# Dimensions of random affine code tree fractals

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+ B. Li

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If $\|T_i\| < \frac{1}{3}$ for all $i$, then $\dim_{\mathrm{H}} E_a = \dim_{\mathrm{p}} E_a = \dim_{\mathrm{B}} E_a = s_0$ for $\mathcal{L}^{Md}$-almost all $\mathbf{a} \in \mathbb{R}^{Md}$, where $p(s_0) = 0$ and $p$ is the pressure.

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- $p$ exists since $\Phi^s$ is submultiplicative

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**Theorem (Jordan, Pollicott, Simon 2007)**

Random perturbation on $\mathbf{a} \implies \|T_i\| < 1$ enough in Falconer's theorem

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- $p$ may not exist
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- $\dim_{\mathrm{H}} E_{\mathbf{a}}^\omega \neq \dim_{\mathrm{p}} E_{\mathbf{a}}^\omega \neq \dim_{\mathrm{B}} E_{\mathbf{a}}^\omega$ possible

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- $P$ ergodic and $\Xi$-invariant
- $\int N_1 \, dP < \infty$, where $N_1$ is the first neck level

Then there exists $s_0$ with $p(s_0) = 0$ almost surely

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Then $P$-almost surely $\dim_{\mathrm{H}} E_a = \dim_{\mathrm{p}} E_a = \dim_{\mathrm{B}} E_a = s_0$ for almost all **a**, where $s_0$ is the zero of the pressure.

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- (1) is necessary but (2) is not