On stochastic completeness of jump processes

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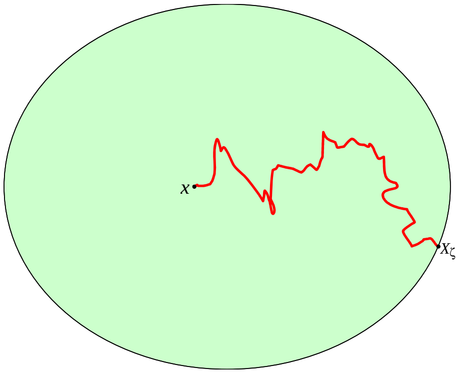
1 Stochastic completeness of a diffusion

Let $\{X_t\}_{t\geq 0}$ be a reversible Markov process on a state space M. This process is called *stochastically complete* if its lifetime is almost surely ∞, that is

$$\mathbb{P}_x \left(X_t \in M \right) = 1.$$

If the process has no interior killing (which will be assumed) then the only way the stochastic incompleteness can occur is if the process leaves the state space in finite time. For example, diffusion in a bounded domain with the Dirichlet boundary condition is stochastically incomplete.



A by far less trivial example was discovered by R.Azencott in 1974: he showed that Brownian motion on a *geodesically complete* non-compact manifold can be stochastically incomplete. In his example the manifold has negative sectional curvature that grows to $-\infty$ very fast with the distance to an origin. The stochastic incompleteness occurs because negative curvature plays the role of a drift towards infinity, and a very high negative curvature produces an extremely fast drift that sweeps the Brownian particle away to infinity in a finite time.

Various sufficient conditions in terms of curvature bounds were obtained by S.-T. Yau 1978, E.P. Hsu 1989, etc. It is somewhat surprising that one can obtain a sufficient condition for stochastic completeness in terms of the volume growth. Let $V(x,r)$ be the volume of the geodesic ball of radius r centered at x. Then

$$V(x,r) \le \exp\left(Cr^2\right) \Rightarrow \text{stochastic completeness.}$$

Moreover,

$$\int^{\infty} \frac{rdr}{\log V(r)} = \infty \Rightarrow \text{stochastic completeness.}$$

Let us sketch the construction of Brownian motion on a Riemannian manifold M and approach to the proof of the volume test for stochastic completeness. Let M be a Riemannian manifold, μ be the Riemannian measure on M and Δ be the Laplace-Beltrami operator on M. By the Green formula, Δ is a symmetric operator on $C_0^\infty(M)$ with respect to μ, which allows to extend Δ to a a self-adjoint operator in $L^2(M, \mu)$. Assuming that M is geodesically complete, it is possible to prove that this extension is unique. Hence, Δ can be regarded as a (non-positive definite) self-adjoint operator in L^2.

By functional calculus, the operator $P_t := e^{t\Delta}$ is a bounded self-adjoint operator for any $t \geq 0$. The family $\{P_t\}_{t\geq 0}$ is called the *heat semigroup* of Δ. It can be used to solve the Cauchy problem in $\mathbb{R}_+ \times M$:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u, \\ u\big|_{t=0} = f, \end{cases}$$

since $u(t, \cdot) = P_t f$ is solution for any $f \in L^2$.

Local regularity theory implies that P_t is an integral operator, whose kernel is $p_t(x, y)$ is a positive smooth function of (t, x, y). In fact, $p_t(x, y)$ is the minimal positive fundamental solution to the heat equation.

The heat kernel can be used to construct a diffusion process $\{X_t\}$ on M with transition density $p_t(x,y)$. For example, in $\mathbb{R}^n$ one has

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

and the process $\{X_t\}$ with this transition density is Brownian motion.

In terms of the heat kernel the stochastic completeness of diffusion $\{X_t\}$ is equivalent to the following identity:

$$\int_M p_t(x,y)\, d\mu(y) = 1,$$

for all $t>0$ and $x \in M$.

