Heat kernels and Green functions on metric measure spaces

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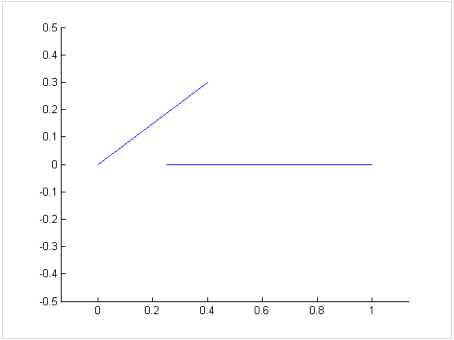
December 10-14, 2012 (Hong Kong)

Programme

- **Background**
- **Conditions**
- **Theorems**.

Metric measure space

- (M, d): a metric space (locally compact, separable).
- μ: a Radon measure (locally finite, inner regular) ($\mu(\Omega) > 0$ for any open $\Omega \neq \emptyset$).
- (M, d, μ): a metric measure space.



A metric space: Hata's tree.

Dirichlet form

- $(\mathcal{E}, \mathcal{F})$: a **Dirichlet form** in $L^2(M, \mu)$ that is regular, strongly local.

$\searrow$

- DF: a closed Markovian symmetric form.
- regular: $C_0(M) \cap \mathcal{F}$ is dense in both $\mathcal{F}$ and $C_0(M)$.
- strongly local: $\mathcal{E}(f, g) = 0$ for any $f, g \in \mathcal{F}$ where f is constant in some neighborhood of $\mathrm{supp}(g)$.

Heat semigroup

- $\{P_t\}_{t\geq 0}$: a **heat semigroup** in $L^2(M,\mu)$:
  (a) strongly cts, contractive, symmetric in L^2;
  (b) Markovian in L^∞:

$$P_t f \geq 0 \text{ if } f \geq 0, \text{ and } P_t f \leq 1 \text{ if } f \leq 1.$$

- $(\mathcal{E},\mathcal{F}) \Leftrightarrow \{P_t\}_{t\geq 0}$:

$$\begin{aligned}\mathcal{E}(f,g) &= \lim_{t\to 0} \mathcal{E}_t(f,g) \\ &:= \lim_{t\to 0} t^{-1}(f - P_t f, g).\end{aligned}$$

Restricted Dirichlet form

- **Restricted DF**: $(\mathcal{E}, \mathcal{F}(\Omega))$, where

$$\mathcal{F}(\Omega) := \overline{C_0(\Omega) \cap \mathcal{F}} \quad \text{in } \mathcal{F}\text{-norm},$$

  for a non-empty open $\Omega \subset M$.
- $(\mathcal{E}, \mathcal{F}(\Omega)) \Leftrightarrow \{P_t^\Omega\}$.
- **Generator:** $\mathcal{L}^\Omega$

$$\mathcal{L}^\Omega f := \lim_{t \to 0} \frac{P_t^\Omega f - f}{t} \quad \text{in } L^2\text{-norm}.$$

Heat kernel

- $\{p_t\}_{t>0}$: a **heat kernel**.

↙

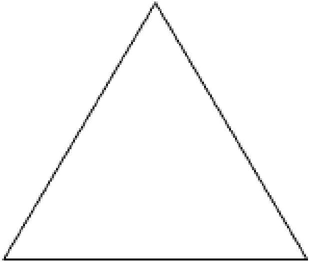
- symmetric: $p_t(x, y) = p_t(y, x)$;
- Markovian: $p_t(x, y) \geq 0$, and $\int_M p_t(x, y) d\mu(y) \leq 1$;
- semigroup property;
- identity approximation.

Heat kernel: examples

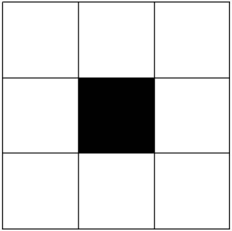
- **Sierpinski gaskets** ('88) and **carpets** ('92, '99)

$$p_t(x,y) \asymp t^{-\alpha/\beta} \exp\left(-c\left(\frac{|x-y|}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right),$$

Gasket



Carpet



Purpose

To find equivalence conditions for the following estimate:

(UE) *Upper estimate:* the heat kernel $p_t(x,y)$ exists, has a Hölder continuous in $x, y \in M$ version, and satisfies

$$p_t(x,y) \le \frac{C}{V(x, \mathcal{R}(t))} \exp\left(-\frac{1}{2} t \Phi\left(c \frac{d(x,y)}{t}\right)\right)$$

for all $t > 0$ and all $x, y \in M$, where $\mathcal{R} := F^{-1}$ and

$$\Phi(s) := \sup_{r>0} \left\{ \frac{s}{r} - \frac{1}{F(r)} \right\}.$$

Interesting case: $F(r) = r^\beta (\beta > 1)$, $V(x,r) \sim r^\alpha$, then $\Phi(s) = cs^{\beta/(\beta-1)}$, and

$$p_t(x,y) \le \frac{C}{t^{\alpha/\beta}} \exp\left(-c \left(\frac{d(x,y)}{t^{1/\beta}}\right)^{\beta/(\beta-1)}\right).$$

Conditions

How ?

