Countable fundamental groups of Rauzy fractals

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Joint work with **Benoît Loridant** and **Jun Luo**

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Substitutions

$$\sigma : \begin{cases} 1 & \mapsto & 12 \\ 2 & \mapsto & 3 \\ 3 & \mapsto & 4 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 1 \end{cases} \qquad M_\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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1

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12

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123

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1234

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12345

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12345112123

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$$1234511212312341234512345112345112123451 \cdots$$

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$$123451121231234123451234511234511212345 1\cdots$$

- Cubic Pisot eigenvalue $\beta \approx 1.325$, with $\beta', \beta'' \approx -0.662 \pm 0.562i$
- Two other eigenvalues $\approx 0.5 \pm 0.866i$

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Action of M_σ on $\mathbb{R}^5$:

- **Expanding line** $\mathbb{E}$ spanned by eigenvector $\mathbf{u}_\beta$

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- **Contracting plane** $\mathbb{P}$ spanned by eigenvectors $\mathbf{u}_{\beta'}, \mathbf{u}_{\beta''}$
- (Supplementary space $\mathbb{H}$ spanned by the other eigenvectors)

Rauzy fractal of $12 \mapsto 1, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

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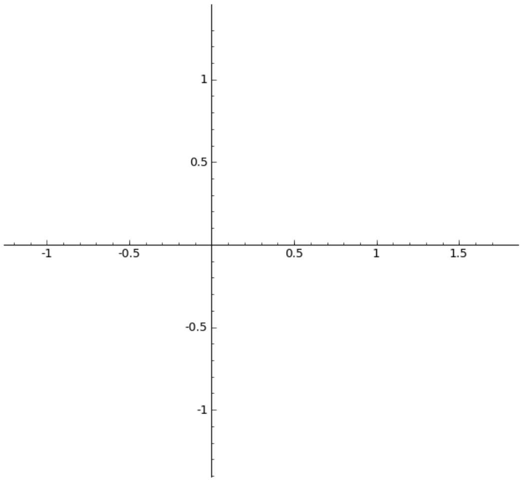
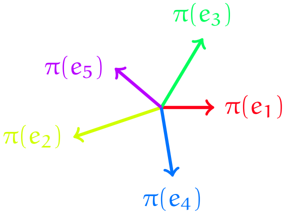
π = projection from $\mathbb{R}^5$ to $\mathbb{P}$ along $\mathbb{E} \oplus \mathbb{H}$





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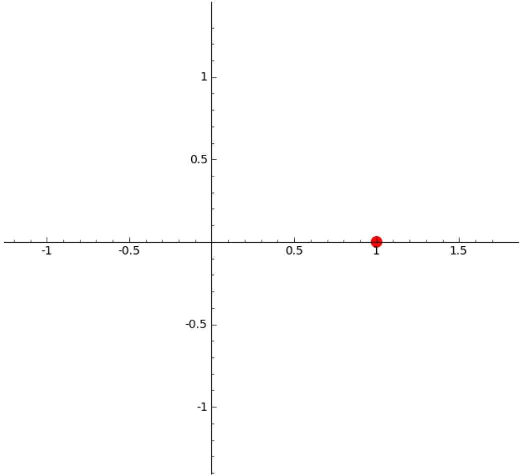
$\pi(e_1)$



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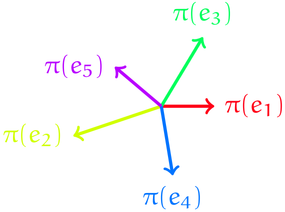

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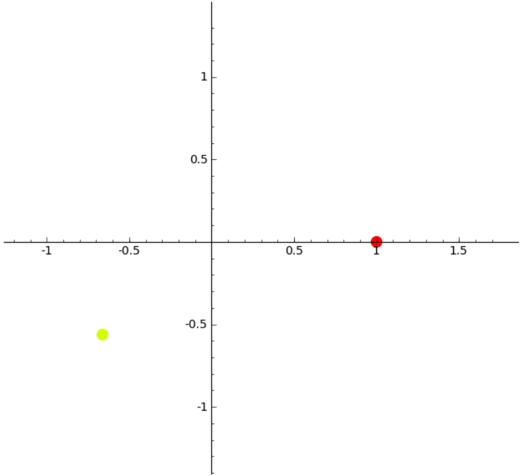
$\pi(e_1) + \pi(e_2)$



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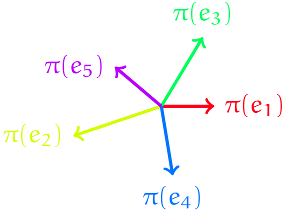

Rauzy fractal of $1 \mapsto 12, 2 \mapsto 3, 3 \mapsto 4, 4 \mapsto 5, 5 \mapsto 1$

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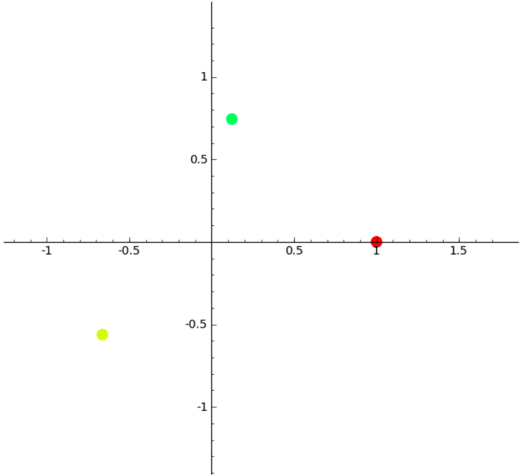
$\pi(e_1) + \pi(e_2) + \pi(e_3)$



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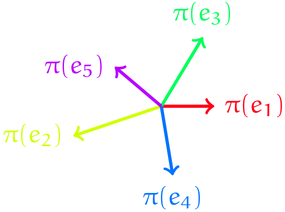

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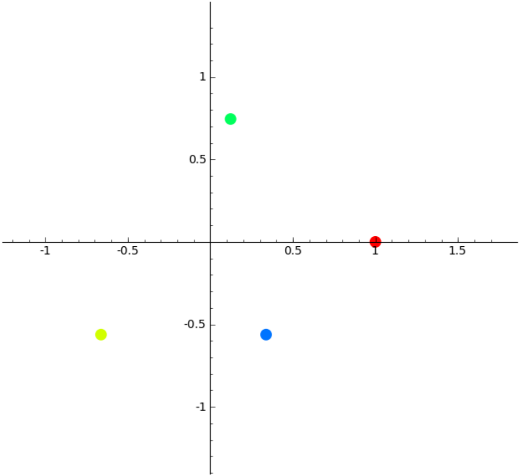
$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4)$



