On exact scaling log-Infinitely divisible cascades

Xiong Jin

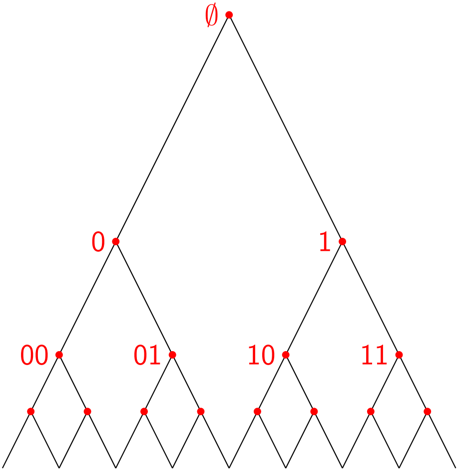
University of St Andrews

xj3@st-andrews.ac.uk

Joint work with Julien Barral

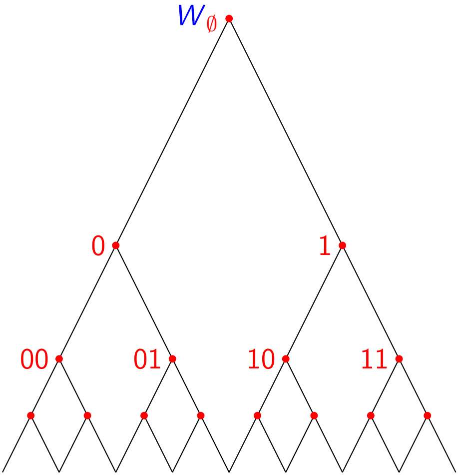
December 11, 2012

Mandelbrot cascades



$\Lambda^* = \{\emptyset\} \cup \{0, 1\} \cup \cdots$: set of all finite dyadic words;

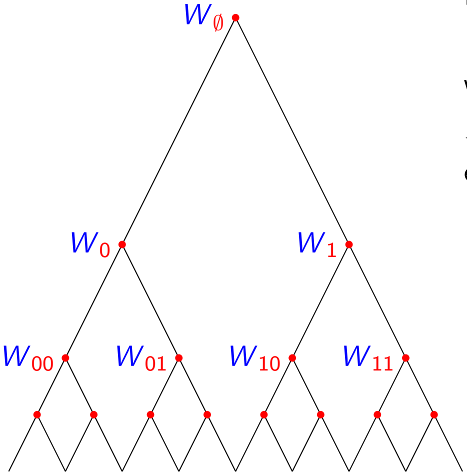
Mandelbrot cascades



$\Lambda^* = \{\emptyset\} \cup \{0, 1\} \cup \cdots$: set of all finite dyadic words;

$W = W_\emptyset \geq 0$: a random variable with expectation 1;

Mandelbrot cascades

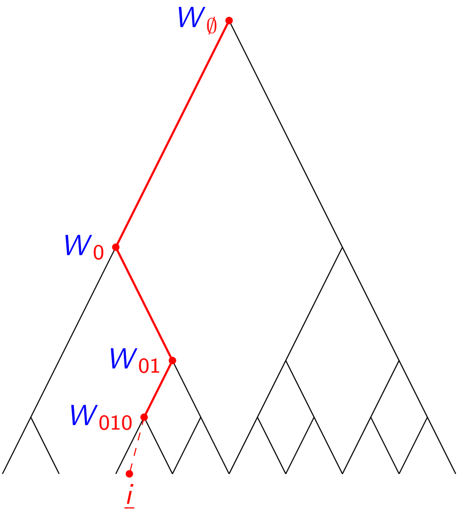


$\Lambda^* = \{\emptyset\} \cup \{0, 1\} \cup \cdots$: set of all finite dyadic words;

$W = W_\emptyset \geq 0$: a random variable with expectation 1;

$\{W_i : i \in \Lambda^*\}$: an i.i.d. sequence encoded by Λ^*;

Mandelbrot cascades



$\Lambda^* = \{\emptyset\} \cup \{0, 1\} \cup \cdots$: set of all finite dyadic words;