Another useful criterion for stochastic completeness is as follows: M is stochastically complete if the homogeneous Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ u|_{t=0} = 0 \end{cases} \tag{1}$$

has a unique solution $u \equiv 0$ in the class of bounded functions (Khas'minskii).

By classical results, in $\mathbb{R}^n$ the uniqueness for (1) holds even in the class

$$|u(t,x)| \leq \exp\left(C|x|^2\right)$$

(Tikhonov class), but not in

$$|u(t,x)| \leq \exp\left(C|x|^{2+\varepsilon}\right).$$

More generally, uniqueness holds in the class

$$|u(t,x)| \leq \exp(f(r))$$

provided function f satisfies

$$\int^{\infty} \frac{r\,dr}{f(r)} = \infty$$

(Täcklind class).

The following result can be regarded as an analogue of the latter uniqueness class.

Theorem 1 (AG, 1986) *Let M be a complete connected Riemannian manifold, and let $u(x,t)$ be a solution to the Cauchy problem* (1). *Assume that, for some $x \in M$ and for some $T > 0$ and all $r > 0$,*

$$\int_0^T \int_{B(x,r)} u^2(y,t)\, d\mu(y) dt \le \exp\left(f(r)\right), \tag{2}$$

where $f(r)$ is a positive increasing function on $(0, +\infty)$ such that

$$\int^{\infty} \frac{r dr}{f(r)} = \infty.$$

Then $u \equiv 0$ in $(0,T) \times M$.

One may wonder why the geodesic balls can be used to determine the stochastic completeness, because the latter condition does not depend on the distance function at all. The reason is that the geodesic distance d is by definition related to the gradient ∇ (and, hence, to the Laplacian) by $|\nabla d| \le 1$. An analogue of this condition will appear later also in jump processes.

If u is a bounded solution, then replacing in (2) u by const we obtain that if

$$V(x,r) \leq \exp(f(r))$$

then $u \equiv 0$, that is, M is stochastic completeness. Setting

$$f(r) = \log V(x,r)$$

we obtain the volume test for stochastic completeness:

$$\int^{\infty} \frac{rdr}{\log V(x,r)} = \infty.$$

The latter condition cannot be further improved: if $W(r)$ is an increasing function such that

$$\int^{\infty} \frac{rdr}{\log W(r)} < \infty$$

then there exists a geodesically complete but stochastically incomplete manifold with $V(x,r) \leq W(r)$.

2 Jump processes

Let (M,d) be a metric space such that all closed metric balls

$$B(x,r) = \{y \in M : d(x,y) \le r\}$$

are compact. In particular, (M,d) is locally compact and separable. Let μ be a Radon measure on M with a full support.

Recall that a *Dirichlet form* $(\mathcal{E},\mathcal{F})$ in $L^2(M,\mu)$ is a symmetric, non-negative definite, bilinear form $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ defined on a dense subspace $\mathcal{F}$ of $L^2(M,\mu)$, that satisfies in addition the following properties:

- Closedness: $\mathcal{F}$ is a Hilbert space with respect to the following inner product:
$$\mathcal{E}_1(f,g) := \mathcal{E}(f,g) + (f,g)\,.$$
- The Markov property: if $f \in \mathcal{F}$ then also $\widetilde{f} := (f \wedge 1)_+$ belongs to $\mathcal{F}$ and $\mathcal{E}(\widetilde{f}) \le \mathcal{E}(f)$, where $\mathcal{E}(f) := \mathcal{E}(f,f)$.

For example, the classical Dirichlet form in $\mathbb{R}^n$ is

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, dx$$

in $\mathcal{F} = W^{1,2}(\mathbb{R}^n)$.