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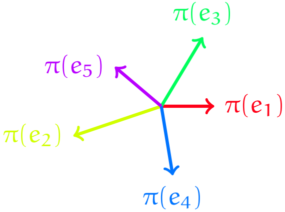

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123451121231234123451234511 23

$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4) + \pi(e_5)$

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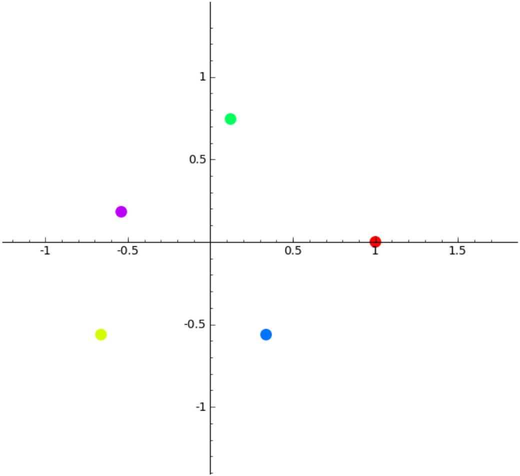

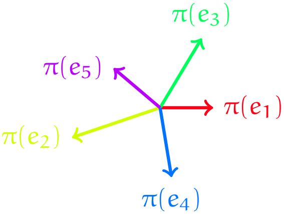

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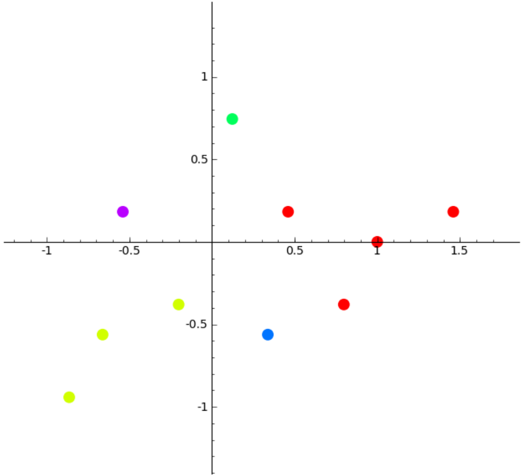
$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4) + \pi(e_5) + \cdots$



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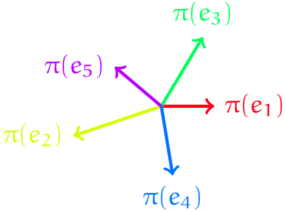

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$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4) + \pi(e_5) + \cdots$

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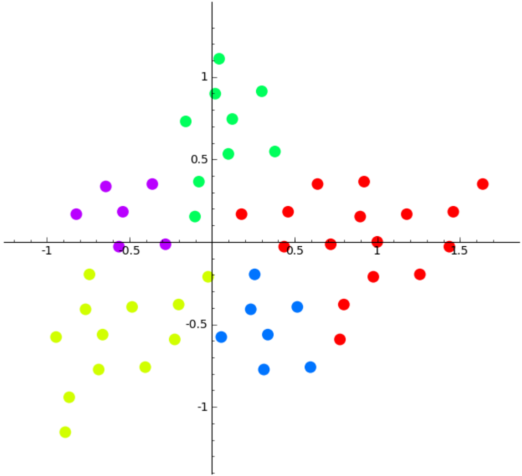

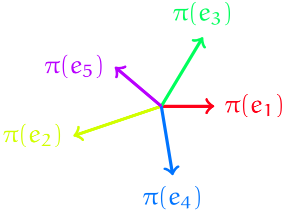

Rauzy fractal of $1 \mapsto 12$, $2 \mapsto 3$, $3 \mapsto 4$, $4 \mapsto 5$, $5 \mapsto 1$

1234511212312341234512345112 3

$\pi(e_1) + \pi(e_2) + \pi(e_3) + \pi(e_4) + \pi(e_5) + \cdots$

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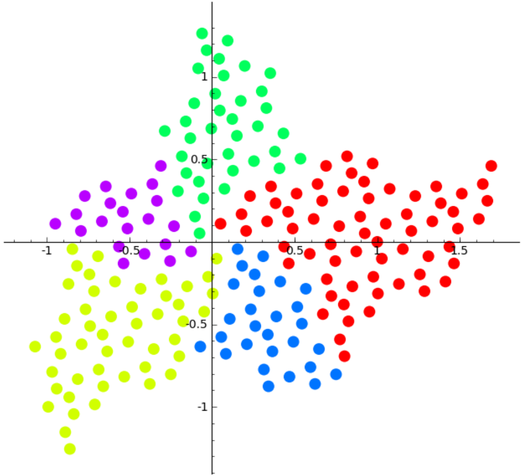

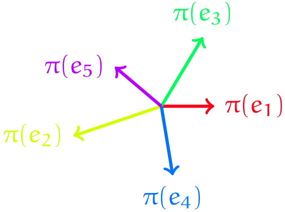

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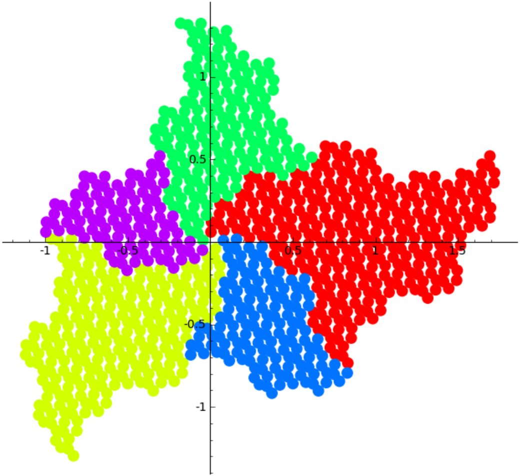