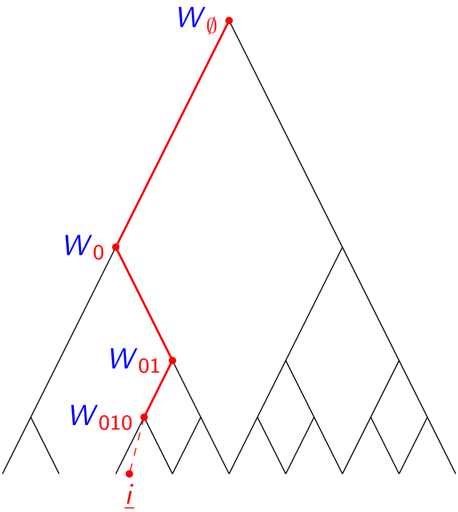
$W = W_\emptyset \geq 0$: a random variable with expectation 1;

$\{W_i : i \in \Lambda^*\}$: an i.i.d. sequence encoded by Λ^*;

For $n \geq 1$ and $\underline{i} \in \Lambda^{\mathbb{N}}$ define

$$Q_n(\underline{i}) = W_{i_1} W_{i_1 i_2} \cdots W_{i_1 \cdots i_n};$$

Mandelbrot cascades



$\Lambda^* = \{\emptyset\} \cup \{0, 1\} \cup \cdots$: set of all finite dyadic words;

$W = W_\emptyset \geq 0$: a random variable with expectation 1;

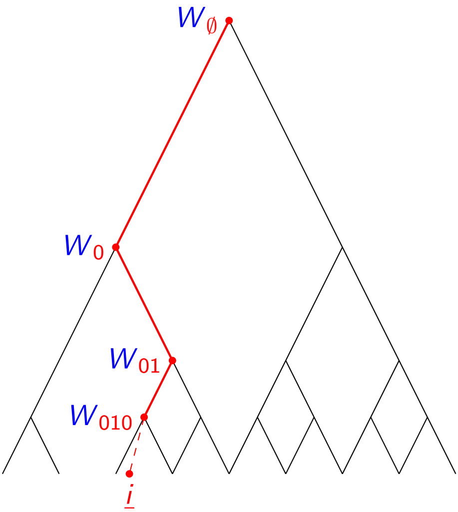
$\{W_i : i \in \Lambda^*\}$: an i.i.d. sequence encoded by Λ^*;

For $n \geq 1$ and $\underline{i} \in \Lambda^{\mathbb{N}}$ define

$$Q_n(\underline{i}) = W_{i_1} W_{i_1 i_2} \cdots W_{i_1 \cdots i_n};$$

For $n \geq 1$: $\mu_n(\mathrm{d}\underline{i}) = Q_n(\underline{i})\, \mathrm{d}\underline{i}$;

Mandelbrot cascades



$\Lambda^* = \{\emptyset\} \cup \{0,1\} \cup \cdots$: set of all finite dyadic words;

$W = W_\emptyset \geq 0$: a random variable with expectation 1;

$\{W_i : i \in \Lambda^*\}$: an i.i.d. sequence encoded by Λ^*;

For $n \geq 1$ and $\underline{i} \in \Lambda^{\mathbb{N}}$ define

$$Q_n(\underline{i}) = W_{i_1} W_{i_1 i_2} \cdots W_{i_1 \cdots i_n};$$

For $n \geq 1$: $\mu_n(\mathrm{d}\underline{i}) = Q_n(\underline{i})\,\mathrm{d}\underline{i}$;

A measure-valued martingale:

$$\mu_n \to \mu.$$

Kahane & Peyrière 76

Let $Z = \|\mu\|$ and $\varphi(q) = \log_2 \mathbb{E}(W^q) - q + 1$ on $I = \{q : \mathbb{E}(W^q) < \infty\}$.

Kahane & Peyrière 76

Let $Z = \|\mu\|$ and $\varphi(q) = \log_2 \mathbb{E}(W^q) - q + 1$ on $I = \{q : \mathbb{E}(W^q) < \infty\}$.

