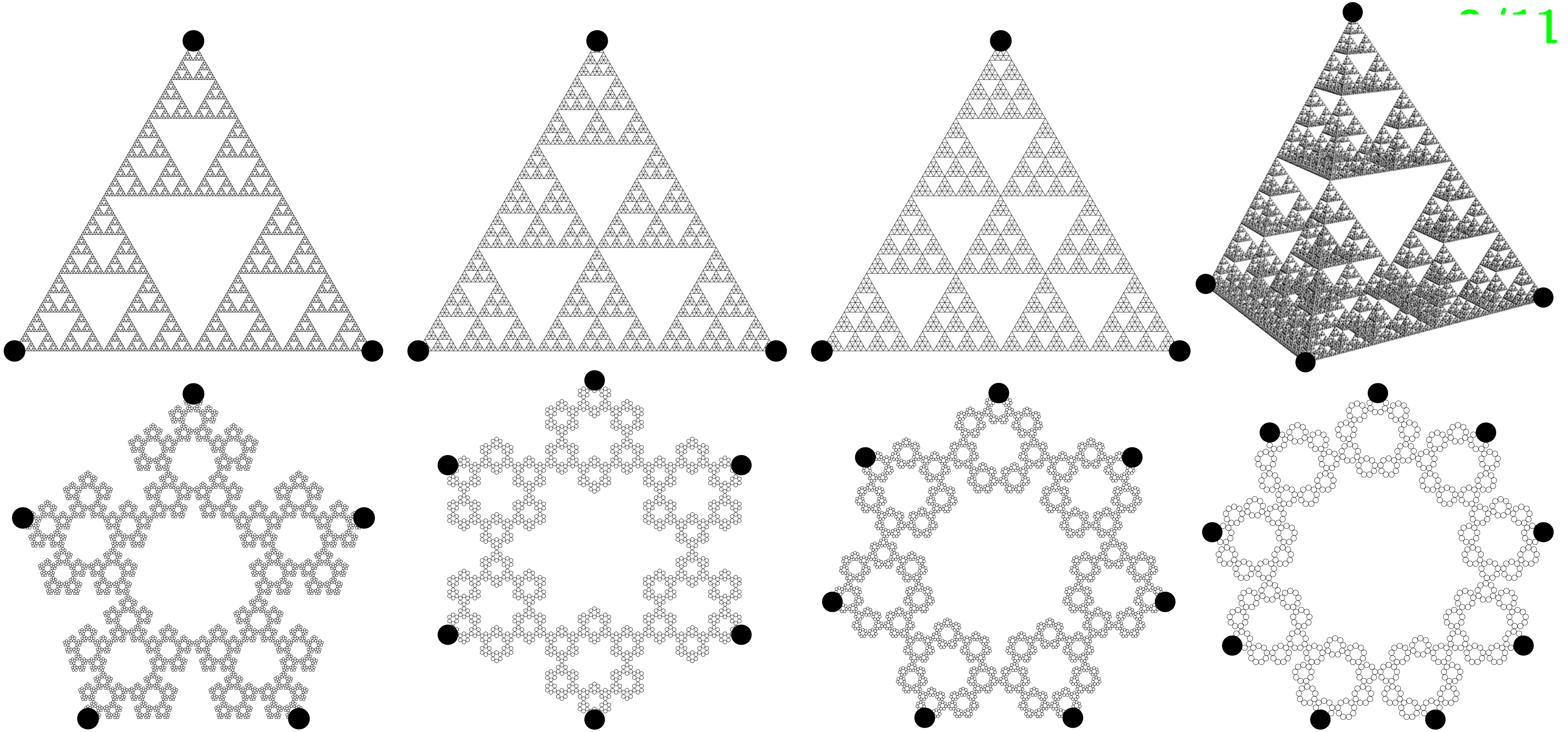
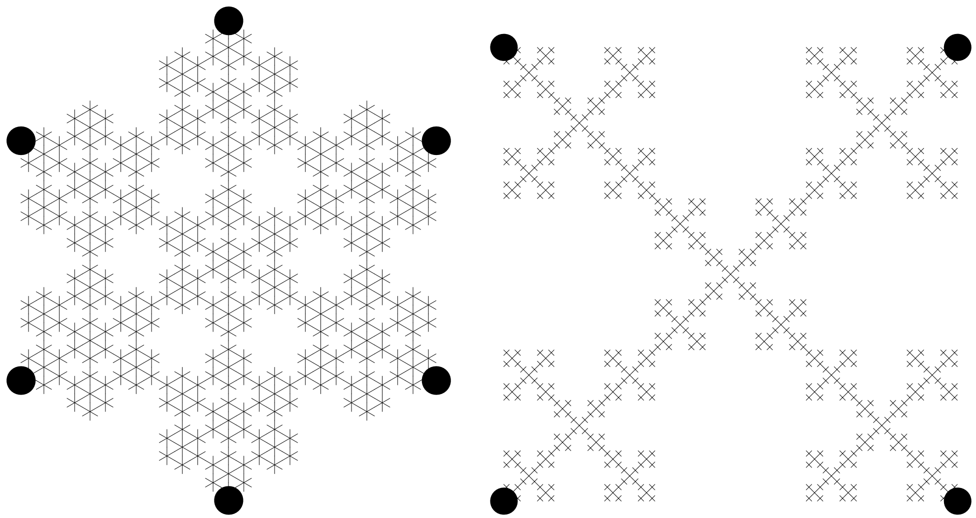




