Iteration of polynomials, functional equations, and fractal zeta functions

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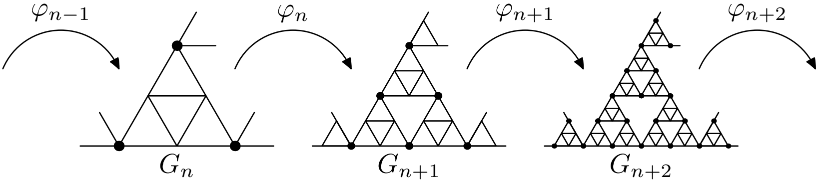
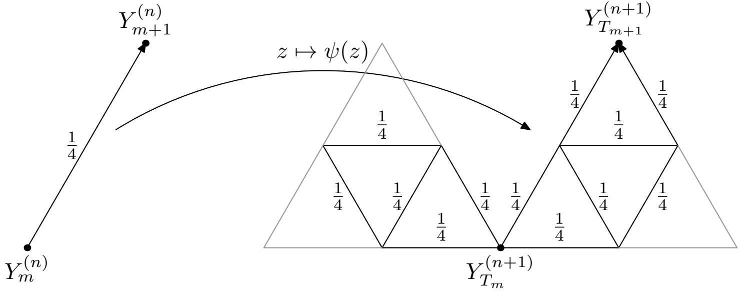
Motivation

For certain fractals, for instance the Sierpiński gasket and its higher dimensional analogues, the eigenfunctions and eigenvalues of the Laplace operator follow a "self-similar" pattern:
the fractal is approximated by a sequence of graphs $(G_n)_{n\in\mathbb{N}}$, which are connected by embeddings of the vertex sets $\varphi_n : V_n \to V_{n+1}$.



The time rescaling factor λ is the fraction between the speed of the particle in G_n and G_{n-1}.

These embeddings φ_n correspond to a rational function ψ, which relates the probability generating function of the random walk on G_n to the probability generating function of the random walk on G_{n+1}.



The time rescaling factor is given by

$$\lambda = \mathbb{E}(T_{m+1} - T_m) = \psi'(1).$$

Spectral decimation

The function ψ also relates the eigenvalues of the discrete Laplacians on G_n and G_{n+1}: every eigenvalue of Δ_{n+1} is a preimage under ψ of an eigenvalue of Δ_n.
For the Laplacian on G, i.e. the limit of the rescaled discrete Laplacians Δ_n this means that every eigenvalue of Δ can be written as

$$\lambda^m \lim_{n\to\infty} \lambda^n \psi^{-n}(z_0),$$

where z_0 is an eigenvalue of Δ_0. The multiplicities a_μ of the eigenvalues depend only on m.
More precisely, we need that the multiplicities of the eigenvalues have a rational generating function.

Poincaré functions

The equation giving the eigenvalues of the Laplacian motivates to study the solutions of the functional equation

$$\Phi(\lambda z) = p(\Phi(z)),$$

where

$$p(z) = \frac{1}{\psi(1/z)},$$

if p is a polynomial.
For instance, this happens for the Sierpiński gaskets.

Φ and the spectrum

The spectrum of the Laplacian can then be described as

$$\Phi^{(-1)}(A)$$

for a finite set A.
The value distribution of Φ therefore encodes the spectrum.

The eigenvalue counting function

$$N(x) = \sum_{\substack{\Delta u = -\mu u \\ \mu < x}} a_\mu$$

the trace of the heat kernel

$$P(t) = \sum_{-\Delta u = \mu u} a_\mu e^{-\mu t} = \int_G p_t(x,x)\, d\mathcal{H}(x),$$

as well as the spectral zeta-function

$$\zeta_\Delta(s) = \sum_{\Delta u = -\mu u} a_\mu \mu^{-s}$$

can be related to Φ.

The spectral zeta function

The spectral zeta function ζ_Δ can be given in the form

$$\zeta_\Delta(s) = \sum_{w \in A} R_w(\lambda^s) \sum_{\substack{\Phi(-\mu)=w \\ \mu \neq 0}} \mu^{-s},$$

where R_w is the rational function encoding the multiplicities of the eigenvalues.

The analytic continuation of the functions

$$\sum_{\substack{\Phi(-\mu)=w \\ \mu \neq 0}} \mu^{-s}$$

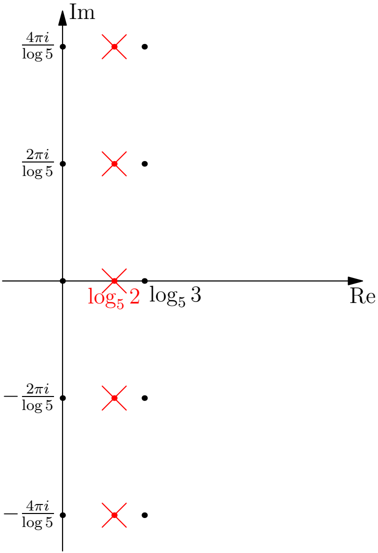
can be obtained from the asymptotic behaviour of Φ at ∞.

