Energy Measures of Harmonic Functions on the Sierpiński Gasket

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Introduction

- Energy: for functions u, v on SG,

$$\mathcal{E}_m(u,v) = \left(\frac{5}{3}\right)^m \sum_{x \underset{m}{\sim} y} (u(x) - u(y))(v(x) - v(y))$$

and

$$\mathcal{E}(u,v) = \lim_m \mathcal{E}_m(u,v).$$

- In general, $\mathcal{E}_m(u,v)$ attains both positive and negative values but $\mathcal{E}_m(u)$ (i.e.$\mathcal{E}_m(u,u)$) is always non-negative.
- We define a function h to be harmonic if its energy on level m

$$\mathcal{E}_m(h) = \sum_{i<j} (h(q_i) - h(q_j))^2$$

is a constant sequence where $\{q_i\}_{i=0}^2$ are vertices of the outermost triangle of SG.

Introduction

- The standard Laplacian Δ_μ, defined by the weak formulation

$$-\mathcal{E}(u,v) = \int (\Delta_\mu u) v d\mu$$

  for all $v \in \mathrm{dom}\mathcal{E}_0$, where μ is standard measure on SG.
- Advantage: Self-similarity

$$5\Delta_\mu(u \circ F_j) = (\Delta_\mu u) \circ F_j.$$

- Disadvantage: For $u \in \mathrm{dom}\Delta_\mu$, $u^2 \notin \mathrm{dom}\Delta_\mu$ is not defined.
- Question: Does there exist another Laplacian which behaves better in terms of functions in the domain of the Laplacian forming an algebra under pointwise multiplication?

Introduction

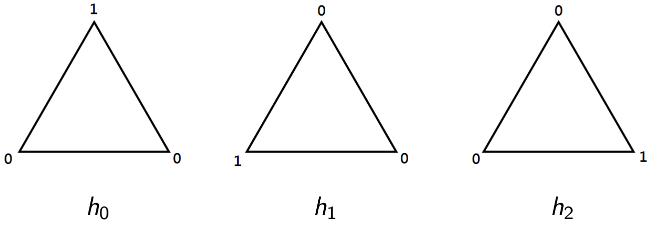
- Energy measure: for a cell $F_w SG$,

$$\nu_{u,v}(F_w SG) = \left(\frac{5}{3}\right)^{|w|} \mathcal{E}(u \circ F_w, v \circ F_w).$$

- $\nu_u := \nu_{u,u}$.
- Similarly, in general, $\nu_{u,v}$ is a signed measure but ν_u is always a positive measure.

Introduction

- The symmetric harmonic functions h_0, h_1, h_2 have values $h_i(q_j) = \delta_{ij}$ on vertices q_j.



- Denote $\nu_i := \nu_{h_i}$, the energy measure of h_i.

Introduction

- The space of energy measures of harmonic functions is three dimensional and $\{\nu_0, \nu_1, \nu_2\}$ form a basis.
- The Kusuoka measure $\nu = \nu_0 + \nu_1 + \nu_2 = 3(\nu_h + \nu_{h^\perp})$ if $\{h, h^\perp\}$ is an orthonormal basis of the space of all harmonic functions modulo constants.
- Fact: Every energy measure is absolutely continuous w.r.t. the Kusuoka measure ν.

Introduction

- We define the "energy Laplacian" by the weak formulation

$$-\mathcal{E}(u, v) = \int (\Delta_\nu u) v d\nu$$

for all v of finite energy vanishing on the boundary of SG, where ν is the Kusuoka measure.

- Whenever $u \in \mathrm{dom}\Delta_\nu$, $u^2 \in \mathrm{dom}\Delta_\nu$ and

$$\Delta_\nu u^2 = 2u\Delta_\nu u + 2\frac{d\nu_u}{d\nu}.$$

Self-Similarity

- Question: Does "energy Laplacian" behave in the sense of self-similarity like the standard Laplacian?
- Answer: We have some results like that but not that nice.
- We first establish the "self-similarity" of the the family $\{\nu_i\}$.

$$\begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{pmatrix} = \sum_{i=0}^{2} M_i \begin{pmatrix} \nu_0 \\ \nu_1 \\ \nu_2 \end{pmatrix} \circ F_i^{-1}$$

for some matrices M_i.

