Progress on self-similar sets with overlaps

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$\{f_i\}_{i\in\Lambda}$ a linear iterated function system on $\mathbb{R}$:

$$f_i(x) = r_i x + a_i \qquad 0 < |r_i| < 1 \quad , \quad a_i \in \mathbb{R}$$

With attractor is the unique compact non-empty set

$$X = \bigcup_{i\in\Lambda} f_i(X)$$

Problem:

What is $\dim(X)$? ? ?

The similarity dimension is the solution s to

$$\sum |r_i|^s = 1$$

Assuming strong separation or the Open Set Condition,

$$\dim(X) = s$$

Without separation we know that

$$\dim(X) \leq \min\{1, s\}$$

Folklore conjecture:

$$\dim(X) < \min\{1, s\} \implies \text{exact overlaps.}$$

For $\mathbf{i} = i_1 \ldots i_n \in \Lambda^n$,

$$f_{\mathbf{i}} = f_{i_1} \circ \ldots \circ f_{i_n}$$
$$r_{\mathbf{i}} = r_{i_1} \cdot \ldots \cdot r_{i_n}$$

Exact overlaps occur if there are $\mathbf{i} \neq \mathbf{j}$ with $f_{\mathbf{i}} = f_{\mathbf{j}}$.
Equivalently: The semigroup generated by $\{f_i\}$ is not free.

The distance between affine maps g, h is

$$d(g,h) = \begin{cases} |g(0) - h(0)| & \text{same contraction ratio} \\ \infty & \text{otherwise} \end{cases}$$

and

$$\Delta_n = \min\{d(f_{\mathbf{i}}, f_{\mathbf{j}}) : \mathbf{i}, \mathbf{j} \in \Lambda^n , \mathbf{i} \neq \mathbf{j}\}$$

Note.

1. Δ_n is decreasing.
2. Exact overlaps occur $\iff$ $\exists n$ such that $\Delta_n = 0$.
3. $\Delta_n \to 0$ exponentially.
   (There exists $0 < \rho < 1$ such that $\Delta_n \leq \rho^n$).

Remark: We can have $\Delta_n \geq \sigma^n$ (even without separation).

Theorem.

$$\dim(X) < \min\{1, s\} \quad \Longrightarrow \quad \Delta_n \to 0 \text{ super-exponentially.}$$

$$\left(-\frac{1}{n} \log \Delta_n \;\to\; \infty \right)$$

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Equivalently,

$$\exists \rho > 0 \quad s.t. \quad \Delta_n > \rho^n \quad \Longrightarrow \quad \dim(X) = \min\{1, s\}$$

Corollary.
The conjecture is true in the class of IFSs with algebraic coefficients: either $\dim(X) = \min\{1, s\}$ or there are exact overlaps.

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Proof.

$$\begin{aligned} \Delta_n &= f_{\mathbf{i}}(0) - f_{\mathbf{j}}(0) \quad \text{for some } \mathbf{i}, \mathbf{j} \in \Lambda^n \\ &= \text{a polynomial of degree } n \text{ in } a_i, r_i. \end{aligned}$$

Since a_i, r_i are algebraic, by general facts about polynomials of algebraic numbers, there exists $\rho > 0$ such that

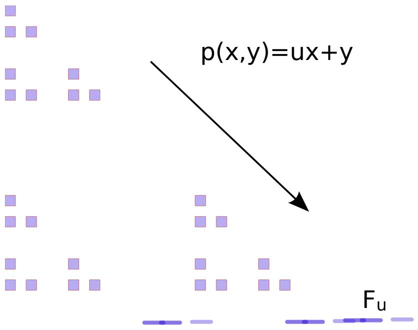
$$\Delta_n = \begin{cases} 0 \\ \geq \rho^n \end{cases}$$

Therefore if there are no exact overlaps $\Delta_n \to 0$ only exponentially, so by the theorem, $\dim(X) = \min\{1, s\}$.

Corollary (Furstenberg's conjecture).
Let $u \in \mathbb{R}$ and F_u the attractor of

$$x \mapsto \frac{1}{3}x \quad , \quad x \mapsto \frac{1}{3}x + 1 \quad , \quad x \mapsto \frac{1}{3}x + u$$

Then $\dim F_u = 1$ for all irrational u.



Remark: $\dim(F_u)$ can be computed also for rational u (Kenyon).

Proof. (Thanks to B. Solomyak and P. Shmerkin)

$$\begin{aligned}\Delta_n &= \left|\sum_{k=1}^{n}(i_k - j_k)3^{-k}\right| && \text{for some } \mathbf{i},\mathbf{j} \in \{0,1,u\}^n \\ &= \left|\frac{a_n}{3^n} - u\cdot\frac{b_n}{3^n}\right| && \text{for integers } |a_n|\,,\ |b_n| \le 3^n\end{aligned}$$

Suppose $\dim(F_u) < 1 =$ **the similarity dimension.**

By the theorem $\Delta_n \to 0$ super-exponentially, so

$$\left|\frac{a_n}{3^n} - u\cdot\frac{b_n}{3^n}\right| = \Delta_n < \frac{1}{100^n} \qquad \text{for large enough } n$$

We can have $b_n = 0$ only finitely often because it implies $|a_n/2^n| < 1/100^n$.