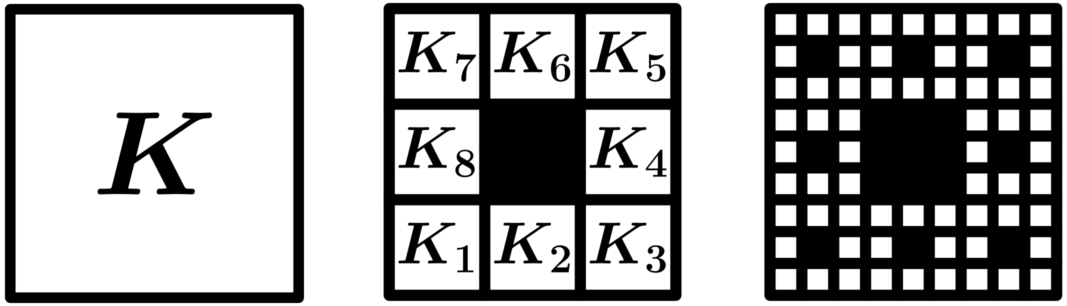
Examples of nested fractals

Solid circles: “Boundary” V_0

- $\#V_0 < \infty$
- **highly symmetric**

1 Dirichlet form and B.M. on Sierpiński carpets

▷ μ : Self-similar measure with weight $\left(\frac{1}{N}, \ldots, \frac{1}{N}\right)$



1 $\quad 1/N$ each $\quad 1/N^2$ each

▷ $^{\exists 1}(\mathcal{E}, \mathcal{F})$: canonical self-sim. Dirich. form on $L^2(K, \mu)$

$\Updownarrow \; T_t f(x) = \mathrm{E}_x[f(X_t)]$

$X = (\{X_t\}_{t \geq 0}, \{\mathrm{P}_x\}_{x \in K})$: μ-symm. conservative diffusion

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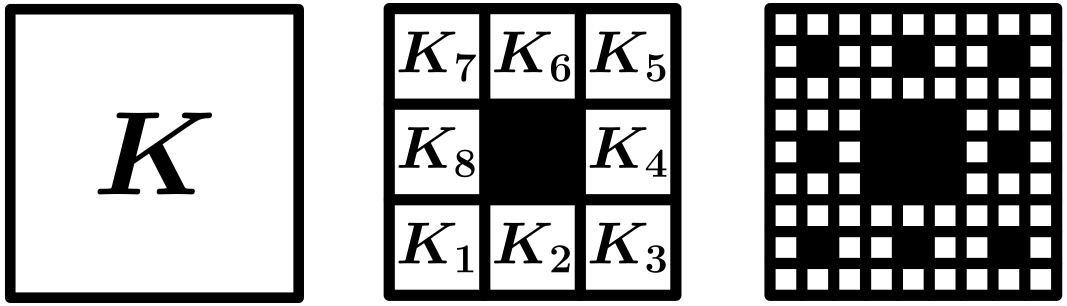

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"$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle dx$"