Self-Similarity

- The "self-similarity" of ν,

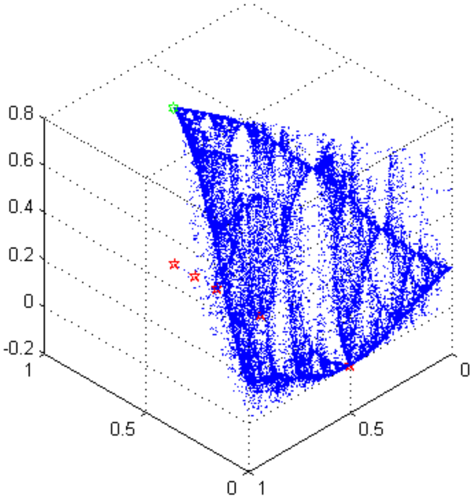
$$\nu = \sum_{i=0}^{2} \left(\left(\frac{1}{15} + \frac{12}{15} \frac{d\nu_i}{d\nu} \right) \nu \right) \circ F_i^{-1}.$$

- For Energy Laplacian, we also have the "self-similarity"

$$\Delta_\nu (u \circ F_j) = \frac{3}{5} \left(\frac{1}{15} + \frac{12}{15} \frac{d\nu_j}{d\nu} \right) (\Delta_\nu u) \circ F_j$$

but with variable weights.

Portraits of the Derivative



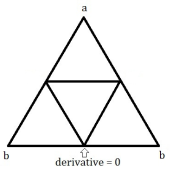
Bounds for Derivatives

Lemma

$$\nu(F_w SG) = \Theta\left(\left(\frac{3}{5}\right)^{|w|}\right)$$

Lemma

If h is symmetric, $\nu_h(F_2 F_1^m SG) = \Theta\left(\left(\frac{1}{15}\right)^m\right)$ and hence $\frac{d\nu_h}{d\nu}(x) = 0$ if $x \in \bigcap_m F_2 F_1^m SG$.



Bounds for Derivatives

Theorem

Let h be a harmonic function. For every cell C,

a)

$$\inf_{x \in C} \frac{d\nu_h}{d\nu}(x) = 0.$$

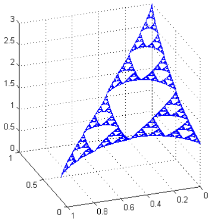
b) *If $\nu_h = a\nu_0 + b\nu_1 + c\nu_2$,*

$$\sup_{x \in C} \frac{d\nu_h}{d\nu}(x) = \frac{2}{3}(a + b + c).$$

In addition, if $\frac{d\nu_h}{d\nu}$ attains the maximum, then $\frac{d\nu_{h^\perp}}{d\nu}$ attains its minimum, and at the same point.

Bounds for Derivatives

Idea of proof:
For a), we look at the edge having an extremum.(existence of extremum proved by K. Dalrymple, R. S. Strichartz and J. P. Vinson, 1999)

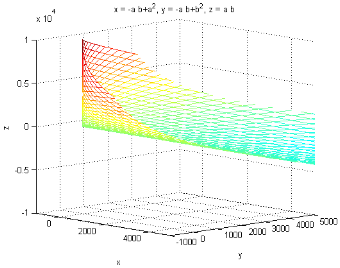


For b) consider $h^{\perp}$ orthogonal to h. Then $\frac{d\nu_h}{d\nu} + \frac{d\nu_{h^{\perp}}}{d\nu} = c$ for normalization constant $c = \frac{2}{3}(a + b + c)$. And

$$\sup \frac{d\nu_h}{d\nu} = c - \inf \frac{d\nu_{h^{\perp}}}{d\nu} = c.$$

Theorem

Take $\{\nu_0, \nu_1, \nu_2\}$ as a basis for signed energy measures of harmonic functions. Then the coefficients of all positive energy measures form a solid, circular cone $\{(x, y, z) \in \mathbb{R}^3 : xy + yz + xz \geq 0\}$. Furthermore, the energy measures ν_h obtained by a single harmonic function form the boundary of the cone.



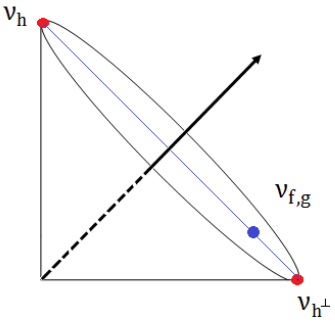
Characterization of Positive Energy Measures

Sketch of proof: The coefficients of the measures ν_h form the cone $\{(x,y,z) \in \mathbb{R}^3 : xy + yz + xz = 0\}$, so it suffices to show each ν_h is on the boundary. Precisely, we show $\nu_h - \varepsilon\nu$ is not a positive measure $\forall \varepsilon > 0$. Suppose, for contradiction, that it gives a positive measure for some $\varepsilon > 0$. Then $\nu_h(C)/\nu(C) > \varepsilon$ for all C, contradicting $\inf_C \frac{d\nu_h}{d\nu} = 0$.

Corollary

Any positive energy measure $\nu_{f,g}$ is precisely a convex combination of two energy measures ν_h and $\nu_{h^\perp}$ for some harmonic h.

Proof. Cut the cone by the plane containing 0, ν and $\nu_{f,g}$. One of the two lines contains the required ν_h and the other contains the required $\nu_{h^\perp}$.



