Fourier and Group Representation Frames

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Outline:

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- The Frame Conjecture (FC)

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- (FC) for Fourier Frames

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Feichtinger's Frame Conjecture

" Every bounded frame is a finite union of Riesz sequences "

Feichtinger Frame Conjecture

Another Formulation of (FC)

For any (orthogonal) projection matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ on $\ell^2(\mathbb{N})$ with main diagonal entries bounded away from 0, there is a finite partition $\mathbb{N} = \cup_{\ell=1}^{L}\Lambda_\ell$ such that the principal submatrice $(a_{ij})_{i,j\in\Lambda_\ell}$ are bounded invertible.

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- PC —— Anderson's Paving Conjecture, 1979
- BTC —— Bourgain-Tzafriri Restrictive Invertibility Conjecture, 1978/91

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- A frame for H is a sequence $\{x_n\}$ such that there exist $C_1, C_2 > 0$ with the property

$$\boxed{C_1||x||^2 \leq \sum_n | < x, x_n > |^2 \leq C_2||x||^2, \quad x \in H}$$

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$$C_1||x||^2 \leq \sum_n | < x, x_n > |^2 \leq C_2||x||^2, \quad x \in H$$

- Parseval frame:

$$\sum_n | < x, x_n > |^2 = ||x||^2, \quad x \in H$$

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- While it is extremely difficult to construct a counterexample (if there exists one, as widely conjectured), it is also extremely difficult to confirm the conjecture for any nice subclass of frames! F9r example, the question remains open for Weil-Heisenberg (Gabor) frames, wavelet frames, Fourier frames.

The Classical Fourier Frames

Let $E \subset [0,1]$ be any (Lebesgue) measurable subset with $\mu(E) > 0$. For $\lambda \in \mathbb{R}$, define $e_\lambda(t) = e^{2\pi i \lambda t}$.

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- The same question can be asked for Fourier frames with respect to some other (probability, fractal) measures, e.g, one-third, one-fourth Cantor measure, Bernoulli Convolution measure, or more generally, fractal measures for iterated function systems (IFS).

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Definition

A subset $\Lambda \subset \mathbb{R}^d$ is called a *spectrum/Riesz spectrum/frame spectrum* if the corresponding set of exponentials

$$E(\Lambda) := \{e_\lambda : \lambda \in \Lambda\}$$

is an orthonormal basis/Riesz basis/frame for $L^2(\mu)$.

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If μ has a spectrum then it is called a *spectral measure*.

Other (Fractal) Measures

Let R be a $d \times d$ expansive integer matrix, let B be a finite subset of $\mathbb{Z}^d$, $0 \in B$, and let $N := \#B$. Define the affine maps

$$\tau_b(x) = R^{-1}(x + b), \quad (x \in \mathbb{R}^d, b \in B)$$

Then $(\tau_b)_{b \in B}$ is called an affine iterated function system (IFS).

Fourier Frames for Fractal Measures

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There exists a unique compact set such that

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There is a unique Borel probability measure μ *on* $\mathbb{R}^d$ *such that*

$$\int f \, d\mu = \frac{1}{N} \sum_{b \in B} \int f \circ \tau_b \, d\mu, \quad (f \in C_c(\mathbb{R}^d))$$

The measure μ *is supported on* X_B*.*

Fourier Frames for Fractal Measures

- $R = 3$ and $B = \{0, 2\} \Rightarrow$ The Middle Third Cantor set.
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- $R = \frac{1}{\alpha}$ with $\alpha \in (0, 1)$ and $B = \{-1, 1\} \Rightarrow$ Bernoulli Convolutions μ_α
- μ_α is spectral if and only if $\alpha = \frac{1}{2n}$ [Jorgensen-Pedersen, Dutkay-H-Jorgensen, Hu-Lau, Xinrong Dai etc.]

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Cantor Measure

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Question

What geometric properties of the measure μ can be deduced if we know a spectrum/ frame spectrum of μ?

Beurling dimension

Definition

Let $Q = [0,1]^d$ be the unit cube. Let Λ be a discrete subset of $\mathbb{R}^d$, and let $\alpha > 0$. Then the α-upper Beurling density is

$$\mathcal{D}_\alpha^+(\Lambda) := \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap (x + hQ))}{h^\alpha}.$$

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or equivalently,

$$D^+(\Lambda) := \inf\{\alpha > 0 : D_\alpha^+(\Lambda) < \infty\}.$$

Sometimes Hausdorff meets Beurling

Theorem (Dutkay-H-Sun-Weber)

(i) If $\{e_\lambda : \lambda \in \Lambda\}$ *is a frame for* $L^2(\mu_3)$*, then (under a technical condition) the upper Beurling dimension of* Λ *is equal to the Hausdorff dimension* $\frac{\ln 2}{\ln 3}$ *of the Cantor set (This also true for general (IFS) induced fractal measures)*

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(ii) For μ_3, there exists Riesz sequences $\{e_\lambda : \lambda \in \Lambda\}$ with Λ having positive Beurling dimensions

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- The existence question of frame spectral for the middle-third cantor measure μ_3 remains open. In fact we even don't know if weighted Fourier frames (or more generally "frame measures") exist or not.