Existence: Barlow-Bass '89, '99, Kusuoka-Zhou '92

Uniqueness: Barlow-Bass-Kumagai-Teplyaev '10

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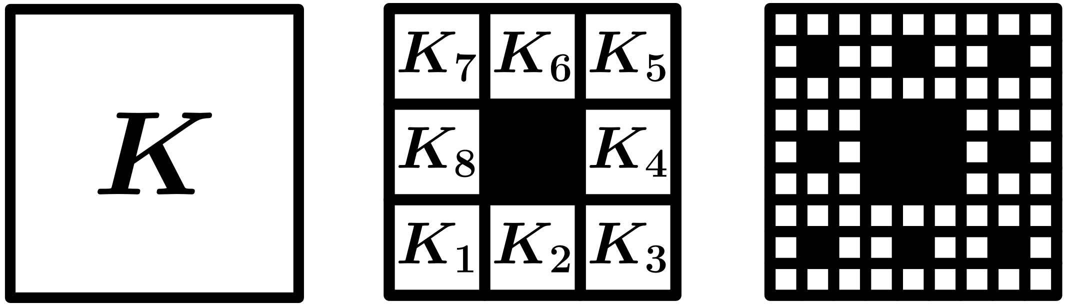

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Dirichlet form		selfad, ≤ 0 "Laplacian"		Markov semigr.
$(\mathcal{E}, \mathcal{F})$	$\mathcal{E}(u,v) = \langle -Au, v\rangle_\mu$ → ← $\mathcal{E}(\sqrt{-A}u, \sqrt{-A}v)$	A	$T_t = e^{tA}$ → ← $A = \lim_{t\downarrow 0} \frac{T_t - I}{t}$	$\{T_t\}_t$

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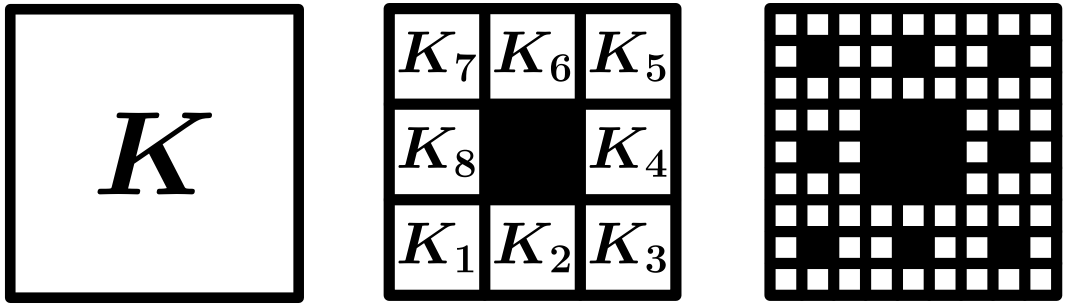

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$(\mathcal{E}, \mathcal{F})$ Dirichlet form	$\mathcal{E}(u,v) = \langle -Au, v\rangle_\mu$ → ← $\mathcal{E}(\sqrt{-A}u, \sqrt{-A}v)$	A selfad, ≤ 0 "Laplacian"	$T_t = e^{tA}$ → ← $A = \lim_{t \downarrow 0} \frac{T_t - I}{t}$	$\{T_t\}_t$ Markov semigr.

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(the "Brownian motion" on K)

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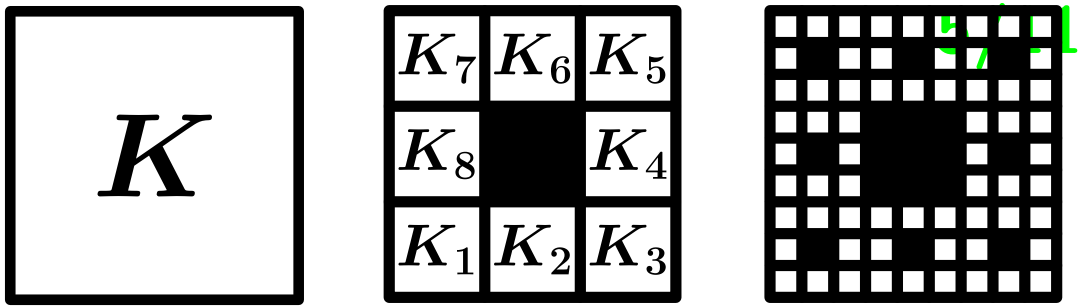

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$$(\mathcal{E},\mathcal{F}) \underset{\mathcal{E}(\sqrt{-A}u,\sqrt{-A}v)}{\overset{\mathcal{E}(u,v)=\langle -Au,v\rangle_\mu}{\rightleftarrows}} A \underset{A=\lim_{t\downarrow 0}\frac{T_t-I}{t}}{\overset{T_t=e^{tA}}{\rightleftarrows}} \{T_t\}_t$$

Dirichlet form — A: selfad, ≤ 0 "Laplacian" — Markov semigr.