A general Dirichlet form $(\mathcal{E}, \mathcal{F})$ has the *generator* $\mathcal{L}$ that is a non-positive definite, self-adjoint operator on $L^2(M,\mu)$ with domain $\mathcal{D} \subset \mathcal{F}$ such that

$$\mathcal{E}(f,g) = (-\mathcal{L}f, g)$$

for all $f \in \mathcal{D}$ and $g \in \mathcal{F}$. The generator $\mathcal{L}$ determines the *heat semigroup* $\{P_t\}_{t\geq 0}$ by $P_t = e^{t\mathcal{L}}$ in the sense of functional calculus of self-adjoint operators. It is known that $\{P_t\}_{t\geq 0}$ is strongly continuous, contractive, symmetric semigroup in L^2, and is *Markovian*, that is, $0 \leq P_t f \leq 1$ for any $t > 0$ if $0 \leq f \leq 1$.

The Markovian property of the heat semigroup implies that the operator P_t preserves the inequalities between functions, which allows to use monotone limits to extend P_t from L^2 to L^∞. In particular, $P_t 1$ is defined.

Definition. The form $(\mathcal{E},\mathcal{F})$ is called *conservative* or *stochastically complete* if $P_t 1 = 1$ for every $t > 0$.

Assume in addition that $(\mathcal{E},\mathcal{F})$ is *regular*, that is, the set $\mathcal{F} \cap C_0(M)$ is dense both in $\mathcal{F}$ with respect to the norm $\mathcal{E}_1$ and in $C_0(M)$ with respect to the sup-norm. By a theory of Fukushima, for any regular Dirichlet form there exists a Hunt process $\{X_t\}_{t\geq 0}$ such that, for all bounded Borel functions f on M,

$$\mathbb{E}_x f(X_t) = P_t f(x) \tag{3}$$

for all $t > 0$ and almost all $x \in M$, where $\mathbb{E}_x$ is expectation associated with the law of $\{X_t\}$ started at x.

Using the identity (3), one can show that the lifetime of X_t is almost surely ∞ if and only if $P_t 1 = 1$ for all $t > 0$, which motivates the term "stochastic completeness" in the above definition.

One distinguishes local and non-local Dirichlet forms. The Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}(f,g) = 0$ for all functions $f, g \in \mathcal{F}$ with disjoint compact support. It is called *strongly local* if the same is true under a milder assumption that $f = \text{const}$ on a neighborhood of $\operatorname{supp} g$.

For example, the following Dirichlet form on a Riemannian manifold

$$\mathcal{E}(f,g) = \int_M \nabla f \cdot \nabla g d\mu$$

is strongly local. The generator of this form the self-adjoint Laplace-Beltrami operator Δ, and the Hunt process is Brownian motion on M.

A well-studied non-local Dirichlet form in $\mathbb{R}^n$ is given by

$$\mathcal{E}(f,g) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x-y|^{n+\alpha}} dxdy \tag{4}$$

where $0 < \alpha < 2$. The domain of this form is the Besov space $B^{\alpha/2}_{2,2}$, the generator is (up to a constant multiple) the operator $-(-\Delta)^{\alpha/2}$, where Δ is the Laplace operator in $\mathbb{R}^n$, and the Hunt process is the symmetric stable process of index α.

By a theorem of Beurling and Deny, any regular Dirichlet form can be represented in the form

$$\mathcal{E} = \mathcal{E}^{(c)} + \mathcal{E}^{(j)} + \mathcal{E}^{(k)},$$

where $\mathcal{E}^{(c)}$ is a strongly local part that has the form

$$\mathcal{E}^{(c)}(f,g) = \int_M \Gamma(f,g)\, d\mu,$$

where $\Gamma(f,g)$ is a so called *energy density* (generalizing $\nabla f \cdot \nabla g$ on manifolds); $\mathcal{E}^{(j)}$ is a jump part that has the form

$$\mathcal{E}^{(j)}(f,g) = \frac{1}{2} \int\int_{X \times X} (f(x) - f(y))(g(x) - g(y))\, dJ(x,y)$$

with some measure J on $X \times X$ that is called a *jump measure*; and $\mathcal{E}^{(k)}$ is a killing part that has the form

$$\mathcal{E}^{(k)}(f,g) = \int_X fg dk$$

where k is a measure on X that is called a *killing measure*.

