Geodesic distances and intrinsic distances on some fractal sets

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1. Introduction

M: a Riemaniann manifold

$d(x, y)$: the intrinsic distance (or the Carnot–Carathéodory distance):

$$d(x,y) := \sup \left\{ f(y) - f(x) \;\middle|\; \begin{array}{l} f\text{: Lipschitz on } M, \\ |\nabla f| \le 1 \text{ a.e.} \end{array} \right\}.$$

This is equal to the geodesic distance $\rho(x, y)$:

$$\rho(x,y) := \inf \left\{ \begin{array}{l} \text{the length of continuous curves} \\ \text{connecting } x \text{ and } y \end{array} \right\}.$$

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Intrinsic distance in the framework of Dirichlet forms

(cf. Biloli–Mosco, Sturm etc.)

(K, λ): a locally compact, separable metric measure space

$(\mathcal{E}, \mathcal{F})$: a strong local regular Dirichlet form on $L^2(K; \lambda)$

- $(\mathcal{E}, \mathcal{F})$ is a closed, nonnegative-definite, symmetric bilinear form on $L^2(K; \lambda)$;
- (Markov property) $^\forall f \in \mathcal{F}$, $\hat{f} := (0 \vee f) \wedge 1 \in \mathcal{F}$ and $\mathcal{E}(\hat{f}, \hat{f}) \le \mathcal{E}(f, f)$;
- (strong locality) For $f, g \in \mathcal{F}$ with compact support, if f is constant on a neighborhood of $\mathrm{supp}[g]$, then $\mathcal{E}(f, g) = 0$.

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Typical example:

$$(K, \lambda) = (\mathbb{R}^d, dx),$$

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (a_{ij}(x) \nabla f(x), \nabla g(x))_{\mathbb{R}^d} \, dx$$

for $f, g \in \mathcal{F} := H^1(\mathbb{R}^d)$,

where $(a_{ij}(x))_{i,j=1}^d$ is symmetric, uniformly positive and bounded.

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$\mu_{\langle f\rangle}$: the energy measure of $f\in\mathcal{F}$

When f is bounded,

$$\int_K \varphi\, d\mu_{\langle f\rangle} = 2\mathcal{E}(f,f\varphi)-\mathcal{E}(f^2,\varphi)\quad {}^\forall\varphi\in\mathcal{F}\cap C_b(K).$$

If $\mathcal{E}(f,g)=\dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d}(a_{ij}(x)\nabla f(x),\nabla g(x))_{\mathbb{R}^d}\,dx$, then

$\mu_{\langle f\rangle}(dx)=(a_{ij}(x)\nabla f(x),\nabla f(x))_{\mathbb{R}^d}\,dx.$

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In this framework, various Gaussian estimates of the transition density have been obtained.

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Is $\mathbf{d}$ identified with the geodesic distance (=shortest path metric)?

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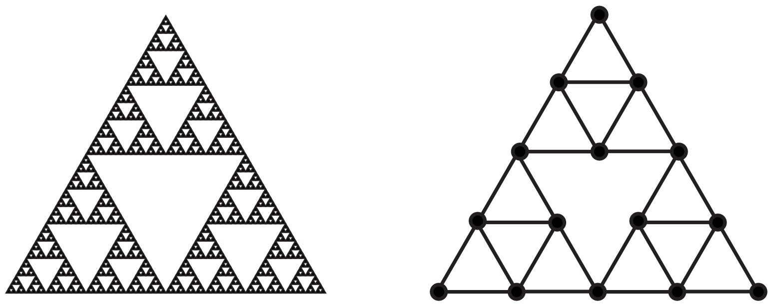
2. Canonical Dirichlet forms on typical self-similar fractals

Case of the 2-dim. standard Sierpinski gasket



$$K \supset V_2$$

V_n: nth level graph approximation

$$\mathcal{E}^{(n)}(f,f) = \left(\frac{5}{3}\right)^n \sum_{x,y \in V_n,\ x \sim y} (f(x) - f(y))^2$$

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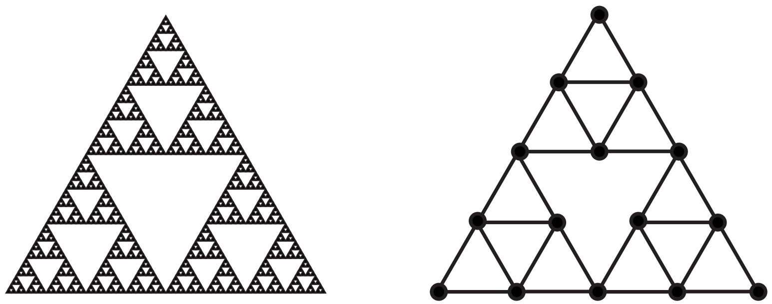

