Multifractal analysis of arithmetic functions

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Multifractal analysis

Purpose of multifractal analysis : Introduce and study classification parameters for data (functions, measures, distributions, signals, images), which are based on regularity

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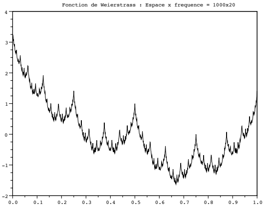

Weierstrass function

$$W_H(x) = \sum_{j=0}^{+\infty} 2^{-Hj} \cos(2^j x)$$

$$0 < H < 1$$

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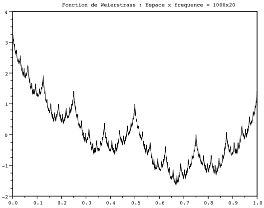

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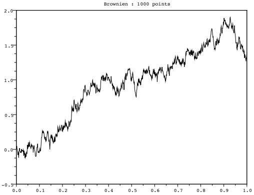
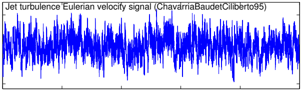
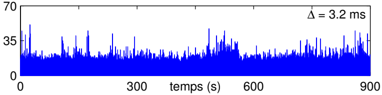


Brownian motion

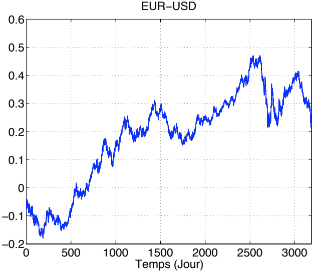
Everywhere irregular signals and images



Fully developed turbulence

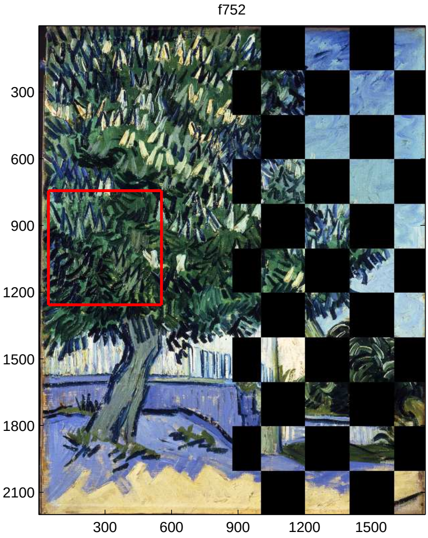


Internet Trafic



Euro vs Dollar (2001-2009)

Van Gogh painting



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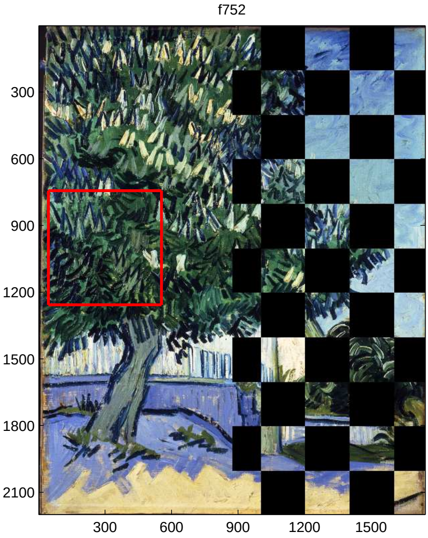

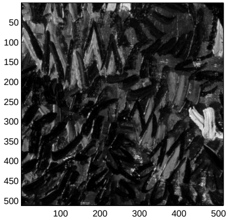

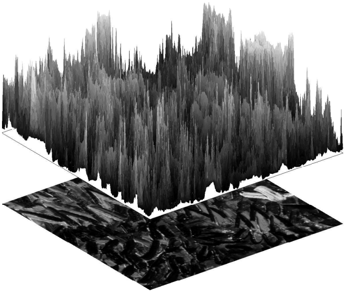

Pointwise regularity

Definition :

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function and $x_0 \in \mathbb{R}^d$;
$f \in C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P such that, for $|x - x_0|$ small enough,

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\alpha$$

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The Hölder exponent of f at x_0 is

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The Hölder exponent of the Weierstrass function W_H is constant and equal to H (Hardy)

The Hölder exponent of Brownian motion is constant and equal to $1/2$ (Wiener)

W_H and B are mono-Hölder function

Multifractal spectrum (Parisi and Frisch, 1985)

The iso-Hölder sets of f are the sets

$$E_H = \{x_0 : \quad h_f(x_0) = H\}$$

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The upper-Hölder sets of f are the sets

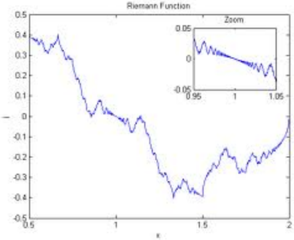
$$\overline{E_H} = \{x_0 : \quad h_f(x_0) \geq H\}$$