In terms of the associated process this means that X_t is in some sense a mixture of diffusion and jump processes with a killing condition.

The log-volume test of stochastic completeness of manifolds can be extended to strongly local Dirichlet forms as follows. Set as before $V(x,r) = \mu(B(x,r))$

Theorem 2 (T.Sturm 1994) *Let $(\mathcal{E},\mathcal{F})$ be a regular strongly local Dirichlet form. Assume that the distance function $\rho(x) = d(x,x_0)$ on M satisfies the condition*

$$\Gamma(\rho,\rho) \leq C,$$

for some constant C. If, for some $x \in M$,

$$\int^{\infty} \frac{rdr}{\log V(x,r)} = \infty$$

then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is stochastically complete.

The method of proof is basically the same as for manifolds because for strongly local forms the same chain rule and product rules are available. The condition $\Gamma(\rho,\rho) \leq C$ is analogous to $|\nabla\rho| \leq 1$ that is automatically satisfied for the geodesic distance on any manifold.

Now let us turn to jump processes. For simplicity let us assume that the jump measure J has a density $j(x,y)$. Namely, let $j(x,y)$ be is a non-negative Borel function on $M \times M$ that satisfies the following two conditions:

(a) $j(x,y)$ is symmetric: $j(x,y) = j(y,x)$;

(b) there is a positive constant C such that

$$\int_M (1 \wedge d(x,y)^2) j(x,y) d\mu(y) \leq C \quad \text{for all } x \in M.$$

Definition. We say that a distance function d is *adapted* to a kernel $j(x,y)$ (or j is adapted to d) if (b) is satisfied.

The condition (b) relates the distance function to the Dirichlet form and plays the same role as $\Gamma(\rho,\rho) \leq C$ does for diffusion.

Consider the following bilinear functional

$$\mathcal{E}(f,g) = \frac{1}{2}\int\int_{X\times X}(f(x)-f(y))(g(x)-g(y))j(x,y)d\mu(x)d\mu\,(y)$$

that is defined on Borel functions f and g whenever the integral makes sense. Define the maximal domain of $\mathcal{E}$ by

$$\mathcal{F}_{\max} = \left\{f \in L^2 : \mathcal{E}(f,f) < \infty\right\},$$

where $L^2 = L^2(M,\mu)$. By the polarization identity, $\mathcal{E}(f,g)$ is finite for all $f,g \in \mathcal{F}_{\max}$. Moreover, $\mathcal{F}_{\max}$ is a Hilbert space with the norm $\mathcal{E}_1$.

Denote by $\mathrm{Lip}_0(M)$ the class of Lipschitz functions on M with compact support. It follows from (b) that

$$\mathrm{Lip}_0(M) \subset \mathcal{F}_{\max}.$$

Define the space $\mathcal{F}$ as the closure of $\mathrm{Lip}_0(M)$ in $(\mathcal{F}_{\max}, \|\cdot\|_{\mathcal{E}_1})$. Under the above hypothesis, $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form in $L^2(M,\mu)$. The associated Hunt process $\{X_t\}$ is a pure jump process with the jump density $j(x,y)$.

Many examples of jump processes in $\mathbb{R}$ are provided by Lévy-Khintchine theorem where the Lévy measure $W(dy)$ corresponds to $j(x,y)d\mu(y)$. The condition (b) appears also in Lévy-Khintchine theorem in the form

$$\int_{\mathbb{R}\setminus\{0\}} \left(1 \wedge |y|^2\right) W(dy) < \infty.$$

Hence, the Euclidean distance in $\mathbb{R}$ is adapted to any Lévy process.

An explicit example of a jump density in $\mathbb{R}^n$ is

$$j(x,y) = \frac{\text{const}}{|x-y|^{n+\alpha}},$$

where $\alpha \in (0,2)$, which defines the Dirichlet form (4).

The next theorem is the main result.