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↑ scaling factor

$$\mathcal{E}^{(n)}(f,f) \nearrow {}^{\exists}\mathcal{E}(f,f) \le +\infty \quad {}^{\forall}f \in C(K).$$

$$\mathcal{F} := \{f \in C(K) \mid \mathcal{E}(f,f) < +\infty\}$$

Then, $(\mathcal{E},\mathcal{F})$ is a strong local regular Dirichlet form on $L^2(K;\lambda)$. (λ: the Hausdorff measure on K)

$\rightsquigarrow \{X_t\}$: "Brownian motion" on K

(invariant under scaling and isometric transformations)

Similar construction is valid for more general finitely ramified fractals.

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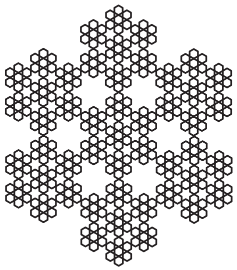 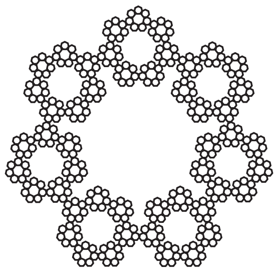 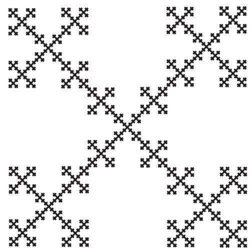 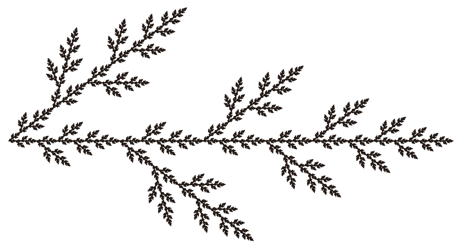

In many examples, $\mu_{\langle f\rangle} \perp \lambda$ (self-similar measure).

Then,

$$\begin{aligned} \mathbf{d}(x,y) &= \sup\{f(y)-f(x) \mid f \in \mathcal{F},\ \mu_{\langle f\rangle} \le \lambda\} \\ &= \sup\{f(y)-f(x) \mid f = \text{const.}\} \\ &= 0. \end{aligned}$$

(This is closely connected with the fact that the heat kernel density has a sub-Gaussian estimate.)

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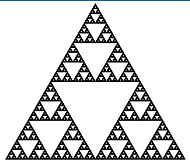
K: 2-dim. Sierpinski gasket 

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(h_i: a harmonic function, $\mathcal{E}(h_i, h_j) = \delta_{i,j}$)

Theorem (Kigami '93, '08, Kajino '12)

- (Ki) $\boldsymbol{h} : K \to \boldsymbol{h}(K) \subset \mathbb{R}^2$ is homeomorphic;
- (Ka) The intrinsic distance $\mathbf{d}$ coincides with the geodesic distance $\rho_{\boldsymbol{h}}$ on $\boldsymbol{h}(K)$ by the identifying K and $\boldsymbol{h}(K)$;
- (Ki, Ka) The transition density $p_t^{\nu}(x, y)$ has a Gaussian estimate w. r. t. $\rho_{\boldsymbol{h}} (= \mathbf{d})$;
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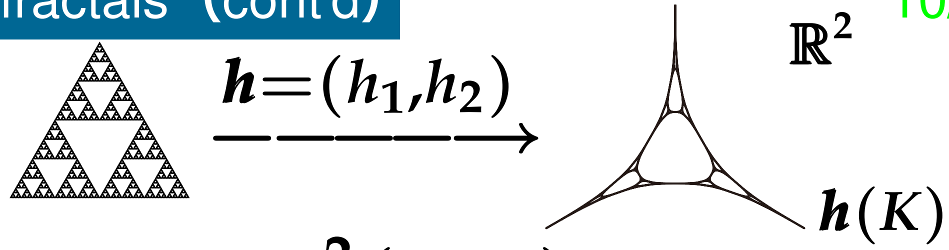