The lower-Hölder sets of f are the sets

$$\underline{E_H} = \{x_0 : \quad h_f(x_0) \leq H\}$$

Riemann's non-differentiable function and beyond

$$\mathcal{R}_2(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$



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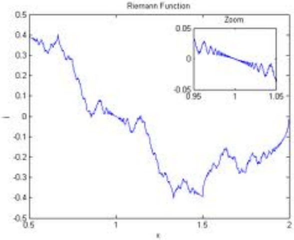

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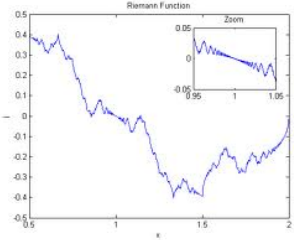

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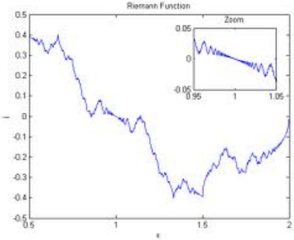

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In a recent paper (arXiv :1208.6533v1) F. Chamizo and A. Ubis consider

$$F(x) = \sum_{n=1}^{\infty} \frac{e^{iP(n)x}}{n^{\alpha}} \qquad deg(P) = k$$

Theorem : (Chamizo and Ubis) : let ν_F be the maximal multiplicity of the zeros of P'. If $1 + \frac{k}{2} < \alpha < k$ and $\frac{1}{k}(\alpha - 1) \leq H \leq \frac{1}{k}\left(\alpha - \frac{1}{2}\right)$, then

$$d_F(H) \geq \max(\nu_f, 2)\left(H - \frac{\alpha - 1}{k}\right)$$

Generalization : Nonharmonic Fourier series

Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of points in $\mathbb{R}^d$; a nonharmonic Fourier series is a function f that can be written

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Theorem : Let $x_0 \in \mathbb{R}^d$. If (λ_n) is separated and $f \in C^\alpha(x_0)$, then $\exists C$ such that $\forall n$,

$$(1) \qquad \text{if } |\lambda_n| \geq \theta_n, \quad \text{then} \quad |a_n| \leq \frac{C}{(\theta_n)^\alpha}.$$

Thus, if

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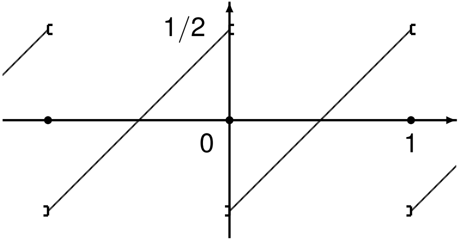
Open problem : Optimality of this result

Davenport series

The sawtooth function is

$$\{x\} = \begin{cases} x - \lfloor x \rfloor - 1/2 & \text{if } x \notin \mathbb{Z} \\ 0 & \text{else} \end{cases}$$



In one variable, Davenport series are of the form

$$F(x) = \sum_{n=1}^{\infty} a_n \{nx\}, \qquad a_n \in \mathbb{R}.$$

Spectrum estimates for Davenport series

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Assuming that $(a_n) \in l^1$, then F is continuous at irrational points and the jump at p/q (if $p \wedge q = 1$) is

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Theorem : Assume that $(n^\beta a_n) \notin l^\infty$ and $\beta > 1$. Then

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Open problem : Sharpen these bounds

Hecke's functions

$$\mathcal{H}_s(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^s}.$$

The function $\mathcal{H}_s(x)$ is a Dirichlet series in the variable s, and its analytic continuation depends on Diophantine approximation properties of x (Hecke, Hardy, Littlewood).

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Theorem : If $Re(s) \geq 2$, the spectrum of singularities of $\mathcal{H}^s$ is

$$\begin{aligned} d(H) &= \frac{2H}{Re(s)} \quad \text{for} \quad H \leq \frac{Re(s)}{2}, \\ &= -\infty \qquad \text{else.} \end{aligned}$$

If $1 < Re(s) < 2$, the spectrum of singularities of Hecke's function $\mathcal{H}^s$ satisfies

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Open problem : Improve the second case

Hecke's functions (continued)

$$\mathcal{H}_s(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^s}.$$

If $Re(s) \leq 1$, the sum is no more locally bounded, however :
if $1/2 < Re(s) < 1$ then $\mathcal{H}_s \in L^p$ for $p < \frac{1}{1-\beta}$

One can still define a pointwise regularity exponent as follows (Calderón and Zygmund, 1961) :

Definition : Let $B(x_0, r)$ denote the open ball centered at x_0 and of radius r ; $\alpha > -d/p$. Let $f \in L^p$. Then f belongs to $T^p_\alpha(x_0)$ if $\exists C, R > 0$ and a polynomial P such that

$$\forall r \leq R, \quad \left(\frac{1}{r^d} \int_{B(x_0,r)} |f(x) - P(x - x_0)|^p dx \right)^{1/p} \leq C r^\alpha.$$

The p-exponent of f at x_0 is : $h^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}$.