Theorem 3 *Assume that j satisfies (a) and (b) and let $(\mathcal{E},\mathcal{F})$ be the jump form defined as above. If, for some $x\in M$, $c>0$ and for all large enough r,*

$$V(x,r)\leq \exp(cr\log r), \tag{5}$$

then the Dirichlet form $(\mathcal{E},\mathcal{F})$ is stochastically complete.

This theorem was proved by AG, Xueping Huang, and Jun Masamune 2010 for $c<\frac{1}{2}$. Then it was observed that a minor modification of the proof works for all c. The latter was also proved slightly differently by Masamune and Uemura, 2011.

For the proof of Theorem 3 we split the jump kernel $j(x,y)$ into the sum of two parts:

$$j'(x,y)=j(x,y)\mathbf{1}_{\{d(x,y)\leq\varepsilon\}} \text{ and } j''(x,y)=j(x,y)\mathbf{1}_{\{d(x,y)>\varepsilon\}} \tag{6}$$

and show first the stochastic completeness of the Dirichlet form $(\mathcal{E}',\mathcal{F})$ associated with j'. For that we adapt the methods used for stochastic completeness for the local form.

The bounded range of j' allows to treat the Dirichlet form $(\mathcal{E}', \mathcal{F})$ as "almost" local: if f, g are two functions from $\mathcal{F}$ such that $d(\operatorname{supp} f, \operatorname{supp} g) > \varepsilon$ then $\mathcal{E}(f, g) = 0$. The condition (b) plays in the proof the same role as the condition $|\nabla d| \leq 1$ in the local case. However, the lack of locality brings up in the estimates various additional terms that have to be compensated by a stronger hypothesis of the volume growth (5).

The tail j'' can be regarded as a small perturbation of j' in the following sense: $(\mathcal{E}, \mathcal{F})$ is stochastically complete if and only if $(\mathcal{E}', \mathcal{F})$ is so. The proof is based on the fact that the integral operator with the kernel j'' is a bounded operator in $L^2(M, \mu)$, because by (b)

$$\int_M j''(x, y)\, d\mu(y) \leq C.$$

It is not yet clear if the volume growth condition (5) in Theorem 3 is sharp.

Let us briefly mention a result about uniqueness class for the heat equation associated with the jump Dirichlet form on graphs satisfying (a) and (b).

Namely, Xueping Huang proved in 2011 that, for any $b < \frac{1}{2}$ the following inequality determines a uniqueness class

$$\int_0^T \int_{B(x,R)} u^2(t,x)\, d\mu(x)\, dt \leq \exp(br \log r)\,. \tag{7}$$

What is more surprising, that for $b > 2\sqrt{2}$ this statement fails even on the graph $\mathbb{Z}$.

The optimal value of b in (7) is unknown. If $b < \frac{1}{2}$ then Huang's result can be used to obtain Theorem 3 on graphs provided the constant c in (5) is smaller than $\frac{1}{2}$. However, in general the stochastic completeness test (5) does not follows from the uniqueness class (7) in contrast to the case for diffusions. The sharpness of (5) in general is also unclear. M.Folz and X.Huang have proved independently in 2012 that on graphs the condition (5) can be replaced by

$$V(x,r) \leq \exp\left(Cr^2\right)$$

or even by the log-volume test.

3 Random walks on graphs

Let us now turn to random walks on graphs. Let (X, E) be a locally finite, infinite, connected graph, where X is the set of vertices and E is the set of edges. We assume that the graph is undirected, simple, without loops. Let μ be the counting measure on X. Define the jump kernel by $j(x, y) = 1_{\{x \sim y\}}$, where $x \sim y$ means that x, y are neighbors, that is, $(x, y) \in E$. The corresponding Dirichlet form is

$$\mathcal{E}(f) = \frac{1}{2} \sum_{\{x,y:x\sim y\}} (f(x) - f(y))^2,$$

and its generator is

$$\Delta f(x) = \sum_{y\sim x} (f(y) - f(x)).$$

The operator Δ is called *unnormalized* (or *physical)* Laplace operator on (X, E). This is to distinguish from the *normalized* or *combinatorial* Laplace operator

$$\hat{\Delta} f(x) = \frac{1}{\deg(x)} \sum_{y\sim x} (f(y) - f(x)),$$

where $\deg(x)$ is the number of neighbors of x. The normalized Laplacian $\hat{\Delta}$ is the generator of the same Dirichlet form but with respect to the degree measure $\deg(x)$.