Limited Continuity

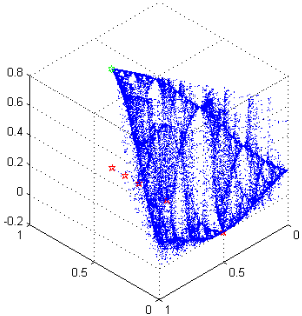
- Because for every cell C,

$$\sup_C \frac{d\nu_h}{d\nu} = \sup_{SG} \frac{d\nu_h}{d\nu}$$

and

$$\inf_C \frac{d\nu_h}{d\nu} = \inf_{SG} \frac{d\nu_h}{d\nu}$$

we see that the function $\frac{d\nu_h}{d\nu}$ is discontinuous, as shown below.

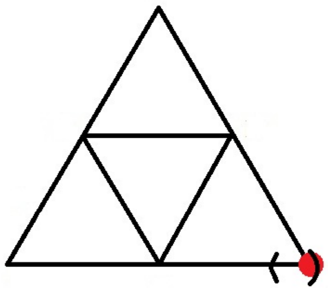


Limited Continuity

Theorem

Let $\nu_{f,g}$ be an energy measure. Given any cell C, the restriction of Radon-Nikodym derivative $\frac{d\nu_{f,g}}{d\nu}$ to the graph is continuous on the three edges of C.

Idea of proof: All we need to prove is the continuity on an edge at one corner point but the proof is technical.



Average Values

- We write the average value of the derivative on a cell as a weighted average of the value on the boundary points of the cell

$$\mathrm{Avg}_C \frac{d\nu_{f,g}}{d\nu} = \sum b_i \frac{d\nu_{f,g}}{d\nu}(p_i)$$

- We find b_i depending on the cell, i.e. depending on a finite word w.
- One set of b_i satisfies all $\nu_{f,g}$.

- Existence and uniqueness: On $F_w SG$

$$\sum b_i^{(w)} \begin{pmatrix} \frac{d\nu_0}{d\nu}(p_i) \\ \frac{d\nu_1}{d\nu}(p_i) \\ \frac{d\nu_2}{d\nu}(p_i) \end{pmatrix} = \begin{pmatrix} \frac{\nu_0(C)}{\nu(C)} \\ \frac{\nu_1(C)}{\nu(C)} \\ \frac{\nu_2(C)}{\nu(C)} \end{pmatrix}$$

- The set of vectors concerning $F_w SG$ and the set of vectors concerning SG differ by an invertible matrix. $b_i^{(\emptyset)}$ exist and unique on SG.

Average Values

- The distance of $b_j^{(w)}$ from $1/3$ is proportional to how "skewed" the Kusuoka measure is on the cell $F_w F_j SG$ relative to $F_w SG$. That is,

$$\frac{1}{5}\left(b_j^{(w)} - \frac{1}{3}\right) = \frac{1}{4}\left(\frac{\nu(F_w F_j SG)}{\nu(F_w SG)} - \frac{1}{3}\right)$$

- $\inf_w \{b_j^{(w)}\} = 0$, $\sup_w \{b_j^{(w)}\} = \frac{2}{3}$, so no boundary point is favored too heavily and no boundary point contributes negatively.

Average Values

If we define the rational maps

$$B_0(x,y,z) = \left(\frac{9x}{13x+y+z}, \frac{2x+2y-z}{13x+y+z}, \frac{2x-y+2z}{13x+y+z}\right)$$
$$B_1(x,y,z) = \left(\frac{2x+2y-z}{x+13y+z}, \frac{9y}{x+13y+z}, \frac{-x+2y+2z}{x+13y+z}\right)$$
$$B_2(x,y,z) = \left(\frac{2x-y+2z}{x+y+13z}, \frac{-x+2y+2z}{x+y+13z}, \frac{9z}{x+y+13z}\right)$$

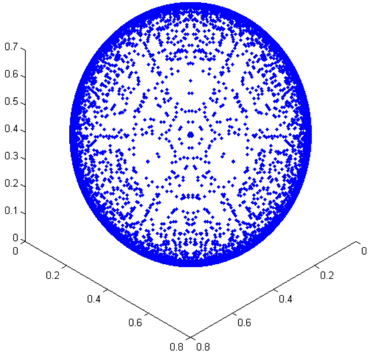
then

$$(b_0^{(w0)}, b_1^{(w0)}, b_2^{(w0)}) = B_0(b_0^{(w)}, b_1^{(w)}, b_2^{(w)})$$
$$(b_0^{(w1)}, b_1^{(w1)}, b_2^{(w1)}) = B_1(b_0^{(w)}, b_1^{(w)}, b_2^{(w)})$$
$$(b_0^{(w2)}, b_1^{(w2)}, b_2^{(w2)}) = B_2(b_0^{(w)}, b_1^{(w)}, b_2^{(w)})$$

- We have the sharp bound:

$$\sum \left(b_j^{(w)} - \frac{1}{3} \right)^2 < \frac{1}{6}$$

- Plot of level 9 of $b^{(w)} = (b_0^{(w)}, b_1^{(w)}, b_2^{(w)})$:



Thank You!