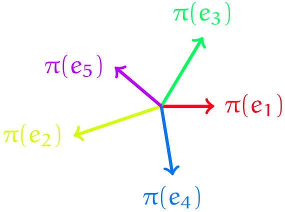

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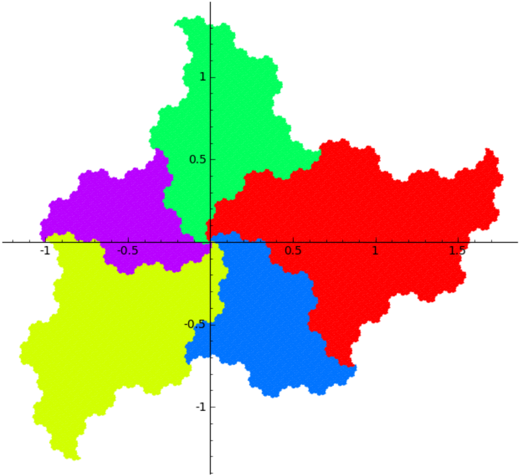
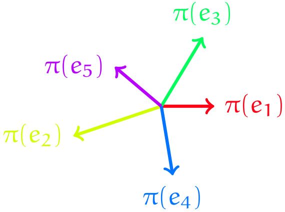
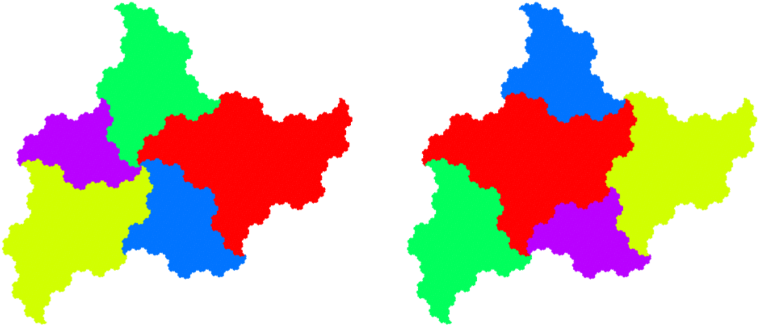




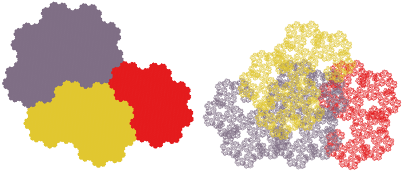
Basic facts

- Dynamical properties (domain exchange), number theory, tilings, ...
- **Topology:**
  - Compact, locally connected
  - Equal to the closure of the interior
  - Graph-directed IFS, self-similar structure
  - $\dim_{\mathrm{H}}(\text{boundary}) \in]1,2[$, computable

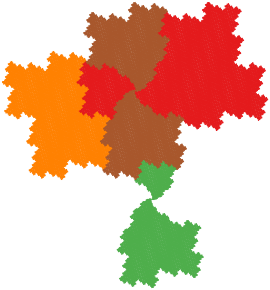


What kind of topological properties do we study?

- Cut points, connectedness, intersection of tiles, etc.



No cut points



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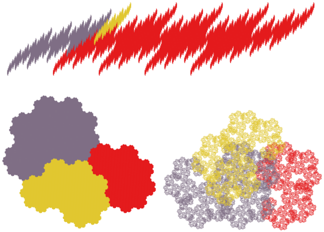
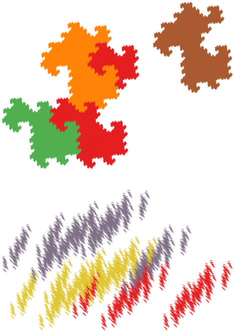


Connected

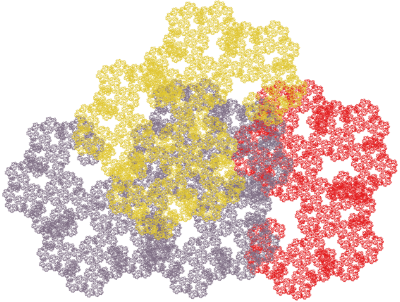


Not connected

What kind of topological properties do we study?

- Cut points, connectedness, intersection of tiles, etc.
- **Simple connectedness**

Disklike



“Many holes”

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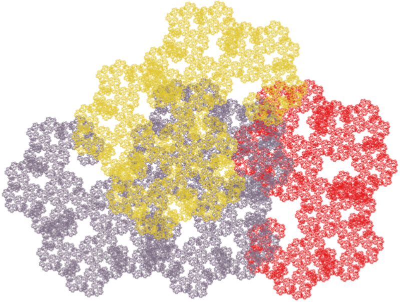
Disklike



"Many holes"

Most of these properties are **decidable** [Siegel-Thuswaldner 2010]

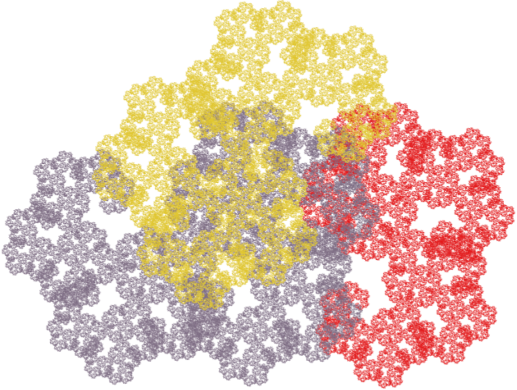
Fundamental group

- Topological space X, point $x_0 \in X$
- $\pi_1(X, x_0)$ is the fundamental group of X at x_0, where:
  - **elements:** loops based at x_0, modulo homotopy
  - **composition law:** concatenation of loops
- $\pi_1(X) := \pi_1(X, x_0)$ if X path-connected (for any x_0)
- Examples:
  - $\pi_1(\text{disc}) \cong \{1\}$
  - $\pi_1(\text{circle}) \cong \mathbb{Z}$
  - $\pi_1(\text{2D torus}) \cong \mathbb{Z}^2$
  - $\pi_1(\infty) \cong \pi_1(8) \cong \pi_1(B) \cong F_2 =$ free group on two elements
  - $\pi_1($❀$) \cong F_6$

Examples

Trivial FG

Examples



Uncountable FG, very hard to describe...

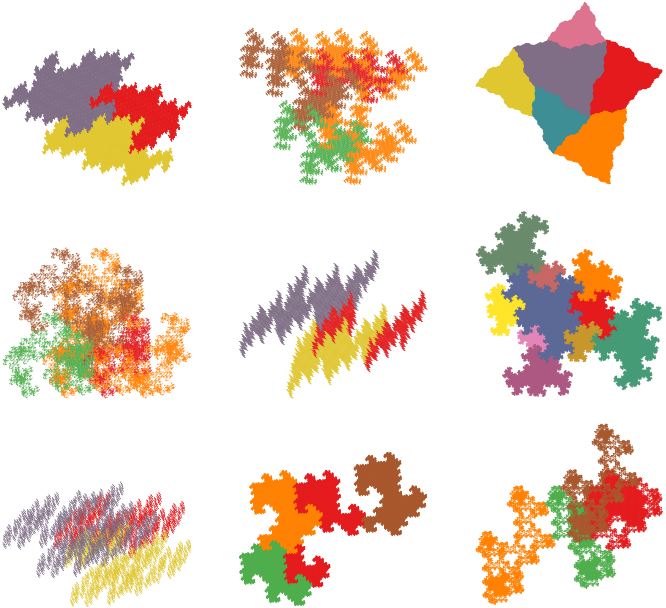
Examples

Doesn't look so bad, but. . .

