Heat kernel estimates on a connected sum along a joint with a capacity growth

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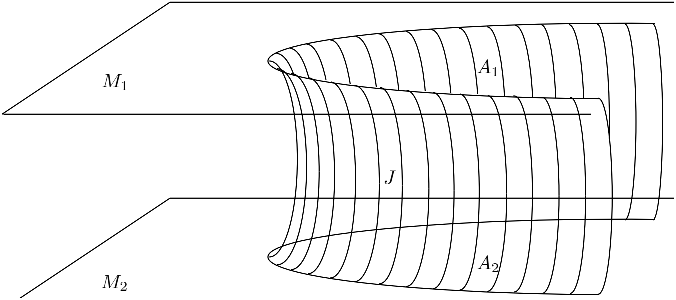
Setting of the problem

Let M_1, M_2: complete non-compact Riemannian manifolds.
For closed subsets $A_1 \subset M_1$, $A_2 \subset M_2$, consider

$$M = M_1 \setminus A_1 \sqcup J \sqcup M_2 \setminus A_2 / \sim_\partial =: M_1 \#_J M_2.$$



Setting of the problem

Problem Long time behavior of the heat kernel $p(t, x, y)$ on M under some geom. or prob. assumptions of M_i, A_i and J?

Assumption of M_i: two sided Gaussian heat kernel estimates (LY):

$$p_i(t, x, y) \sim \frac{C}{V_i(x, \sqrt{t})} e^{-bd_i(x,y)^2/t}.$$

Problem $M = M_1 \#_J M_2$: also satisfy (LY)?
If the joint is narrow, the probability of the BM from $x \in M_1 \setminus A_1$ to $y \in M_2 \setminus A_2$ is significantly small so that (LY) is failed.
The failure of (LY) is characterized by the capacity growth of A_i.

Known Result 1
Connected sum along a compact joint

Theorem (Grigor'yan, Saloff-Coste, 2009)

Assume that A_1, A_2 and J are compact. Then for $x \in M_1 \setminus A_1$, $y \in M_2 \setminus A_2$ outside of a neighborhood of J, and for $t > T$,

$$p(t,x,y) \sim$$
$$C\left(\frac{1}{V_1(x,\sqrt{t})}\frac{d_2(y,J)^2}{V_2(y,d_2(y,J))} + \frac{1}{V_2(y,\sqrt{t})}\frac{d_1(x,J)^2}{V_1(x,d_1(x,J))}\right)e^{-bd(x,y)^2/t}.$$

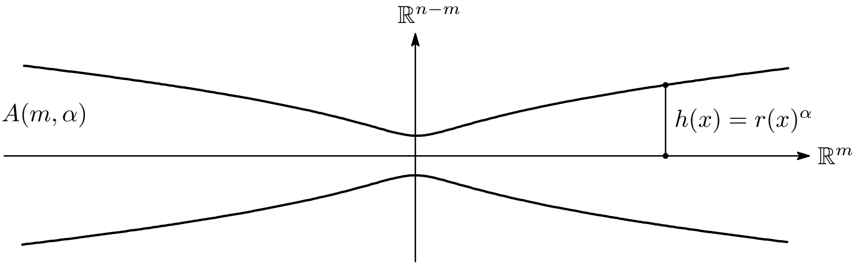
In particular, when $M_1 = M_2 = \mathbb{R}^n$,

$$p(t,x,y) \sim Ct^{-n/2}\left(\frac{1}{d(x,J)^{n-2}} + \frac{1}{d(y,J)^{n-2}}\right)e^{-bd(x,y)/t}.$$

Known Result 2
Connected sum of $\mathbb{R}^n$ along surface of a revolution

Let $M_1 = M_2 = \mathbb{R}^n$. For $0 \le m \le n-3$, $0 \le \alpha < 1$, define

$$A_1 = A_2 = \left\{ (x_1, \ldots, x_n) \mid \sqrt{\sum_{i=m+1}^{n} x_i^2} \le \left(\sqrt{1 + \sum_{i=1}^{m} x_i^2} \right)^{\alpha} \right\}.$$



Consider $M = M_1 \#_J M_2 = \mathbb{R}^n \#_J \mathbb{R}^n$, where $J \sim \partial A_1 \times [0,1]$.

Known Result 2
Connected sum along surface of a revolution

Theorem (Grigor'yan, I, 2012)

For $x \in M_1 \setminus A_1$, $y \in M_2 \setminus A_2$ *outside of the conical neighborhood of* J, *and for* $t > T(d(x,J)^2 + d(y,J)^2)$,

$$p(t,x,y) \sim Ct^{-n/2}\left(\frac{1}{d(x,J)^{(1-\alpha)(n-m-2)}} + \frac{1}{d(y,J)^{(1-\alpha)(n-m-2)}}\right)e^{-bd(x,y)^2/t}.$$

We note that

$$\operatorname{cap}(B(o,r) \cap \partial A_i) \sim Cr^{m+\alpha(n-m-2)}.$$

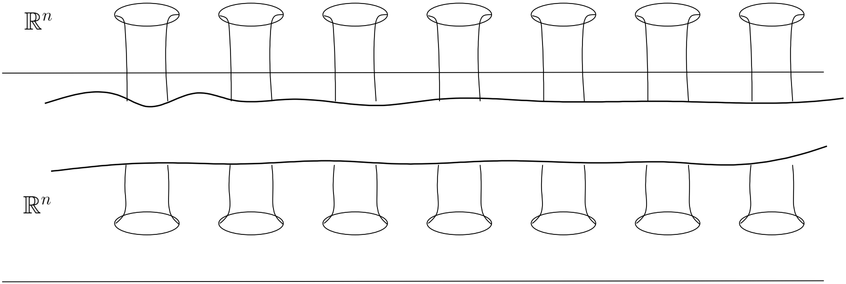
Main Problem

Can we predict the heat kernel estimate by the capacity growth of the joint?