Both Δ and $\hat{\Delta}$ generate the heat semigroups $e^{t\Delta}$ and $e^{t\hat{\Delta}}$ and, hence, associated continuous time random walks on X. It is easy to prove that $\hat{\Delta}$ is a bounded operator in $L^2(X, \deg)$, which then implies that the associated random walk is always stochastically complete. On the contrary, the random walk associated with the unnormalized Laplace operator can be stochastically incomplete.

We say that the graph (X, E) is stochastically complete if the heat semigroup $e^{t\Delta}$ is stochastically complete.

Denote by $\rho(x, y)$ the graph distance on X, that is the minimal number of edges in an edge chain connecting x and y. Let $B_\rho(x, r)$ be closed metric balls with respect to this distance ρ and set $V_\rho(x, r) = |B_\rho(x, r)|$ where $|\cdot| := \mu(\cdot)$ denotes the number of vertices in a given set.

Theorem 4 *If there is a point $x_0 \in X$ and a constant $c > 0$ such that*

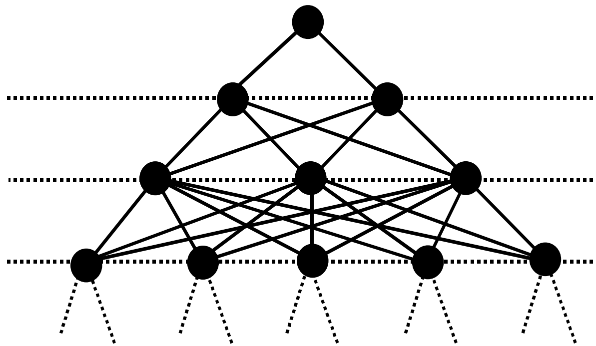
$$V_\rho(x_0, r) \leq cr^3 \log r \tag{8}$$

for all large enough r, then the graph (X, E) is stochastically complete.

Note that the function $r^3 \log r$ is sharp here in the sense that it cannot be replaced by $r^3 \log^{1+\varepsilon} r$. For any non-negative integer r, set

$$S_r = \{x \in X : \rho(x_0, x) = r\}.$$

R.Wojciechowski considered the graph where every vertex on S_r is connected to all vertices on S_{r-1} and S_r:



Anti-tree of Wojciechowski

He proved in 2008 that for such graphs the stochastic incompleteness is equivalent to the following condition:

$$\sum_{r=1}^{\infty} \frac{V_\rho(x_0, r)}{|S_{r+1}|\,|S_r|} < \infty. \tag{9}$$

Taking $|S_r| \simeq r^2 \log^{1+\varepsilon} r$ we obtain $V_\rho(x_0, r) \simeq r^3 \log^{1+\varepsilon}$ so that the condition (9) is satisfied and, hence, the graph is stochastically incomplete.

The proof of Theorem 4 is based on the following ideas. Observe first that the graph distance ρ is in general not adapted. Indeed, the integral in (b) is equal to

$$\sum_y \left(1 \wedge \rho^2 (x, y)\right) j (x, y) = \sum_y j (x, y) = \deg (x)$$

so that (b) holds if and only if the graph has uniformly bounded degree, which is not interesting as all graphs with bounded degree are automatically stochastically complete.