Non-degeneracy (ND) [Kahane]

$\mathbb{E}(Z) > 0 \Leftrightarrow \mathbb{E}(Z) = 1 \Leftrightarrow \varphi'(1^-) < 0.$

Kahane & Peyrière 76

Let $Z = \|\mu\|$ and $\varphi(q) = \log_2 \mathbb{E}(W^q) - q + 1$ on $I = \{q : \mathbb{E}(W^q) < \infty\}$.

Non-degeneracy (ND) [Kahane]

$\mathbb{E}(Z) > 0 \Leftrightarrow \mathbb{E}(Z) = 1 \Leftrightarrow \varphi'(1^-) < 0.$

Moments of positive orders (MP) [Kahane]

For $q > 1$, $0 < \mathbb{E}(Z^q) < \infty \Leftrightarrow q \in I \;\&\; \varphi(q) < 0.$

Kahane & Peyrière 76

Let $Z = \|\mu\|$ and $\varphi(q) = \log_2 \mathbb{E}(W^q) - q + 1$ on $I = \{q : \mathbb{E}(W^q) < \infty\}$.

Non-degeneracy (ND) [Kahane]

$\mathbb{E}(Z) > 0 \Leftrightarrow \mathbb{E}(Z) = 1 \Leftrightarrow \varphi'(1^-) < 0.$

Moments of positive orders (MP) [Kahane]

For $q > 1$, $0 < \mathbb{E}(Z^q) < \infty \Leftrightarrow q \in I$ & $\varphi(q) < 0$.

Finite moments of all positive orders (FMP) [Kahane]

$(\uparrow) \Leftrightarrow \|W\|_\infty \leq 2$ & $\mathbb{P}(W = 2) < \frac{1}{2} \Rightarrow \lim_{q\to\infty} \frac{\log \mathbb{E}(Z^q)}{q \log q} = \log_2 \|W\|_\infty$.

Kahane & Peyrière 76

Let $Z = \|\mu\|$ and $\varphi(q) = \log_2 \mathbb{E}(W^q) - q + 1$ on $I = \{q : \mathbb{E}(W^q) < \infty\}$.

Non-degeneracy (ND) [Kahane]

$\mathbb{E}(Z) > 0 \Leftrightarrow \mathbb{E}(Z) = 1 \Leftrightarrow \varphi'(1^-) < 0.$

Moments of positive orders (MP) [Kahane]

For $q > 1$, $0 < \mathbb{E}(Z^q) < \infty \Leftrightarrow q \in I \ \& \ \varphi(q) < 0.$

Finite moments of all positive orders (FMP) [Kahane]

$(\uparrow) \Leftrightarrow \|W\|_\infty \leq 2 \ \& \ \mathbb{P}(W = 2) < \frac{1}{2} \Rightarrow \lim_{q\to\infty} \frac{\log \mathbb{E}(Z^q)}{q \log q} = \log_2 \|W\|_\infty$.

Hausdorff dimension (HD) [Peyrière], [Kahane 87]

Almost surely $\dim_H \mu = -\varphi'(1^-)$.

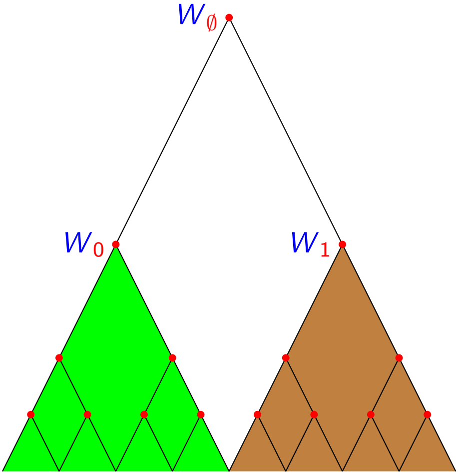
Guivarc'h 90

Infinite moments of some positive orders (IMP)