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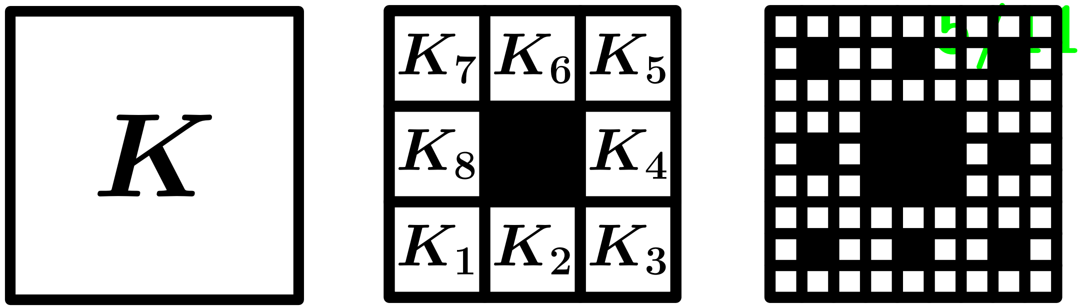

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$\Updownarrow$ $T_tf(x)=\mathbf{E}_x[f(X_t)]$

$X=(\{X_t\}_{t\geq 0},\{\mathrm{P}_x\}_{x\in K})$: μ-symm. conservative diffusion

▷ $p_t(x,y)$: Heat kernel

$$T_tf(x)=\mathbf{E}_x[f(X_t)]=\int_K p_t(x,y)f(y)d\mu(y)$$

Sub-Gaussian bound of $p_t(x,y)$

Thm (Barlow-Bass '92, '99). For $t \in (0,1]$, $x, y \in K$,

$$p_t(x,y) \asymp \frac{c_1}{t^{d_s/2}} \exp\left(-c_2\left(\frac{|x-y|^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right).$$

- $d_s := 2d_f/d_w, \quad d_f := \dim_{H,Euc} K$
- $d_w > 2$ (Barlow-Bass '90, '92, '99)

$\Rightarrow \quad c_3 \leq t^{d_s/2} p_t(x,x) \leq c_4, \quad t \in (0,1], x \in K.$

Q. $\exists \lim_{t \downarrow 0} t^{d_s/2} p_t(x,x)$? If not, HOW it oscillates?

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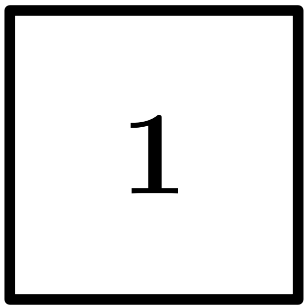
2 Thm 1. $p_t(x,x)$ NOT vary reg. & non-periodic

Thm (K.). ${}^{\exists}c_5 \in (0,\infty)$, ${}^{\exists}N \subset K$ Borel, $\nu_{\mathrm{q}}(N) = 0$ for any self-similar measure ν_{q}, and ${}^{\forall}x \in K \setminus N$:

(NRV) $\boxed{p_{(\cdot)}(x,x) \text{ does NOT vary regularly at } 0}$,

and hence $\nexists \lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$.

▷ ν_{q} : Self-similar measure with weight $\mathrm{q} = (q_i)_{i=1}^{N}$ ($q_i > 0$, $\sum_{i=1}^{N} q_i = 1$)



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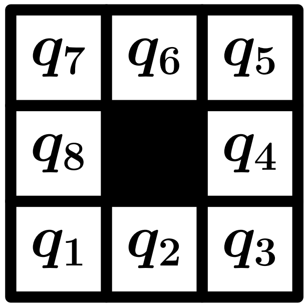
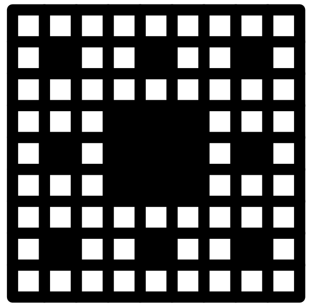


q_i on K_i



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- $f : (0,\infty) \to (0,\infty)$ *varies regularly at* 0

$\overset{\text{def}}{\Longleftrightarrow}$ $^{\forall}\alpha \in (0,\infty)$, $^{\exists}\lim\limits_{t\downarrow 0} f(\alpha t)/f(t) \in (0,\infty)$.