Dividing by $3^n/b_n$,

$$|u - \frac{a_n}{b_n}| \; < \; \frac{3^n}{b_n}\Delta_n \; < \; \frac{1}{30^n} \qquad \text{for large enough } n$$

Subtracting successive n,

$$|\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}}| \; < \; \frac{2}{30^n} \qquad \text{for large enough } n$$

But

$$|\frac{a_n}{b_n} - \frac{a_{n+1}}{b_{n+1}}| = \frac{|a_n b_{n+1} - a_{n+1} b_n|}{|b_n b_{n+1}|} = \begin{cases} 0 & \text{if } = 0 \\ \geq 1/9^n & \text{if } \neq 0 \end{cases}$$

but $1/9^n > 2/30^n$ so **this difference must be 0 for large n.**

$\implies$ $\exists a, b \in \mathbb{N}$ **with** $|u - a/b| < 30^{-n}$, which implies $u = a/b$.

Corollary.
Let ν_λ be the Bernoulli convolution with parameter $1/2 < \lambda < 1$:

$$\nu_\lambda \quad = \quad \text{distribution of } \sum_{k=1}^{\infty} \pm\lambda^k \;\; \text{(signs i.i.d. and uniform)}$$

Then there is a set $E \subseteq [1/2, 1]$ of packing dimension 0 such that $\dim \nu_\lambda = 1$ for $\lambda \notin E$.

More generally this is true for any 1-parameter family of self-similar sets or measures as long as the parametrization is real-analytic, under a mild non-degeneracy condition.

Theorem.

$$\dim(X) < \min\{1, s\} \implies \Delta_n \to 0 \text{ super-exponentially.}$$

For simplicity assume:

- Uniform contraction 1/2.
- $0 \in X \subseteq [0, 1]$.

Then $s = \log_2 |\Lambda|$.

Define the n-th approximation of X:

$$X_n := \{f_{\mathbf{i}}(0) \ : \ \mathbf{i} \in \Lambda^n\}$$

We have the relation

$$X_{m+n} \ = \ X_m + 2^{-m} \cdot X_n$$

Here

$$\begin{aligned} A + B \ &= \ \{a + b \ : \ a \in A \, , \ b \in B\} \\ cA \ &= \ \{ca \ : \ a \in A\} \end{aligned}$$

$$N_n(Y) := \min \left\{ k \ : \ Y \subseteq \bigcup_{i=1}^{k} Y_i \ \text{ with } \ \operatorname{diam}(Y_i) \leq 2^{-n} \right\}$$

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Sumset theorem.

If $\dim(X) < 1$ then $\forall \varepsilon > 0 \ \exists \delta > 0$ such that for m large enough,

$$N_m(X_m + A) \leq (N_m(X_m))^{1+\delta} \quad \Longrightarrow \quad N_m(A) \leq 2^{\varepsilon m}$$

Remark. There exist finite sets Y with $N_m(Y + Y) \approx N_m(Y)$ and $|Y| \approx 2^{cm}$. So self similarity here is important.

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Sumset theorem for dimension.
If $\dim(X) < 1$ then $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\dim(A) > \varepsilon \implies \dim(X + A) > \dim(X) + \delta.$$

Assume $\dim(X) < \min\{1, s\}$. We must prove

$$\Delta_n \to 0 \quad \text{super-exponentially}$$

Equivalently,

$$\text{For every } q > 1 \quad : \quad \Delta_n < 2^{-qn} \text{ for all large } n.$$

Assume no exact overlaps. Then

$$\Delta_n \;=\; \text{minimal distance between points in } X_n$$

So we need to prove:

$$\text{For every } q > 1 \quad : \quad \begin{cases} \text{For all large enough } n, \\ |X_n \cap I| \geq 2 \text{ for some} \\ \text{interval } I \text{ of length } < 2^{-qn} \end{cases}$$

It is well known that,

$$N_n(X) \approx 2^{n \dim(X)}$$

One can also show

$$N_{qn}(X \cap I_{n+1}(x)) \approx 2^{(q-1) n \dim(X)}$$

where $x \in X$ and

$$I_n(x) = [x - 2^{-n}, x + 2^{-n}]$$

$$X_{qn} = X_n + 2^{-n} X_{(q-1)n}$$

Therefore

$$X_{qn} \cap I_n(x) \supseteq (X_n \cap I_{n+1}(x)) + 2^{-n} X_{(q-1)n}$$

Therefore

$$\begin{aligned} N_{qn}\left((X_n \cap I_{n+1}(x)) + 2^{-n} X_{(q-1)n}\right) &\leq N_{qn}(X_{qn} \cap I_n(x)) \\ &\approx 2^{(q-1)n \dim(X)} \\ &\approx N_{qn}(2^{-n} X_{(q-1)n}) \end{aligned}$$

Equivalently (scaling everything by 2^n),

$$N_{(q-1)n}\left(2^n (X_n \cap I_n(x)) + X_{(q-1)n}\right) \lesssim N_{(q-1)n}(X_{(q-1)n})$$

By the sumset theorem

$$N_{qn}(X_n \cap I_n(x)) \;=\; 2^{o(n)} \qquad \text{as } n \to \infty$$

Since

$$\frac{|X_n|}{N_n(X_n)} \;\approx\; \frac{|\Lambda|^n}{2^{n\,\dim(X)}} \;=\; 2^{n(s-\dim(X))}$$

there must be some (even many) $x \in X$ such that

$$|X_n \cap I_n(x)| \;\gtrsim\; 2^{n(s-\dim(X))}$$

The first and third relations show that there are many points in $X_n \cap I_n(x)$ within distance 2^{-qn} of each other. **QED**.

Multidimensional generalizations: In progress...

Open questions

1. The conjecture we started with!
2. Dimension of Bernoulli convolutions.
3. Analogous results for absolute continuity?
4. What are implications of $\Delta_n \to 0$ superexponentially?
5. Nonlinear setting?

Preprint on arxiv will available from Tuesday afternoon HK time.

Thank you.