If there exists $\xi \in (1, \infty) \cap I$ s.t. $\varphi(\xi) = 0$ and the distribution of $\log(W)$ is non-arithmetic, then there exists a constant $0 < d < \infty$ such that

$$\lim_{x \to \infty} x^{\xi} \mathbb{P}(Z > x) = d.$$

Key ingredient: a functional equation



$$2Z = W_0 Z_0 + W_1 Z_1$$

A sketched history

- Mandelbrot 1972 - 1974

A sketched history

- Mandelbrot 1972 - 1974
- Kahane & Peyrière 1976

A sketched history

- Mandelbrot 1972 - 1974
- Kahane & Peyrière 1976
- Kahane 1985

A sketched history

- Mandelbrot 1972 - 1974
- Kahane & Peyrière 1976
- Kahane 1985
- Fan 1997

A sketched history

- Mandelbrot 1972 - 1974
- Kahane & Peyrière 1976
- Kahane 1985
- Fan 1997
- Barral & Mandelbrot 2002

A sketched history

- Mandelbrot 1972 - 1974
- Kahane & Peyrière 1976
- Kahane 1985
- Fan 1997
- Barral & Mandelbrot 2002
- Bacry & Muzy 2003

Independently scattered random measures (ISRM)

$\mathbb{H} = \mathbb{R} \times \mathbb{R}_+$: upper half-plane; λ the measure on $\mathbb{H}$ with

$$\lambda(\mathrm{d}x\mathrm{d}y) = y^{-2}\mathrm{d}x\mathrm{d}y.$$

ψ: a characteristic Lévy exponent given by

$$\psi : q \in \mathbb{R} \mapsto iaq - \frac{1}{2}\sigma^2 q^2 + \int_{\mathbb{R}} \left(e^{iqx} - 1 - iqx\mathbf{1}_{|x|\leq 1}\right) \nu(\mathrm{d}x).$$

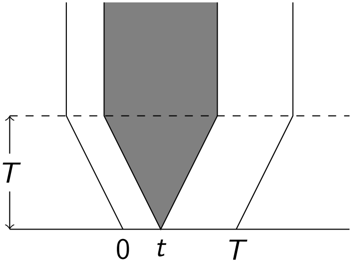
Λ: a (ψ, λ) ISRM: that is for any $B \in \mathcal{B}$ with $\lambda(B) < \infty$,

$$\mathbb{E}(e^{iq\Lambda(B)}) = e^{\psi(q)\lambda(B)}.$$

In particular, for any two disjoint B_1, B_2, the random variable $\Lambda(B_1)$ and $\Lambda(B_2)$ are independent.
Assumption: $\psi(-i) = 0$ and

$$[0,1] \subset I_\nu := \{q \in \mathbb{R} : \int_{|x|\geq 1} e^{qx}\, \nu(\mathrm{d}x) < \infty\}.$$

Log-infinitely divisible cascades



Fix $T > 0$. For $t \in [0, T]$ take

$$V^T(t) = \text{the gray cone.}$$

For $\epsilon > 0$ let

$$V_\epsilon^T(t) = V^T(t) \cap \{y > \epsilon\}.$$

Then let

$$\mu_\epsilon(\mathrm{d}t) = e^{\Lambda(V_\epsilon^T(t))}\,\mathrm{d}t.$$

A measure-valued martingale:

$$\mu_\epsilon \to \mu.$$

Log-infinitely divisible cascades

Let $T = 1$, $Z = \|\mu\|$ and $\varphi(q) = \psi(-iq) - q + 1$ on I_ν.

Barral & Mandelbrot 02; Bacry & Muzy 03

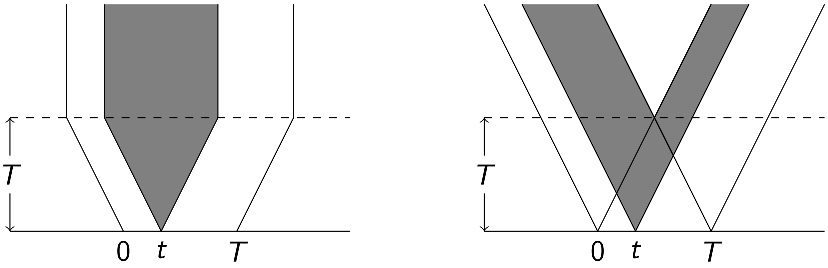
ND	MP	FMP	HD	IMP
almost	almost	not known	almost	not known
$\Rightarrow \varphi'(1^-) \leq 0$	$\Rightarrow \varphi(q) \leq 0$		$\varphi'(1)$ exists	

Log-infinitely divisible cascades

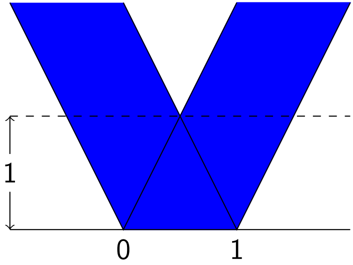
Let $T = 1$, $Z = \|\mu\|$ and $\varphi(q) = \psi(-iq) - q + 1$ on I_ν.