Frame Measures

Definition

A frame measure for μ is a Borel measure ν on $\mathbb{R}$ such that for every $f \in L^2(\mathbb{R})$

$$\int_{\mathbb{R}} |\hat{f}(t)|^2 d\nu(t) \approx \int_{\mathbb{R}} |f(x)|^2 d\mu(x)$$

where $\hat{f}(t) = \int_{\mathbb{R}} f(x) e^{-2\pi i t x} d\mu(x)$.

Frame Measures

Theorem (Dutkay-H-Weber)

Existence of frame measure $\Rightarrow$ *Existence of weighted Fourier frame and the "Beurling dimension"' of* $\nu \leq$ *Hausdorff dimension* $\frac{\ln 2}{\ln 3}$

Fourier Frames

Observation: Let $\pi : \mathbb{Z} \to B(L^2(E))$ be the unitary group representation defined by $\pi(n) = M_{e^{2\pi int}}$ (the multiplication unitary operator by $e^{2\pi int}$).

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Question: What about frame representations for some other groups? (e.g. ICC groups, free groups) Why do we care?

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- $\mathcal{U}(H)$— group of unitary operators on a separable Hilbert space H
- unitary representation π of G on H — group homomorphism from G to $\mathcal{U}(H)$
- λ – the left regular representation of G on $\ell^2(G)$: $\lambda(g)e_h = e_{gh}$
- subrepresentation $\pi = \lambda|_P, \in \lambda(G)'$ an orthogonal projection

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- every frame representation is unitarily equivalent to a subrepresentation $\lambda|_P$

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Question III: Find ONE group such that every frame vector satisfies (FC).

Remark: Affirmative answer to Question III for a group $G \Rightarrow$ Affirmative answer to Question III for every subgroup of G.

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About Question II

Theorem (Dutkay-H-Picioroaga,)

Let G be an ICC group, and $\pi = \lambda|_p$ with $p \in \lambda(G)'$ and $tr(p) = \frac{1}{N}$. If there is a normal subgroup H such that $[G : H] = N$, then there is a Parseval frame vector η such that $\{\sqrt{N}\pi(g)\eta : g \in G\}$ is the union of N-orthonormal sequences. Moreover, the associated partition of G is given by cosets of a subgroup of G.

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The free group case

Corollary

Let $G = \mathcal{F}_n$ $(n \geq 2)$. Then every frame representation of G admits a Riesz decomposible frame vector. Moreover, the associated partition of G is given by cosets of a subgroup of G

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- Remark: Using Bourgain-Tzafriri and Halpern-Kaftal- Weiss example, there exists a frame vector for $\mathcal{F}_n$ satisfying (FC), but it can not be decomposed by the cosets of a subgroup of $\mathcal{F}_n$

Question I

Theorem (H-Larson)

Let $\pi = \lambda|_P$ with $P \in \lambda(G)'$. Then the following are equivalent:

(i) $P \in \lambda(G)' \cap \lambda(G)''$,

(ii) for every two frame vectors η and ξ for π, there exits an invertible operator $S \in \pi(G)'$ such that $\eta = S\xi$

Consequence: Question I has a positive answer for frame representations $\pi = \lambda|_P$ with $p \in \lambda(G)' \cap \lambda(G)''$.

Question I

Theorem (H-Larson)

Let π be a frame representation and ξ be a Parseval frame vector for π. Then
(i) $\eta \in H$ is a Parseval frame vector for π if and only if there exists a unitary operator $U \in \pi(G)''$ such that $\eta = \xi$
(ii) $\eta \in H$ is a frame vector for π if and only if there exits an invertible operator $S \in \pi(G)''$ such that $\eta = S\xi$

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Theorem (Dutkay-H-Picioroaga)

Let π be a frame representation for a free group G. Then there exist decomposable Parseval frame vectors ξ_i $(i = 1, ..., N)$ (where N is the cyclic multiplicity of $\pi(G)'$)

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Let π be a frame representation for a free group G. Then there exist decomposable Parseval frame vectors ξ_i $(i = 1, \ldots, N)$ (where N is the cyclic multiplicity of $\pi(G)'$) such that for any Parseval frame vector η for π, there exist operators $A_1, \ldots, A_N \in \pi(G)'$ such that $\sum_{i=1}^{N} A_i A_i^* = I$ and $\eta = \sum_{i=1}^{N} A_i \xi_i$.

THANK YOU!