Examples

Doesn't look so bad, but... uncountable FG

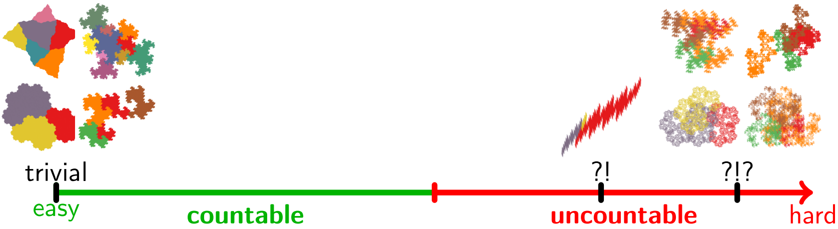
Examples



Examples

There seems to be a dichotomy:

trivial vs. uncountable



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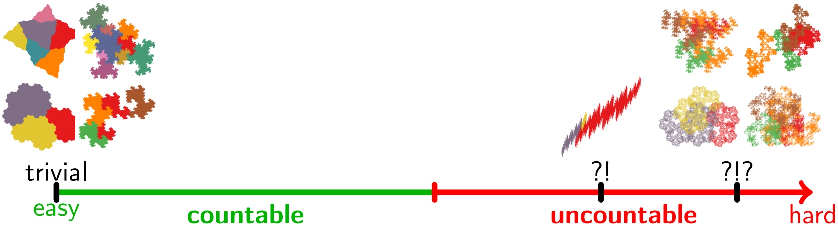
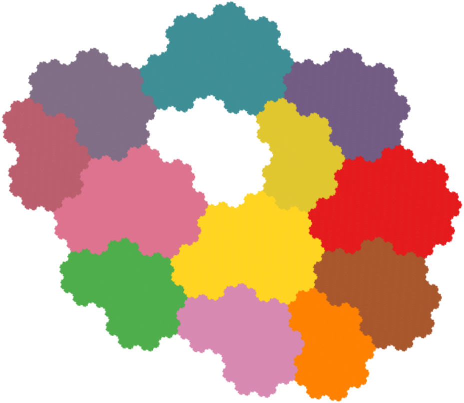


But. . .

Example with nontrivial countable FG



$\pi_1(X) \cong F_1 \cong \mathbb{Z}$

$$
\begin{array}{rclrcl}
1 & \mapsto & 11, 1, 3, 8 & 2 & \mapsto & 11, 1, 3, 8 \\
3 & \mapsto & 5, 2, 4, 10, 12 & 4 & \mapsto & 6, 9, 7 \\
5 & \mapsto & 5, 2, 4, 10, 12 & 6 & \mapsto & 5, 2, 4, 10, 12 \\
7 & \mapsto & 6, 9, 7, 11, 1, 3, 8 & 8 & \mapsto & 6, 9, 7, 11, 1, 3, 8 \\
9 & \mapsto & 11, 1, 3, 8 & 10 & \mapsto & 11, 1, 3, 8 \\
11 & \mapsto & 5, 2, 4, 10, 12 & 12 & \mapsto & 5, 2, 4, 10, 12, 6, 9, 7, 11, 1, 3, 8.
\end{array}
$$

How to prove that $\pi_1 \Big($  $\Big) \cong F_1$?

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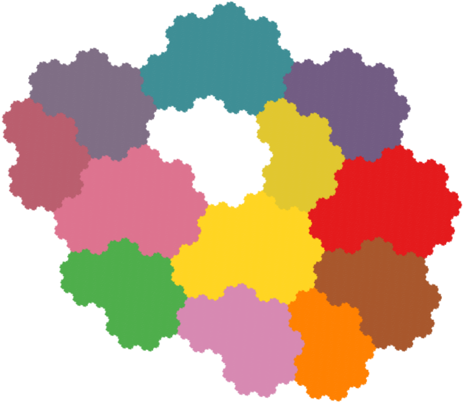

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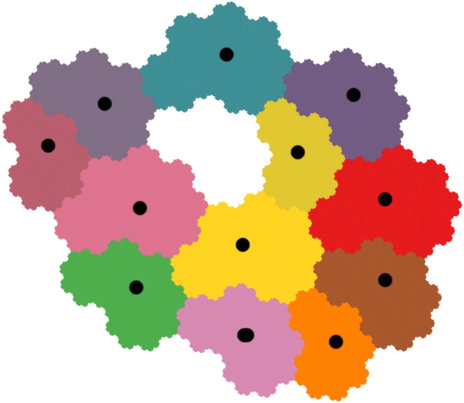

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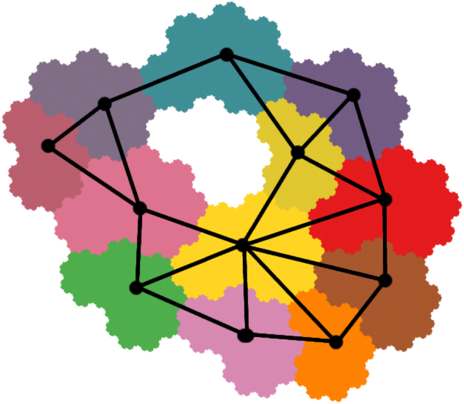

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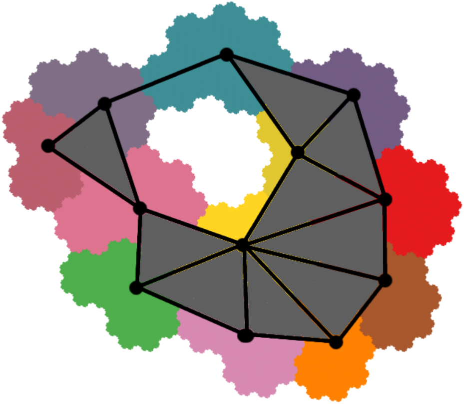
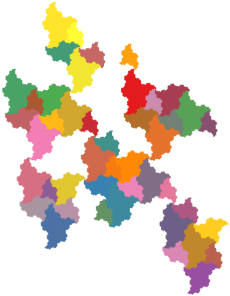
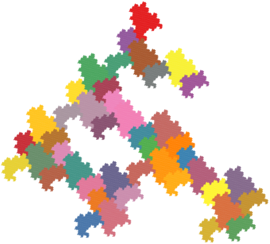


Other examples with nontrivial countable FG



F_1



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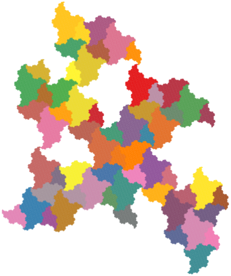
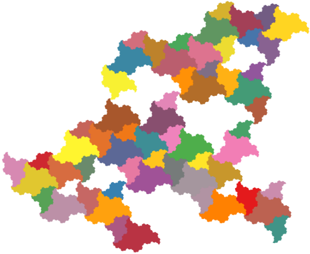