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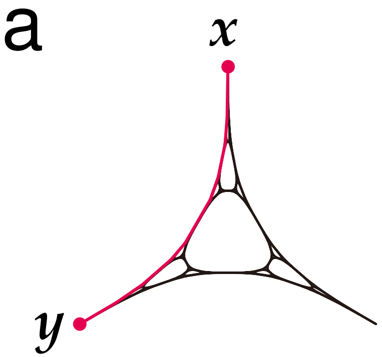

3. General framework

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$N \in \mathbb{N}$, $\boldsymbol{h} = (h_1, \ldots, h_N) \in \mathcal{F}^N \cap C(K \to \mathbb{R}^N)$

$$\nu := \mu_{\langle \boldsymbol{h} \rangle} := \sum_{j=1}^{N} \mu_{\langle h_j \rangle}$$

The intrinsic distance $\mathrm{d}_{\boldsymbol{h}}(x, y)$ based on $(\mathcal{E}, \mathcal{F})$ and $\boldsymbol{h}$ is defined as

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4. Results

Theorem 1 $\rho_h(x,y) \leq \mathbf{d}_h(x,y)$ if the following hold:

(A1) (Finitely ramified cell structure) There exists an increasing sequence of finite subsets $\{V_m\}_{m=0}^{\infty}$ of K such that

(i) $\bigcup_{m=0}^{\infty} V_m$ is dense in K;

(ii) For each m, $K \setminus V_m$ is decomposed as a finite number of connected components $\{U_\lambda\}_{\lambda \in \Lambda_m}$;

(iii) $\lim_{m\to\infty} \max_{\lambda\in\Lambda_m} \operatorname{diam} U_\lambda = 0$.

(A2) $\mathcal{F} \subset C(K)$.

(A3) $\mathcal{E}(f,f) = 0$ if and only if f is a constant function.

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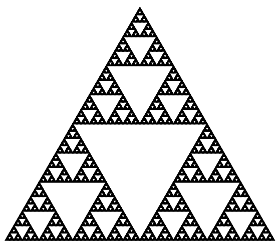 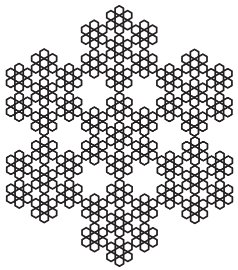 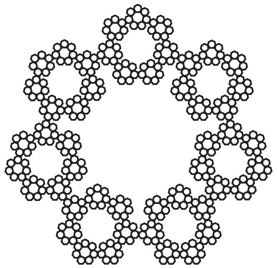 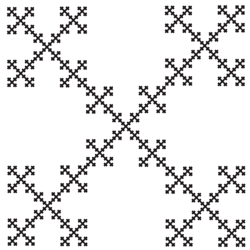 

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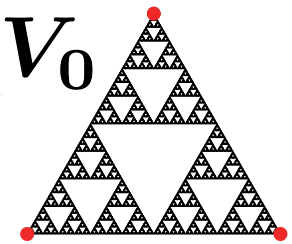
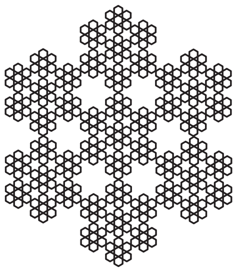
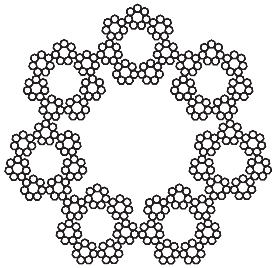
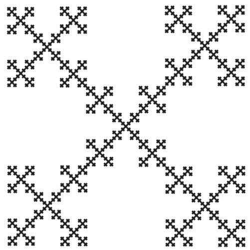
V_0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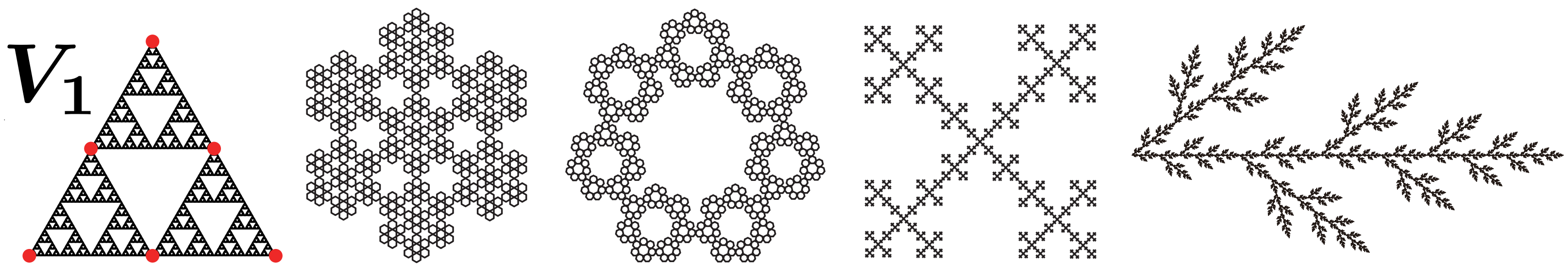

