Hausdorff dimension of affine random covering sets in torus

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AFRT, Hong Kong 2012

Joint work with E. Järvenpää, H. Koivusalo, B. Li and V. Suomala

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- Fan and Kahane (1993), Fan (2002), Barral and Fan (2005)

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- Li, Shieh and Xiao

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Proposition

Almost surely

$$\dim_H E = \min\{s_0, d\},$$

where $s_0 = \inf\{s \geq 0 \mid \sum_{n=1}^{\infty} \rho_n^s < \infty\}$.

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The mass transference priciple gives $\mathcal{H}^s(\limsup_{n\to\infty} B_n) = \infty$, which leads to $\dim_H E \geq \min\{s_0, d\}$.

Higher dimensional case: main theorem

- Given a contractive linear injection $L : \mathbb{R}^d \to \mathbb{R}^d$, let $0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1$ be the singular values of L.

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- Given a contractive linear injection $L : \mathbb{R}^d \to \mathbb{R}^d$, let $0 < \alpha_d(L) \leq \cdots \leq \alpha_1(L) < 1$ be the singular values of L.
- For $0 < s \leq d$, define the *singular value function* by

$$\Phi^s(L) = \alpha_1(L) \cdots \alpha_{m-1}(L) \alpha_m(L)^{s-m+1},$$

where m is the integer such that $m - 1 < s \leq m$.

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Theorem

Almost surely $\dim_H E = s_0$.

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- The verification of the upper bound: Falconer.

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- This gives $P(\dim_H E^\omega \geq s) > 0$.
- The Kolmogorov zero-one law implies that $P(\dim_H E \geq s) = 1$.