F_2



F_6

Countable FGs of Rauzy fractals, in general

Theorem [J-Loridant-Luo]

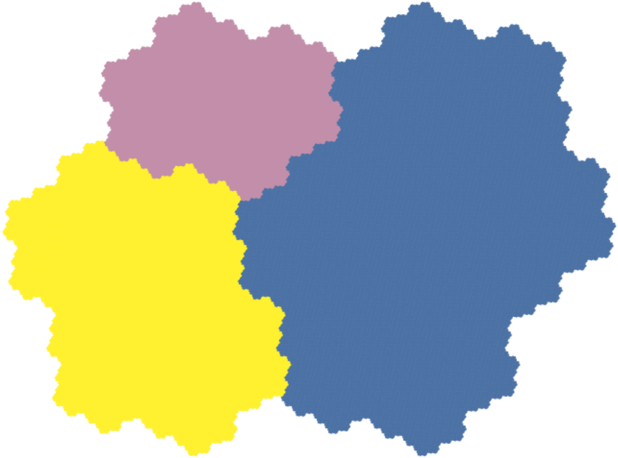
- A nontrivial countable Rauzy fractal FG is always $\cong F_k$ ($k \in \mathbb{N}$).
- k can be computed if all the tiles are discs (and intersect well).

Can we get all F_k?

Difficulty: control the fractal by working on σ.

Can we get all F_k? Possible approach: drill holes

- **Step 1:** cut the tiles into smaller tiles (on σ: "state splittings")



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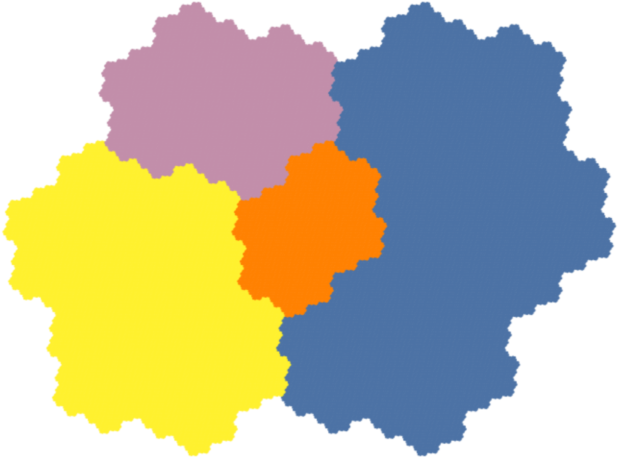

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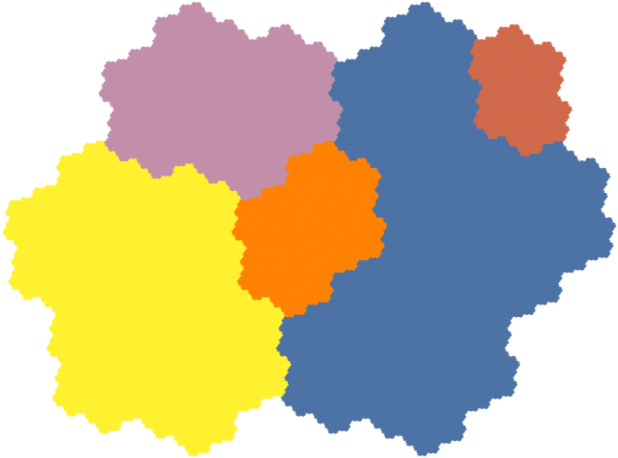

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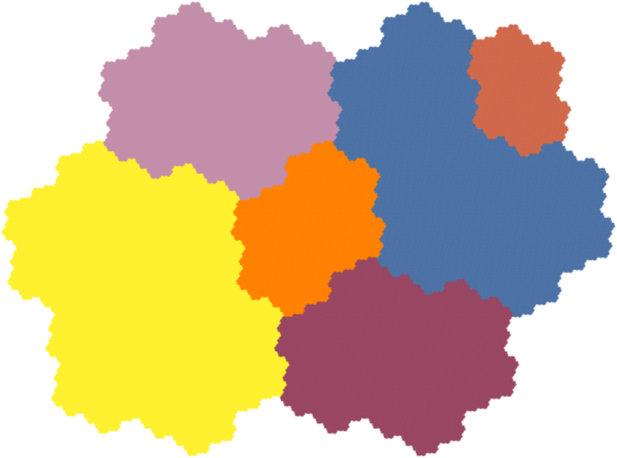

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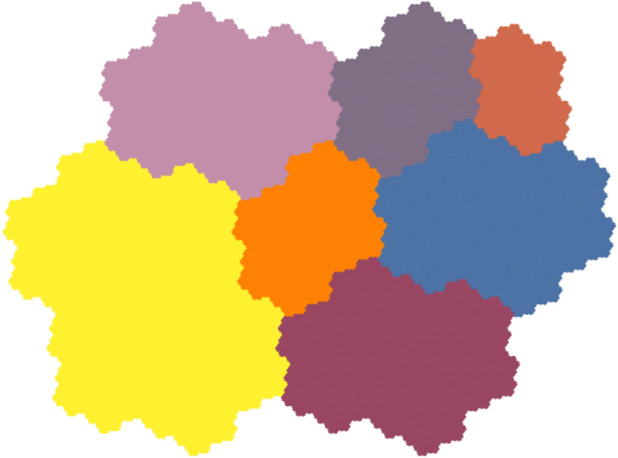

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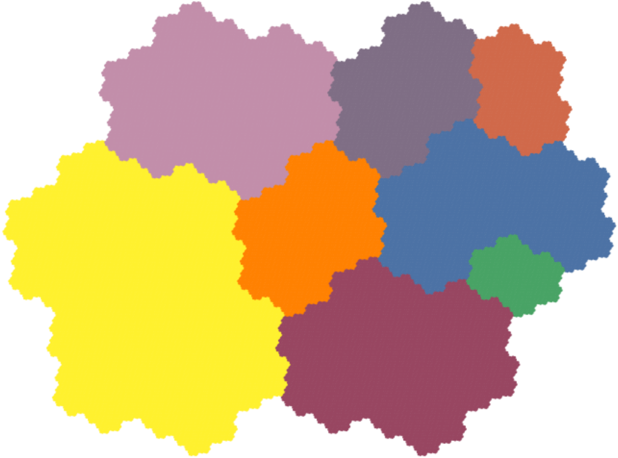