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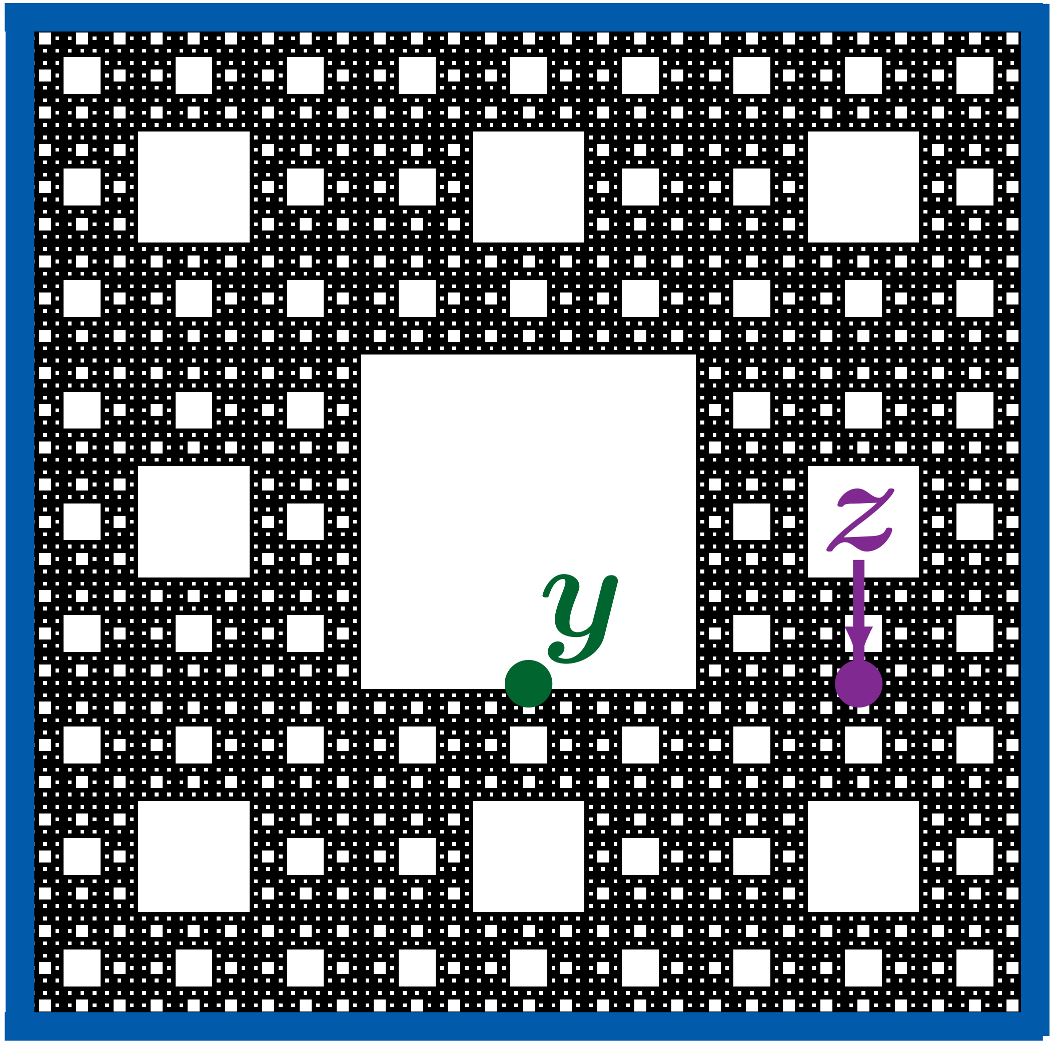
(NP) $\boxed{\limsup_{t\downarrow 0} \left| t^{d_{\mathrm{s}}/2} p_t(x,x) - G(-\log t) \right| \geq c_5}$

for any periodic $G : \mathbb{R} \to \mathbb{R}$.