- **Volume doubling condition**: for all $x \in M, r > 0$,

$$V(x, 2r) \leq C_D V(x, r), \qquad (VD)$$

where $V(x, r) := \mu(B(x, r))$. Then, for all $0 < r_1 \leq r_2$,

$$\frac{V(x, r_2)}{V(x, r_1)} \leq c \left(\frac{r_2}{r_1} \right)^{\alpha}.$$

- **Reverse volume doubling condition**: for all $x \in M$ and $0 < r_1 \leq r_2$,

$$\frac{V(x, r_2)}{V(x, r_1)} \geq c^{-1} \left(\frac{r_2}{r_1} \right)^{\alpha'}. \qquad (RVD)$$

If M is connected and unbounded, then $(VD) \Rightarrow (RVD)$.

Conditions

- The (uniform elliptic) Harnack inequality: for any function $u \in \mathcal{F}$ that is **harmonic** and **non-negative** in $B(x_0, r)$,

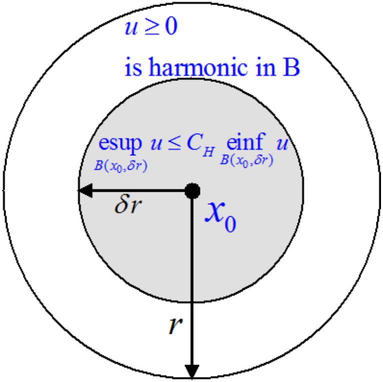
$$\operatorname*{esup}_{B(x_0,\delta r)} u \leq C_H \operatorname*{einf}_{B(x_0,\delta r)} u, \qquad (H)$$

  where the constants C_H and δ are **independent** of the ball $B(x_0, r)$ and the function u.

A function $u \in \mathcal{F}$ is **harmonic** in Ω if

$$\mathcal{E}(u, \varphi) = 0 \text{ for any } \varphi \in \mathcal{F}(\Omega).$$

Harnack inequality:



Harmonic function u is **nearly constant** in $B(x_0, \delta r)$.

Conditions

- The **resistance condition** (R_F):

$$\mathrm{res}\,(B, KB) \simeq \frac{F\,(r)}{\mu\,(B)}, \qquad (R_F)$$

where $K > 1$, r is the radius of B, and F is continuous increasing such that for all $0 < r_1 \leq r_2$,

$$C^{-1}\left(\frac{r_2}{r_1}\right)^{\beta} \leq \frac{F\,(r_2)}{F\,(r_1)} \leq C\left(\frac{r_2}{r_1}\right)^{\beta'} \ (\beta > 1).$$

The **resistance** and **capacity** are defined by

$$\mathrm{res}\,(A, \Omega) := \frac{1}{\mathrm{cap}(A, \Omega)},$$

$$\mathrm{cap}(A, \Omega) := \inf\left\{\mathcal{E}\,(\varphi) : \varphi \text{ is a cutoff function of } (A, \Omega)\right\}$$

for any $A \Subset \Omega$.

Conditions

Interesting case: $F(r) = r^{\beta}(\beta > 1), V(x,r) \sim r^{\alpha}$, then condition (R_F) becomes

$$\mathrm{res}\left(B, KB\right) \simeq \frac{F\left(r\right)}{\mu\left(B\right)} \simeq r^{\beta-\alpha}.$$

Conditions

- **Condition** (G_F) : the Green function g^B exists and is jointly continuous off the diagonal, and

$$g^B(x_0, y) \leq C \int_{\frac{d(x_0,y)}{K}}^{R} \frac{F(s)\,ds}{sV(x,s)} \quad (y \in B \setminus \{x_0\}), \quad (G_F \leq)$$

$$g^B(x_0, y) \geq C^{-1} \int_{\frac{d(x_0,y)}{K}}^{R} \frac{F(s)\,ds}{sV(x,s)} \quad (y \in K^{-1}B \setminus \{x_0\}), \quad (G_F \geq)$$

  where $K > 1$ and $C > 0$, and $B := B(x_0, R)$.

The **Green function** g^Ω is defined by

$$G^\Omega f(x) = \int_\Omega g^\Omega(x, y) f(y) d\mu(y),$$

and the **Green operator** G^Ω:

$$\mathcal{E}(G^\Omega f, \varphi) = (f, \varphi), \ \forall \varphi \in \mathcal{F}(\Omega).$$

Conditions

- **Condition** (E_F) : for any ball B of radius r,

$$\underset{B}{\operatorname{esup}} E^B \leq CF(r), \qquad (E_F \leq)$$

$$\underset{\delta_1 B}{\operatorname{einf}} E^B \geq C^{-1} F(r). \qquad (E_F \geq)$$

  where $C > 1$ and $\delta_1 \in (0, 1)$.

The **function** E^B is defined by

$$E^B(x) = G^B \mathbf{1}(x) = \mathbb{E}_x(\tau_B),$$

where τ_B is the **first exit time** from B.