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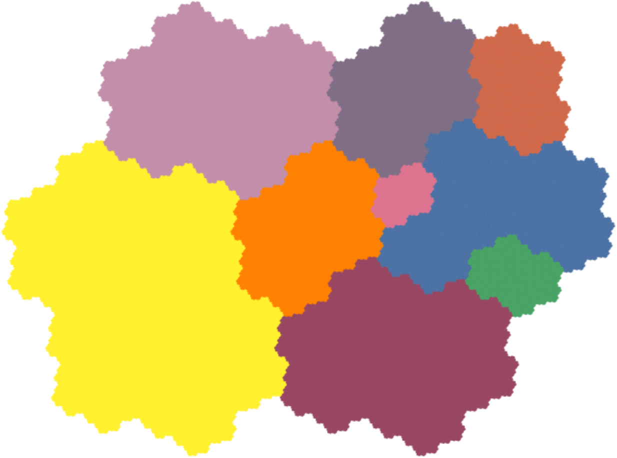

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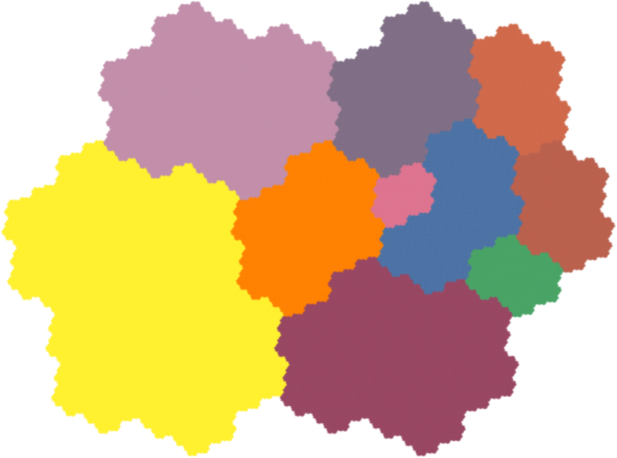

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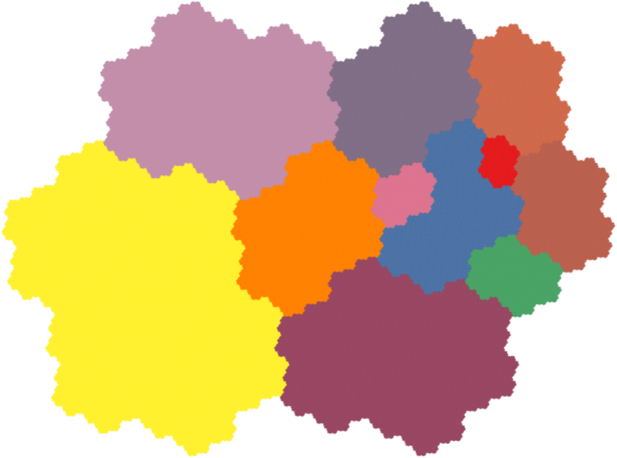

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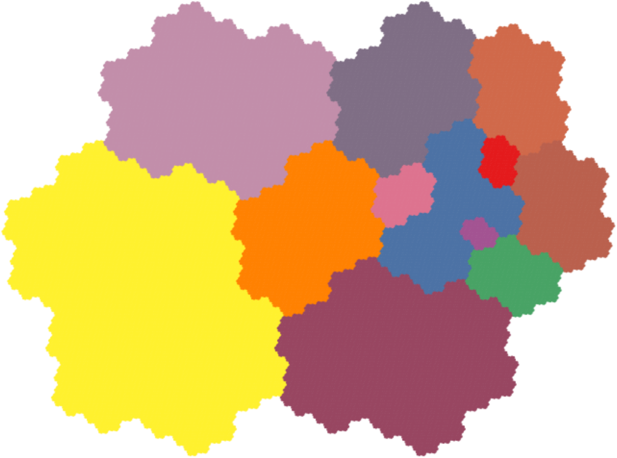

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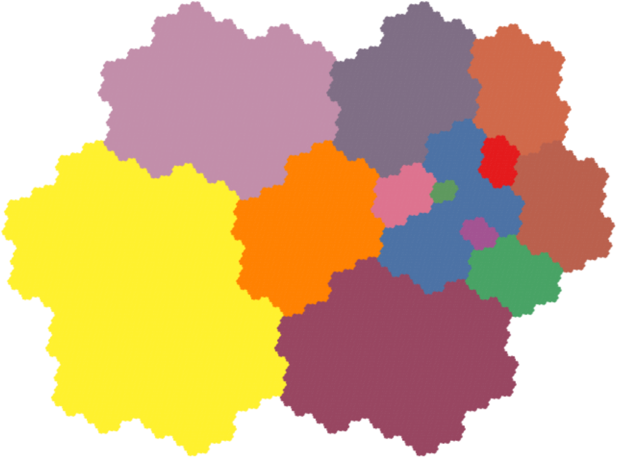

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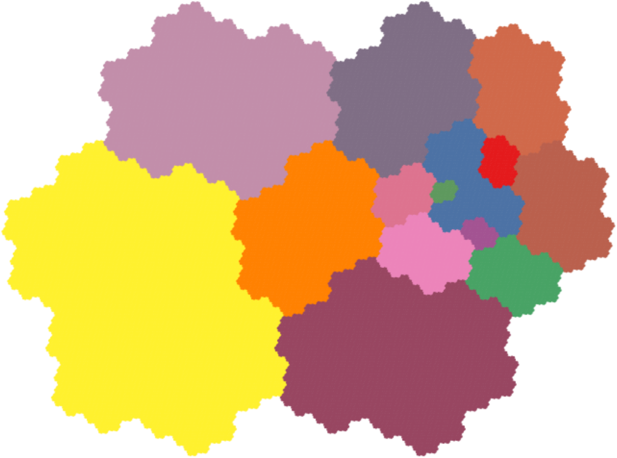

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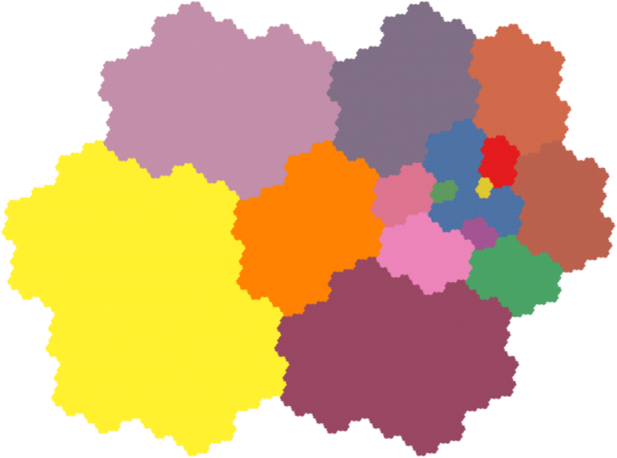