Barral & J. 2012 (arXiv:1208.2221)

ND	MP	FMP	HD	IMP
✓	✓	✓	✓	✓

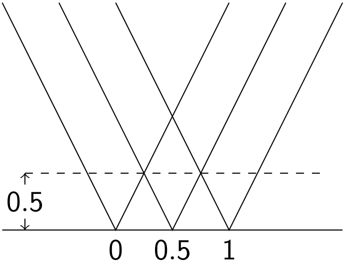

T
0
t
T
T
0
t
T

A “non-independent” functional equation



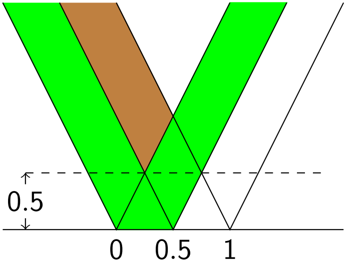
Z

A "non-independent" functional equation



Z

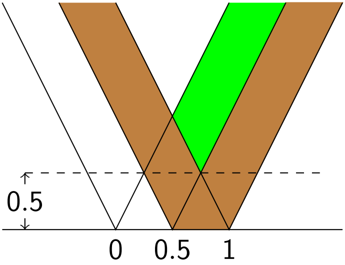
A "non-independent" functional equation



Z

W_0 and Z_1

A "non-independent" functional equation

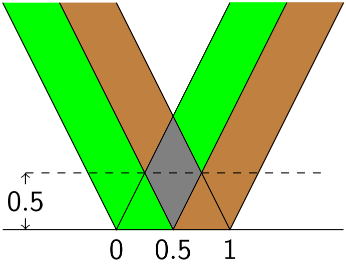


Z

W_0 and Z_0

W_1 and Z_1

A "non-independent" functional equation



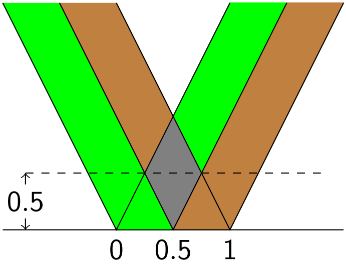
Z

W_0 and Z_0

W_1 and Z_1

$$2Z = W_0 Z_0 + W_1 Z_1$$

A "non-independent" functional equation



$$Z$$

$$W_0 \text{ and } Z_0$$

$$W_1 \text{ and } Z_1$$

$$2Z = W_0 Z_0 + W_1 Z_1$$

For the critical value $\xi > 1$ with $\varphi(\xi) = 0$, though

$$\mathbb{E}(Z^{\xi}) = \infty,$$

but

$$\mathbb{E}({Z_0}^{\xi-1} Z_1) < \infty!$$

Goldie's implicit renewal theory

Goldie 91

Suppose there exists $\kappa > 0$ such that

$$\mathbb{E}(A^{\kappa}) = 1, \quad \mathbb{E}(A^{\kappa} \log^{+} A) < \infty, \tag{1}$$

and suppose that the conditional law of $\log A$, given $A \neq 0$, is non-arithmetic. For

$$\widetilde{R} = AR + B,$$

where $\widetilde{R}$ and R have the same law, and A and R are independent, we have that if

$$\mathbb{E}\left((AR + B)^{\kappa} - (AR)^{\kappa}\right) < \infty,$$

then

$$\lim_{t \to \infty} t^{\kappa} \mathbb{P}(R > t) = \frac{\mathbb{E}\left((AR + B)^{\kappa} - (AR)^{\kappa}\right)}{\kappa \mathbb{E}(A^{\kappa} \log A)} \in (0, \infty).$$

Thanks!