General Case

Let us consider that the connected sum of two copies of $\mathbb{R}^n$ along

$$A_1 = A_2 = B\left(\mathbb{Z}, \frac{1}{3}\right) \text{ by } J \sim \partial B\left(\mathbb{Z}, \tfrac{1}{3}\right) \times [0, 1].$$



$$\mathrm{cap}(B(o, r) \cap \partial A_i) \sim r.$$

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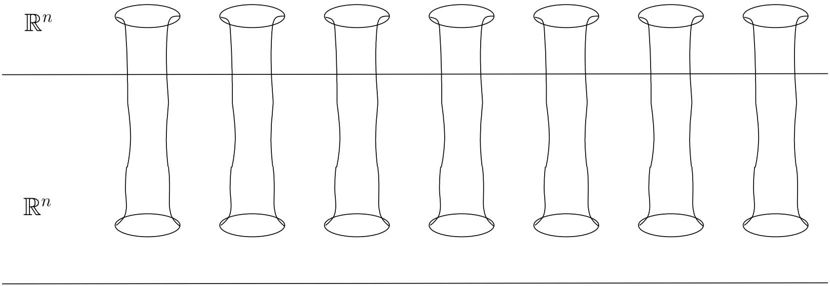

$$p(t, x, y) \sim ct^{-n/2}\left(\frac{1}{d(x, J)^{n-3}} + \frac{1}{d(y, J)^{n-3}}\right)e^{-bd(x,y)^2/t}.$$

General Case

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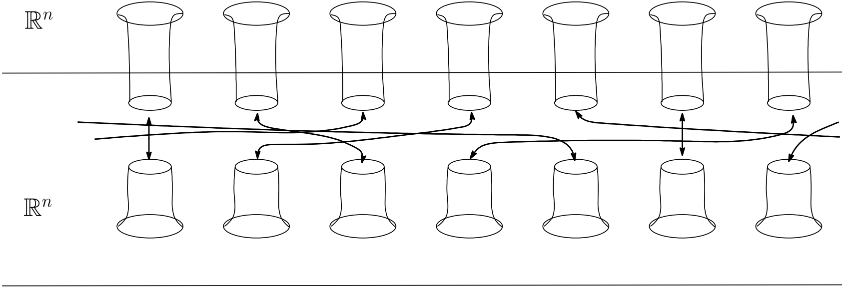


$$p(t, x, y) \sim ???$$

To avoid such difficulty, we assume that $d \sim d_i$ on $M_i \setminus A_i \sqcup J$, namely

$$cd_i(x,y) \leq d(x,y) \leq d_i(x,y) \quad x,y \in M_i \setminus A_i \sqcup J.$$

Theorem

Suppose that $M_i \setminus A_i \sqcup J$, $i = 1,2$ satisfies the Gaussian upper bound for Neumann heat kernel. Assume also that $d \sim d_i$. Then the connected sum $M = M_1 \#_J M_2$ also admits the Gaussian heat kernel upper bound.

Theorem (Gyrya, Saloff-Coste, 2011)

Suppose that M satisfies (LY). Let $U \subset M$ be an inner uniform domain. Then the Neumann heat kernel on U also satisfies (LY).

Main Result

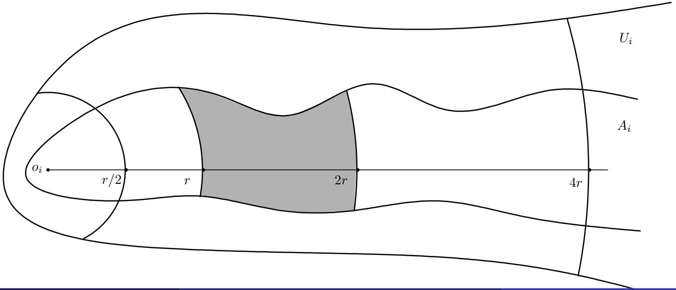
Suppose that M_i, $i = 1, 2$ satisfies

$$\left(\frac{R}{r}\right)^{\beta_i} \leq \frac{V_i(x,R)}{V_i(x,r)} \leq \left(\frac{R}{r}\right)^{\beta_i'} \quad R > r > 0$$

for some $2 < \beta_i \leq \beta_i'$.

For fixed $o_i \in A_i$, we assume that there exist an open subset U_i containing A_i and $0 \leq \alpha_i < \beta_i - 2$ so that

$$\mathrm{cap}\,(A_i \cap B(o_i, 2r) \setminus B(o_i, r), U_i \cap B(o_i, 4r) \setminus B(o_i, r/2)) \leq Cr^{\alpha_i}.$$



Main Result

Assume that the Neumann heat kernel on $M_i \setminus A_i \sqcup J$ satisfies the Gaussian upper bound.
Let $M = M_1 \#_J M_2$ be a connected sum of M_1 and M_2 along the boundary of $A_1 \subset M_1$ and $A_2 \subset M_2$ by J. We assume that $d \sim d_i$.

Theorem

For $x \in M_1 \setminus A_1$, $y \in M_2 \setminus A_2$ outside of a conical neighborhood of U_i,

$$p(t,x,y) \leq C\left(\frac{1}{V_1(x,\sqrt{t})}\frac{d(y,U_2)^{2+\alpha_2}}{V_2(y,d(y,U_2))} + \frac{1}{V_2(y,\sqrt{t})}\frac{d(x,U_1)^{2+\alpha_1}}{V_1(x,d(x,U_1))}\right)e^{-bd(x,y)^2/t}.$$