The p-spectrum of f is : $d^p_f(H) = \dim \left(\{x_0 : h^p_f(x_0) = H\}\right)$

Open problem : Determine the p-spectrum of Hecke's functions

The Lebesgue-Davenport function

Let $t \in [0, 1)$ and

$t = (0; t_1, t_2, \dots, t_n, \dots)_2$

be its proper expansion in basis 2.

Then $\mathcal{L}(t) = (x_3(t), y_3(t))$ where

$$\begin{cases} x_3(t) = & (0; t_1, t_3, t_5, \dots)_2 \\ y_3(t) = & (0; t_2, t_4, t_6, \dots)_2. \end{cases}$$

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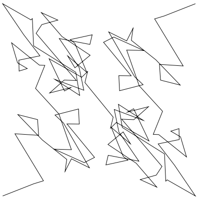

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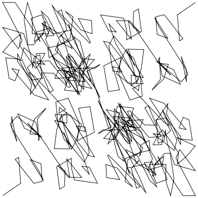

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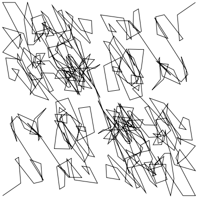

The Lebesgue-Davenport function $\mathcal{L}$ has the following expansion

$$x_3(t) = \frac{1}{2} + \sum a_n \{2^n t\} \quad \text{where } a_{2n} = 2^{-n} \quad \text{and} \quad a_{2n+1} = -2^{-n-1}$$

$$y_3(t) = \frac{1}{2} + \sum b_n \{2^n t\} \quad \text{where } b_{2n} = -2^{-n} \quad \text{and} \quad b_{2n+1} = 2^{-n}.$$

The spectrum of singularities of $\mathcal{L}$ is

$$\begin{cases} d_{\mathcal{L}}(H) & = 2H \quad \text{if} \quad 0 \le H \le 1/2 \\ & = -\infty \quad \text{else.} \end{cases}$$

Davenport series in several variables

Davenport series in several variables are of the form

$$f(x) = \sum_{n \in \mathbb{Z}^d} a_n \{n \cdot x\}$$

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Discontinuities of Davenport series

For $p \in \mathbb{Z}$ and $q \in \mathbb{Z}^d_*$, let

$$H_{p,q} = \{x \in \mathbb{R}^d \mid p = q \cdot x\}$$

Let us assume that $(a_n)_{n \in \mathbb{Z}^d}$ is an odd sequence in ℓ^1. Then,
The Davenport series is continuous except on the set $\bigcup H_{p,q}$ where it has a jump of magnitude $|A_q|$ with

$$A_q = 2 \sum_{l=1}^{\infty} a_{lq}$$

Upper bound on the Hölder exponent of a Davenport series

For each $q \in \mathbb{Z}^d$, let $\mathcal{P}_q = \{p \in \mathbb{Z} \mid \gcd(p, q) = 1\}$.
For $x_0 \in \mathbb{R}^d$, let

$$\delta_q^{\mathcal{P}}(x_0) = dist\left(x_0, \bigcup_{p \in \mathcal{P}_q} H_{p,q}\right)$$

Let f be a Davenport series with jump sizes $(A_q)_{q \in \mathbb{Z}^d}$. Then,

$$\forall x_0 \in \mathbb{R}^d \qquad h_f(x_0) \leq \liminf_{\substack{q \to \infty \\ A_q \neq 0}} \frac{\log |A_q|}{\log \delta_q^{\mathcal{P}}(x_0)}.$$

Connection with Diophantine approximation :

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Corollary : If the jumps A_q satisfy : $|A_q| \geq C/q^a$ for all q in one direction at least, then

$$\forall x, \;\; h_f(x) \leq a/2 \;\; \text{and} \;\; d(\underline{E_H}) \leq d - 1 + \frac{2H}{a}$$

Sparse Davenport series

A Davenport series with coefficients given by a sequence $(a_n)_{n\in\mathbb{Z}^d}$ is sparse if

$$\lim_{R\to\infty} \frac{\log \#\{|n| < R \mid a_n \neq 0\}}{\log R} = 0.$$

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Theorem : Let f be a Davenport series with coefficients $a = (a_n)_{n\in\mathbb{Z}^d}$. Let

$$\gamma_a := \sup\{\gamma > 0 \mid (a_n)_{n\in\mathbb{Z}^d} \in \mathcal{F}^\gamma\}$$

We assume that f is sparse and that $0 < \gamma_a < \infty$. Then,

$$\forall H \in [0, \gamma_a] \qquad d_f(H) = d - 1 + \frac{H}{\gamma_a},$$

$$\text{else} \quad d_f(H) = -\infty$$

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Thank you for your (fractal ?) attention !