Key to the proof of Thm 1



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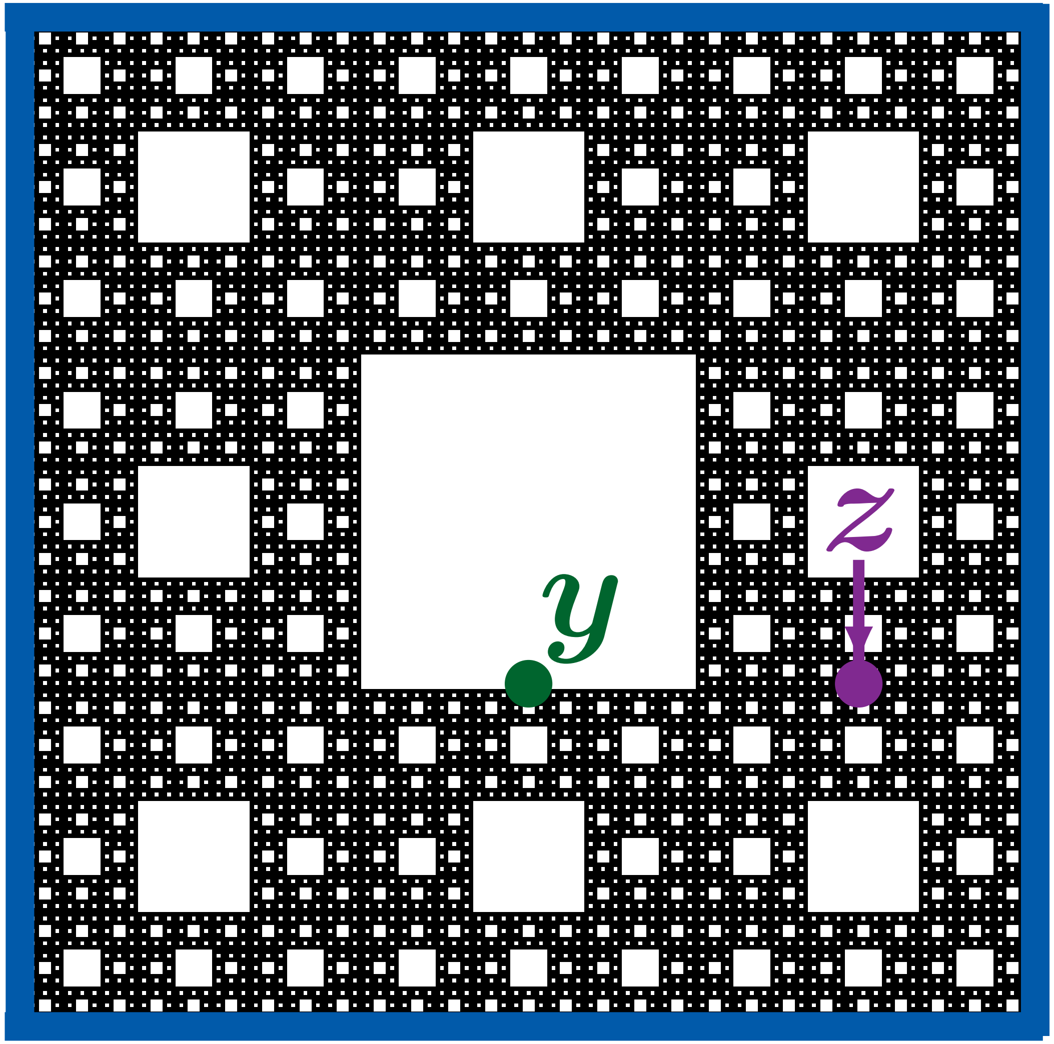

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▷ $\mathcal{Z}_K(t) := \sum_{n=1}^{\infty} e^{-\lambda_n^K t} = \int_K p_t(x,x)\,d\mu(x)$

▷ $\tau \in (1,\infty)$: the time scaling factor for $\{X_t\}_{t\geq 0}$

Thm (K.). $^\exists G_k : \mathbb{R} \longrightarrow \mathbb{R}$ continuous $\log\tau$-periodic for $0 \leq k \leq d$, $G_0, G_1 > 0$ and, as $t \downarrow 0$,

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($d_0 = \dim_{\mathrm{H}} K$, $d_1 = \dim_{\mathrm{H}} \partial K$, $d_{d-1} = 1$ and $d_d = 0$)

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($d_0 = \dim_{\mathrm{H}} K$, $d_1 = \dim_{\mathrm{H}} \partial K$, $d_{d-1} = 1$ and $d_d = 0$)

3 Thm 2. Periodic asymp. expansion of $\mathcal{Z}_K(t)$

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- NOT known whether G_0 and G_k are non-const.