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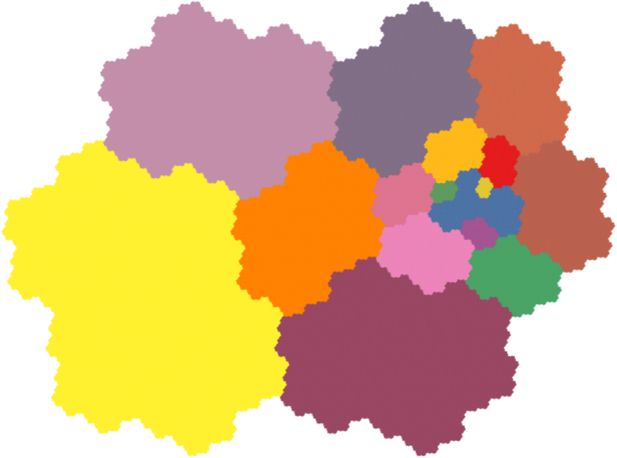

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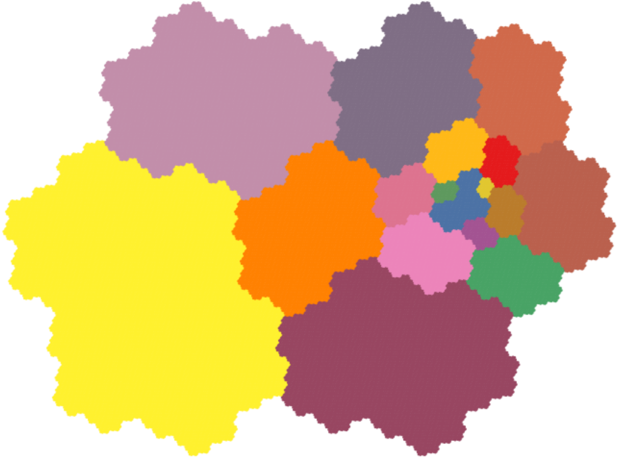

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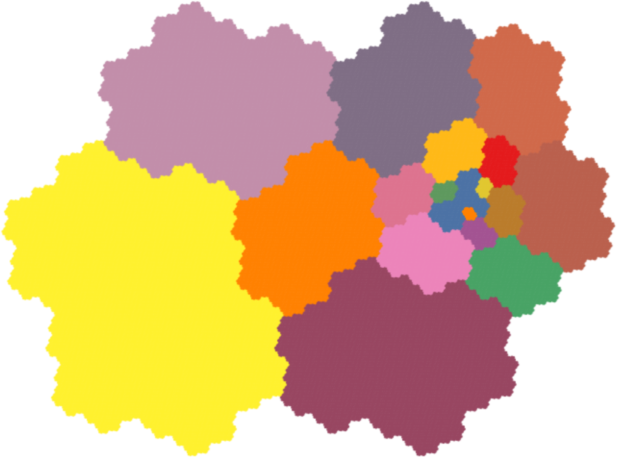

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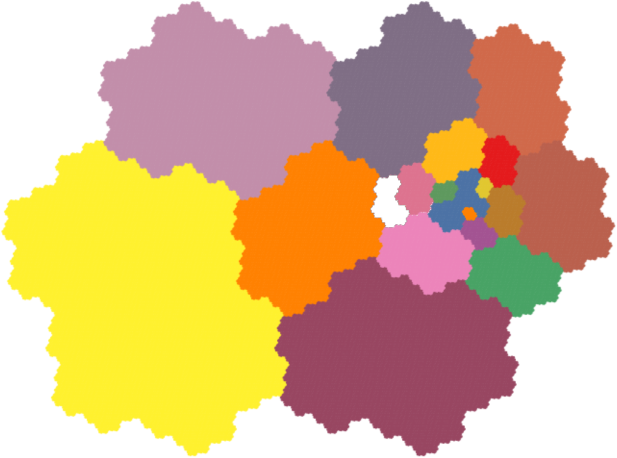

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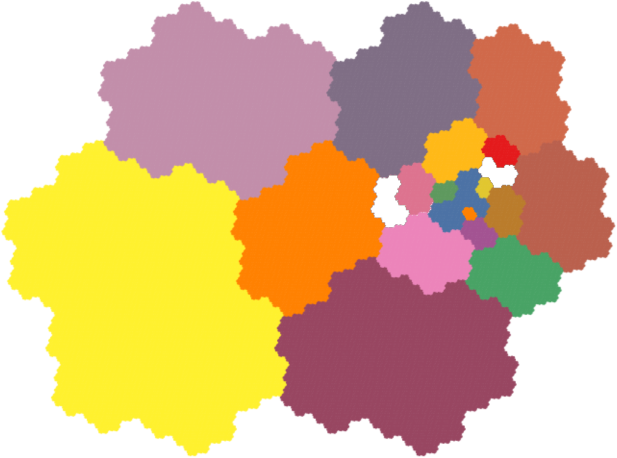

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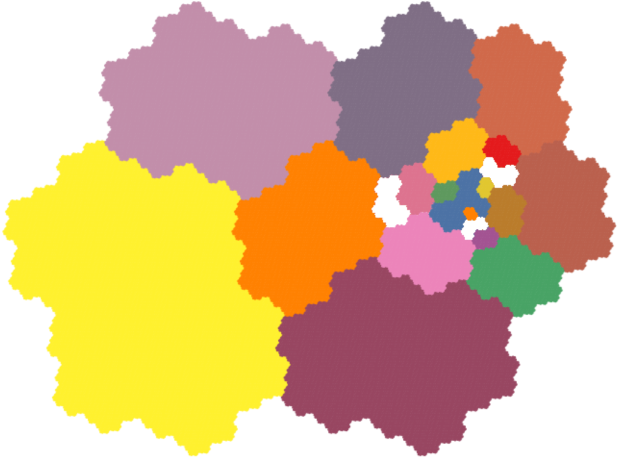

Free group automorphisms $\leftrightarrow$ Rauzy fractals

Compare σ and $\tau^{-1}\sigma\tau$, with

Free group automorphisms ↔ Rauzy fractals

Compare σ and $\tau^{-1}\sigma\tau$, with $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$ and $\tau : 1 \mapsto 1, 2 \mapsto 32, 3 \mapsto 3$

$$\tau^{-1}\sigma\tau : \begin{cases} 1 & \mapsto 1 & \mapsto 12 & \mapsto 132 \\ 2 & \mapsto 3^{-1}2 & \mapsto 1^{-1}13 & \mapsto 3 \\ 3 & \mapsto 3 & \mapsto 1 & \mapsto 1 \end{cases}$$