Conditions

Namely, function E^B satisfies the **Poisson-type** equation:

$$-\mathcal{L}^B E^B = 1 \quad \text{weakly},$$

that is, $\mathcal{E}(E^B, \varphi) = \int_B \varphi d\mu$ for any $\varphi \in \mathcal{F}(B)$.

Note: if the Green function g^B exists, then

$$E^B(x) = \int_B g^B(x, y) d\mu(y).$$

Note: **Condition** (E_F) can be written

$$C^{-1} F(r) \le \operatorname*{esup}_B E^B \le C \operatorname*{einf}_{\delta_1 B} E^B.$$

Theorem 1

Theorem 1(Grigor'yan, Hu, 2012): Assume that

- (M, d, μ): a metric measure space.
- $(\mathcal{E}, \mathcal{F})$: a regular, strongly local DF in $L^2(M, \mu)$.
- (VD) and (RVD) hold.

  Then we have the following equivalences:

$$(H) + (R_F) \Leftrightarrow (G_F) \Leftrightarrow (H) + (E_F) \, .$$

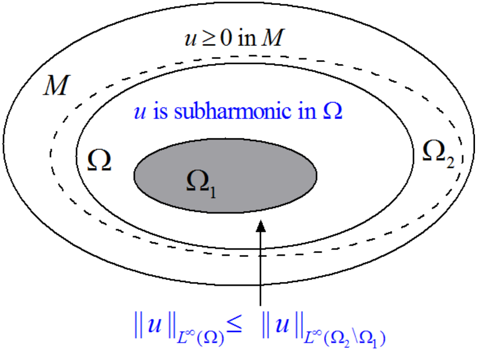
Remark: Condition (RVD) is needed **only** for

$$(H) + (E_F) \Rightarrow (R_F \geq) \, .$$

Theorem 1

Ideas of the proof:

- **Maximum principles** for **subharmonic** functions. (*Subharmonic*: $\mathcal{E}(u,\varphi) \le 0$ for any $\varphi \in \mathcal{F}(\Omega)$)



If u is **continuous on** $\overline{\Omega}$, then

$$\operatorname*{esup}_{\overline{\Omega}} u = \sup_{\partial\Omega} u.$$

Theorem 1

Ideas of the proof:

- The **Riesz measure** associated with a **superharmonic** function: if $0 \leq f \in \mathrm{dom}(\mathcal{L}^\Omega)$ is superharmonic in Ω, then

$$-\mathcal{L}^\Omega f \, d\mu(x) = d\nu_f(x),$$

a **non-negative Borel measure** on Ω, namely,

$$\mathcal{E}(u, \varphi) = \int_\Omega \varphi(x) \, d\nu_f(x) \quad \text{for any } \varphi \in C_0(\Omega) \cap \mathcal{F}.$$

Consequently, if f is **harmonic** in $\Omega \setminus S$ for a compact set S,

$$f(x) = \int_S g^\Omega(x, y) d\nu_f(y) \quad (x \in \Omega).$$

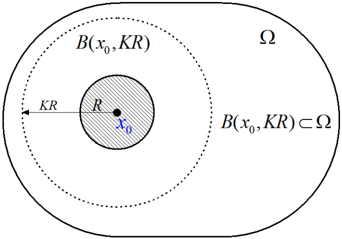
Theorem 1

The hardest part of the proof:

- The annulus Harnack for the Green function from (H):

$$\sup_{\partial B} g^{\Omega}(x_0,\cdot) = \sup_{\Omega\setminus B} g^{\Omega}(x_0,\cdot) \le C \inf_{B} g^{\Omega}(x_0,\cdot) = C \inf_{\partial B} g^{\Omega}(x_0,\cdot),$$

where $C > 0$ is **independent** of the ball $B = B(x_0, R)$ and Ω.



$$\sup_{\Omega\setminus B} g^{\Omega}(x_0,\bullet) \le C \inf_{B} g^{\Omega}(x_0,\bullet)$$

One more condition

- **Near-diagonal lower estimate:** The heat kernel $p_t(x, y)$ exists, has a Hölder continuous in $x, y \in M$ version, and satisfies

$$p_t(x, y) \geq \frac{c}{V(x, \mathcal{R}(t))}, \qquad (NLE)$$

for all $t > 0$ and all $x, y \in M$ such that $d(x, y) \leq \eta \mathcal{R}(t)$, where $\eta > 0$ is a sufficiently small constant.

Recall that $\mathcal{R} = F^{-1}$, for example,

$$\mathcal{R}(t) = t^{1/\beta} \quad (\beta > 1).$$

Theorem 2

Theorem 2 (Grigor'yan, Hu, 2012): Assume that

- (M, d, μ): a metric measure space.
- $(\mathcal{E}, \mathcal{F})$: a regular, strongly local DF in $L^2(M, \mu)$.
- (VD) and (RVD) hold.

Then we have the following three equivalences:

$$\begin{aligned} (H) + (R_F) \quad &\Leftrightarrow \quad (G_F) \Leftrightarrow (H) + (E_F) \\ &\Leftrightarrow \quad (UE) + (NLE)\,. \end{aligned}$$

The End of Talk