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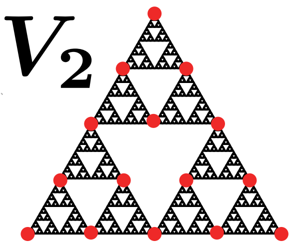

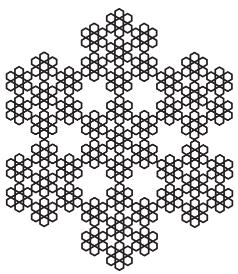

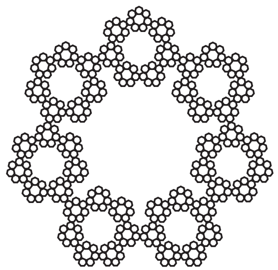

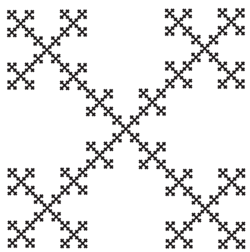

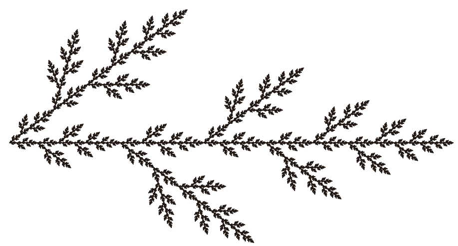

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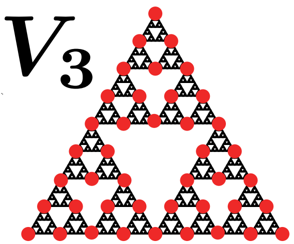
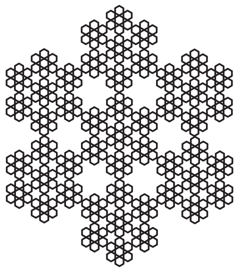
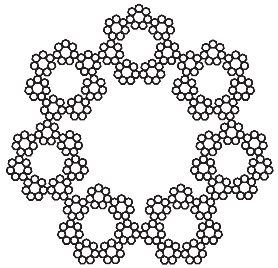
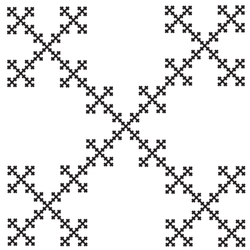
- (i) $\bigcup_{m=0}^{\infty} V_m$ is dense in K;
- (ii) For each m, $K \setminus V_m$ is decomposed as a finite number of connected components $\{U_\lambda\}_{\lambda \in \Lambda_m}$;
- (iii) $\lim_{m\to\infty} \max_{\lambda\in\Lambda_m} \operatorname{diam} U_\lambda = 0$.

V_3









4. Results

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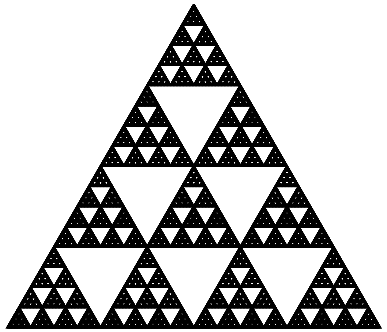
(A2) $\mathcal{F} \subset C(K)$.

(A3) $\mathcal{E}(f, f) = 0$ if and only if f is a constant function.

Theorem 2 $\rho_{\boldsymbol{h}}(x, y) \geq \mathbf{d}_{\boldsymbol{h}}(x, y)$ if

- K: a 2-dimensional (generalized) Sierpinski gasket that is also a nested fractal;



- (λ: the normalized Hausdorff measure;)
- $(\mathcal{E}, \mathcal{F})$: the self-similar Dirichlet form associated with the Brownian motion on K;
- $\boldsymbol{h} = (h_1, \ldots, h_d)$; each h_i is a harmonic function;
- The harmonic structure associated with $(\mathcal{E}, \mathcal{F})$ is nondegenerate. (That is, for any nonconstant harmonic functions g, g is not constant on any nonempty open sets.)