The poles of ζ_Δ

The functions

$$\sum_{\substack{\Phi(-\mu)=w\\ \mu\neq 0}} \mu^{-s}$$

have poles on the line $\Re s = \log_5 2$, which cancel in the sum forming ζ_Δ. This is a general phenomenon for fully symmetric fractals, as was shown recently by Steinhurst and Teplyaev.



Zero counting and the harmonic measure

The function Φ has infinitely many zeros, which come in geometric progressions with factor λ by

$$\Phi(\lambda z) = p(\Phi(z)).$$

Let

$$N_\Phi(r) = \sum_{\substack{|z|<r \\ \Phi(z)=0}} 1$$

denote the zero counting function.
Then the following are equivalent

- $\lim\limits_{r\to\infty} r^{-\rho} N_\Phi(r)$ exists
- $\lim\limits_{t\to 0} t^{-\rho} \mu(B(0,t))$ exists.

Applications

The existence of an analytic continuation of ζ_Δ to the whole complex plane allows for the definition and computation of an according Casimir energy:
Consider the operator

$$P = -\frac{\partial^2}{\partial \tau^2} - \Delta$$

on $(\mathbb{R}/\frac{1}{\beta}\mathbb{Z}) \times G$, where $\beta = 1/(kT)$.
The eigenvalues of P are then given by

$$\frac{4k^2\pi^2}{\beta^2} + \lambda_n.$$

Zeta function of P

The zeta function of P is then given by

$$\zeta_P(s) = \frac{1}{\Gamma(s)} \int_0^\infty K(t) \sum_{n\in\mathbb{Z}} e^{-\frac{4n^2\pi^2}{\beta^2}t} t^{s-1}\, dt.$$

Using the theta relation

$$\sum_{n\in\mathbb{Z}} e^{-\frac{4\pi^2 n^2}{\beta^2}t} = \frac{\beta}{2\sqrt{\pi t}} \sum_{n\in\mathbb{Z}} e^{-\frac{\beta^2 n^2}{4t}}$$

we obtain

$$\zeta_P(s) = \frac{\beta}{2\sqrt{\pi}\Gamma(s)} \Gamma\left(s - \frac{1}{2}\right) \zeta_\Delta\left(s - \frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}\Gamma(s)} \int_0^\infty K(t) \sum_{n=1}^\infty e^{-\frac{\beta^2 n^2}{4t}} t^{s-\frac{3}{2}}\, dt$$

Regularised determinant of P

The regularised determinant ("product of eigenvalues") of P is given by

$$\det(P) = \exp\left(-\zeta_P'(0)\right).$$

From the expression obtained before, we get

$$\zeta_P'(0) = -\beta\zeta_\Delta\left(-\frac{1}{2}\right) + \frac{\beta}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\int_0^\infty e^{-\frac{\beta^2 n^2}{4t}-\lambda_j t}\, t^{-\frac{3}{2}}\, dt.$$

The integral and the summation over n can be evaluated explicitly, which gives

$$\zeta_P'(0) = -\beta\zeta_\Delta\left(-\frac{1}{2}\right) - 2\sum_{j=1}^{\infty}\ln\left(1 - e^{-\beta\sqrt{\lambda_j}}\right).$$

Casimir energy

The energy of the system is then given by

$$E = -\frac{1}{2}\frac{\partial}{\partial\beta}\zeta_P'(0) = \frac{1}{2}\zeta_\Delta\left(-\frac{1}{2}\right) + \sum_{j=1}^{\infty}\frac{\sqrt{\lambda_j}}{e^{\beta\sqrt{\lambda_j}}-1}.$$

The zero point or Casimir energy of the system is then obtained by letting the temperature tend to 0, which is equivalent to letting β tend to ∞. This gives

$$E_{\mathrm{Cas}} = \frac{1}{2}\zeta_\Delta\left(-\frac{1}{2}\right).$$

Numerical computations

The very explicit procedure used for the analytic continuation of ζ_Δ allows for the numerical computation of $\zeta_\Delta(-\frac{1}{2})$ to arbitrary precision.
We computed

$$E_{\mathrm{Cas}}^{\mathrm{D}} = 0.5474693544\ldots$$

for the Casimir energy of the two-dimensional Sierpiński gasket with Dirichlet boundary conditions.

$$E_{\mathrm{Cas}}^{\mathrm{N}} = 2.134394089264\ldots$$

for Neumann boundary conditions.