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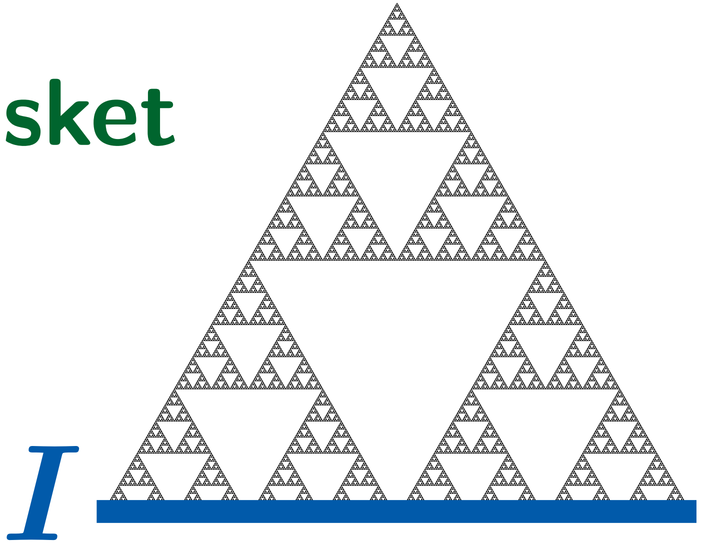
Extension of Thm 2 for nested fractals

▷ $K :=$ the standard Sierpiński gasket

▷ $I :=$ the bottom line of K

▷ $d_{\mathrm{f}} := \log_2 3$, $d_{\mathrm{w}} := \log_2 5$



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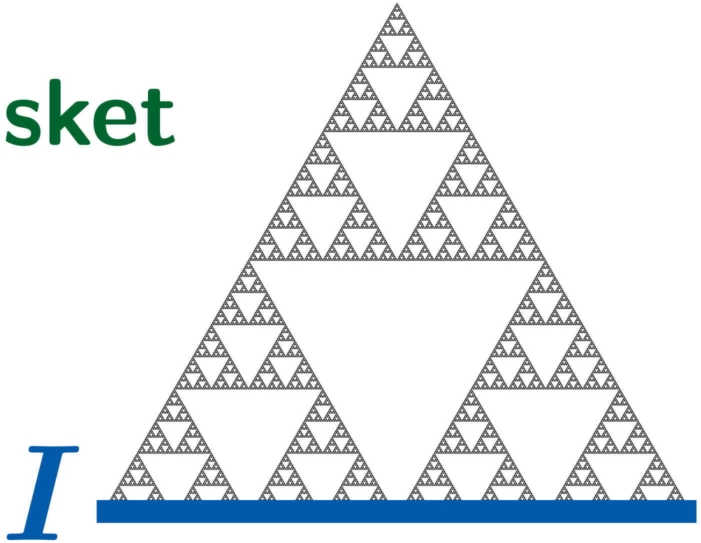

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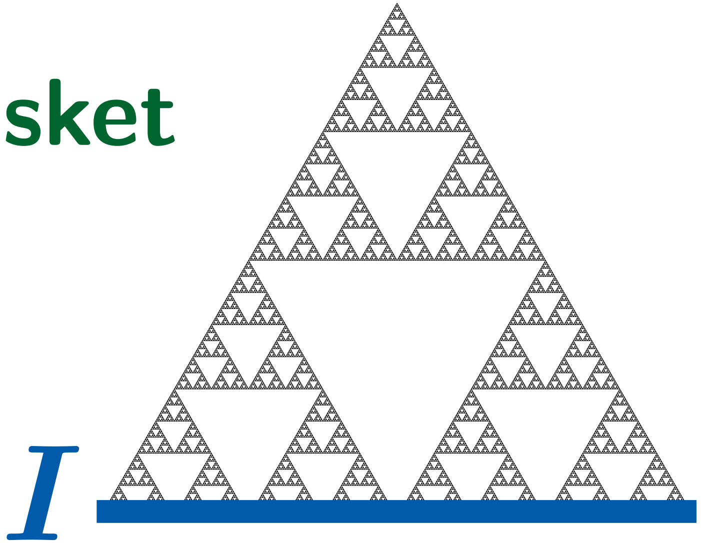

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