Let us construct an adapted distance as follows. For any edge $x \sim y$ define first its length $\sigma(x, y)$ by

$$\sigma(x,y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}}.$$

Then, for all $x, y \in X$ define $d(x, y)$ as the smallest total length of all edges in an edge path connecting x and y. It is easy to verify that d satisfies (b):

$$\sum_y \left(1 \wedge d^2(x,y)\right) j(x,y) \leq \sum_y \left(\frac{1}{\deg(x)} \wedge \frac{1}{\deg(y)}\right) j(x,y) \leq \sum_{y \sim x} \frac{1}{\deg(x)} = 1.$$

Then one proves that (8) for ρ-balls implies that the d-balls have at most quadratic exponential volume growth, so that the stochastic completeness follows by the result of Folz and Huang.

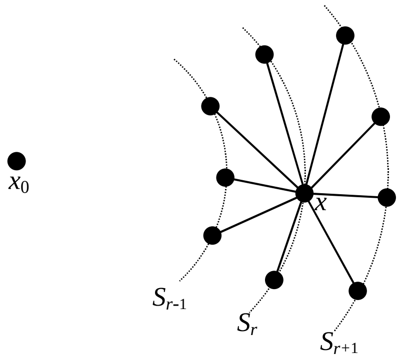
To see that, let us consider a more restrictive hypothesis

$$|S_r| \leq C r^2 \log r \quad \text{for } r >> 1. \tag{10}$$

(clearly, (10) is a bit stronger hypothesis than (8)).

Any point $x \in S_r$ admits a trivial estimate of the degree as follows:

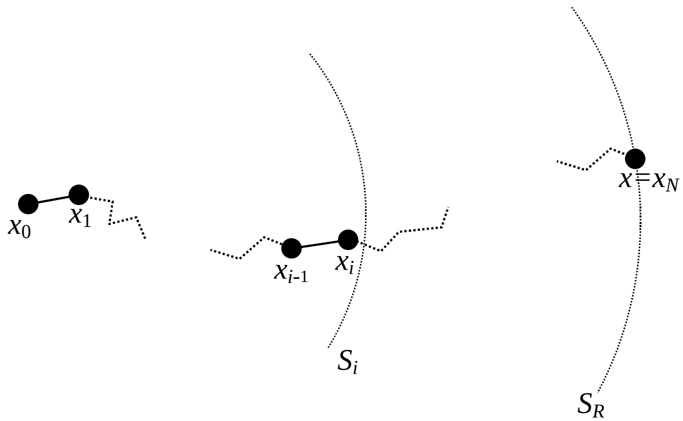
$$\deg(x) \le |S_{r-1}| + |S_r| + |S_{r+1}| \le C_1 r^2 \log r.$$



Therefore, if x, y are two neighboring vertices in $B_\rho(x_0, r)$, then

$$\sigma(x,y) = \frac{1}{\sqrt{\deg(x)}} \wedge \frac{1}{\sqrt{\deg(y)}} \ge \frac{c_1}{r\sqrt{\log r}}. \tag{11}$$

Fix a vertex $x \in S_R$ and let $\{x_i\}_{i=0}^N$ be a path connecting x_0 to x with the minimal σ-length:



Clearly, $\rho(x_0, x_i) \le i$ so that by (11) $\sigma(x_{i-1}, x_i) \ge \frac{c_1}{i\sqrt{\log i}}$ whence

$$d(x_0, x) = \sum_{i=1}^{N} \sigma(x_{i-1}, x_i) \ge c_1 \sum_{i=1}^{R} \frac{1}{i\sqrt{\log i}} \ge c_2\sqrt{\log R}.$$

Denoting by B_d the d-balls, we obtain

$$B_d(x_0, c_2\sqrt{\log R}) \subset B_\rho(x_0, R).$$

Changing variables $r = c_2\sqrt{\log R}$ and denoting by V_d the volume of B_d, we obtain using (8) that

$$V_d(x_0, r) \leq V_\rho(x_0, e^{Cr^2}) \leq \exp\left(C' r^2\right)$$

which was claimed.

In the general case (8) does not imply (10) for all $r >> 1$, but nevertheless (10) holds for sufficiently many values of r, which can be used to prove the estimates of d as above.