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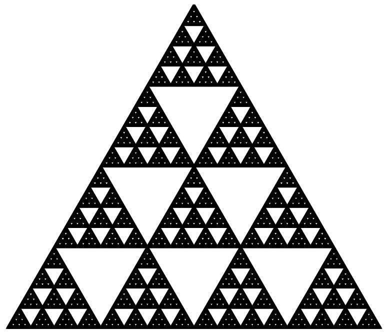

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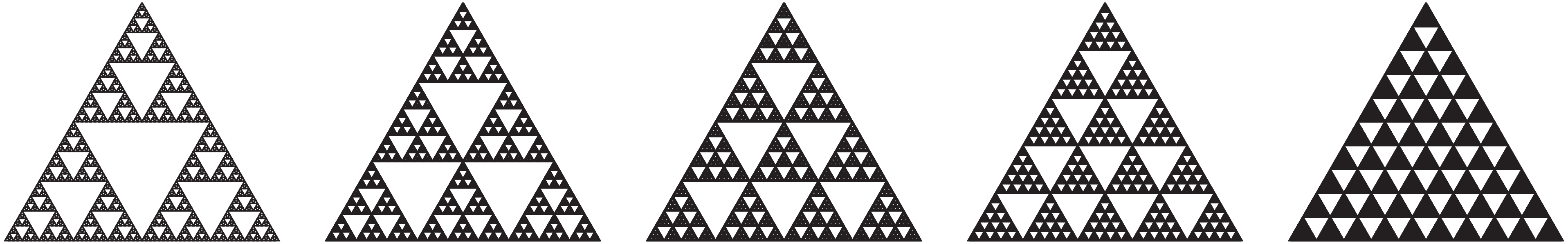

(level l S. G. with $l = 2, 3, 4, 5, 10$)

Remark Theorem 2 is valid under more general situations. Essential assumptions (for the current proof) are:

- # the vertex set $= 3$;
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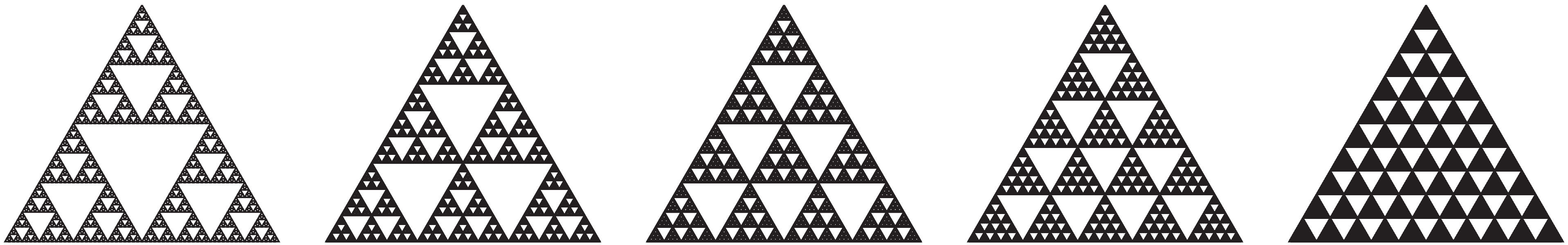

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Some ingredients for the proof

- A version of Rademacher's theorem
- An alternative of the fundamental theorem of calculus
- Proof of better nondegeneracy

Remark The classical case:

K: closure of a bdd domain of $\mathbb{R}^N$ with smooth boundary

$$\mathcal{E}(f,g) = \frac{1}{2}\int_K (\nabla f, \nabla g)_{\mathbb{R}^N} dx, \quad \mathcal{F} = H^1(K)$$

$h_i(x) := x_i \ (i = 1, \ldots, N)$

Then, ρ_h is the usual geodesic distance on K, and

$\mu_{\langle h \rangle}(dx) = \sum_{i=1}^{N} dx = N\, dx$. Therefore, $\rho_h = \sqrt{N}\mathrm{d}_h$.

Probably, $\frac{1}{p(x)}\mu_{\langle h \rangle}(dx)$ is the correct measure to define the intrinsic distance in general.

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