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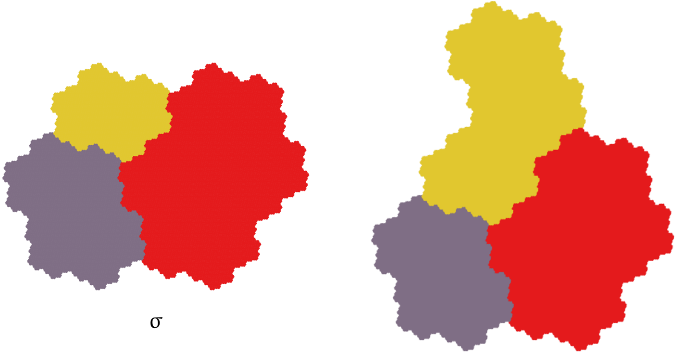

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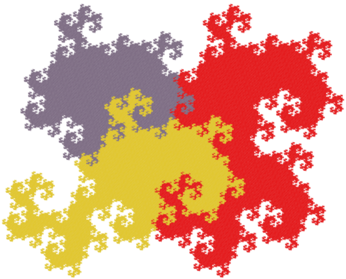

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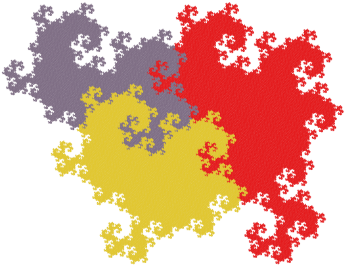

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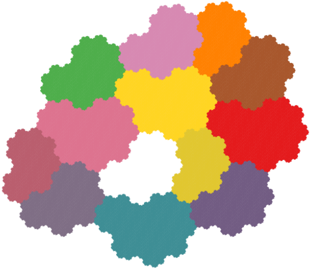

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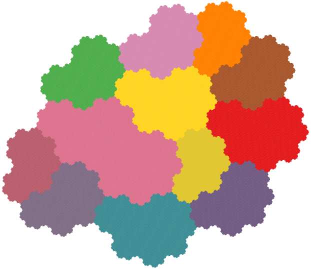

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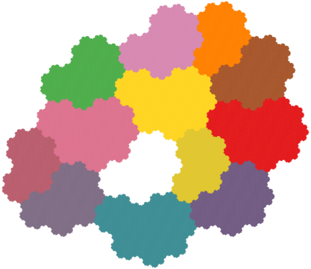

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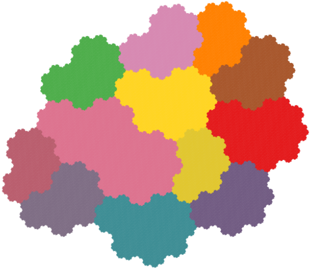


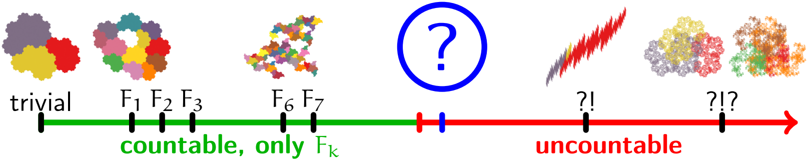
$\tau^{-1}\sigma\tau$

Question [Gähler, Arnoux-Berthé-Hilion-Siegel]

Links between σ and $\tau^{-1}\sigma\tau$?

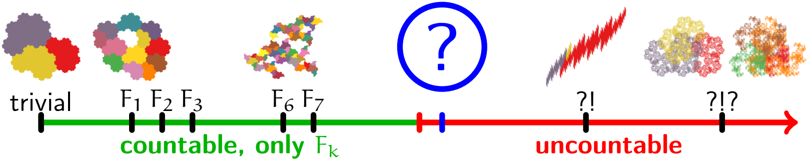
➥ **The fund. group is not preserved!**

Uncountable, but manageable FG?



A Rauzy fractal with **uncountable** FG that we can **describe in detail**?

Uncountable, but manageable FG?



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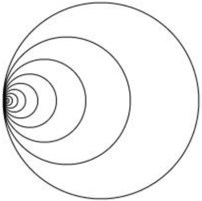
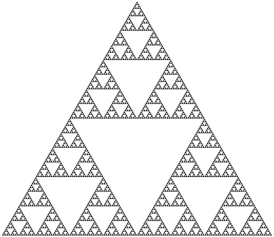
Like:

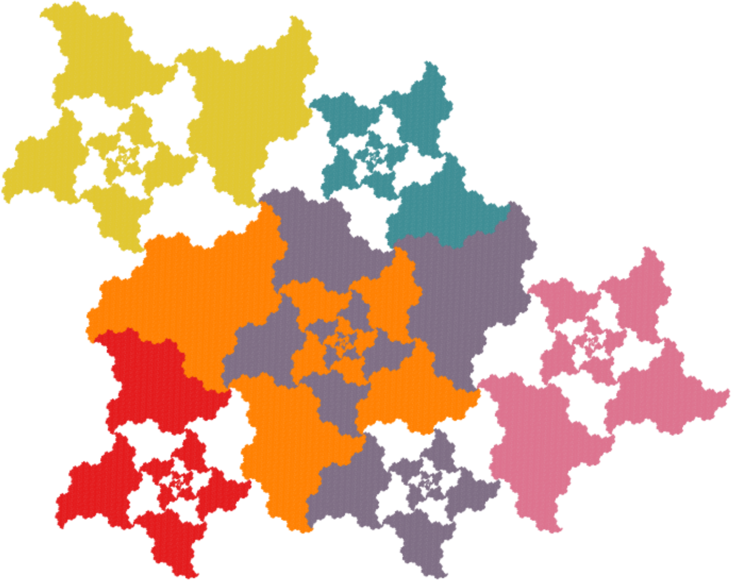


Hawaiian Earring
(Cannon-Conner 2000)

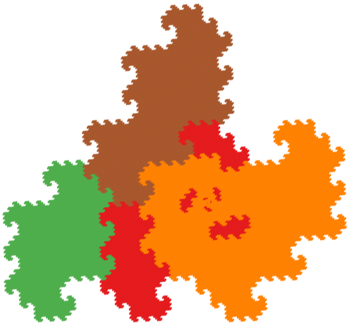


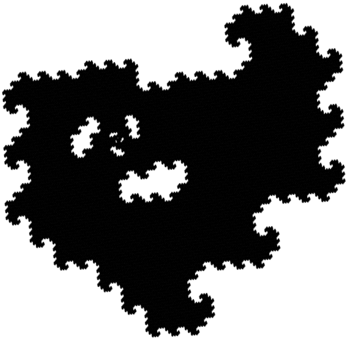
Sierpiński gasket
(Akiyama-Dorfer
-Thuswaldner-Winkler 2009)

Uncountable, but manageable FG? — Candidate 1



Uncountable, but manageable FG? — Candidate 2





Summary

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- "Inverse" IFS problems
- Operations on $\sigma \rightsquigarrow$ fractal
- New examples

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唔該 for your attention