Measure of Self-Affine Sets and Associated Densities

Jean-Pierre Gabardo

McMaster University
Department of Mathematics and Statistics
gabardo@mcmaster.ca

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Joint work with Xiaoye Fu

Definition

(a) An $n \times n$ real matrix B is expansive if all of its eigenvalues λ_i satisfy $|\lambda_i| > 1$.

(b) A set $\mathcal{D} = \{d_1, d_2, \ldots, d_m\} \subseteq \mathbb{R}^n$ of m distinct vectors with $0 \in \mathcal{D}$ is called a digit set.

(c) Given B and $\mathcal{D}$ as above, the self-affine set $K(B, \mathcal{D})$ is the unique compact set $K \subset \mathbb{R}^n$ satisfying the set-valued equation

$$BK = \bigcup_{i=1}^{m} (K + d_i).$$

- Note that $K \subset BK$ since $0 \in \mathcal{D}$.

- For example, if we take $n = 1$, $B = 2$, $\mathcal{D} = \{0, 1\}$, then $K = [0, 1]$.
- If we take $n = 1$, $B = 3$, $\mathcal{D} = \{0, 2\}$, we get K is the ternary Cantor set.

Definition

(a) Given B and $\mathcal{D}$ as above, we can define the maps $f_i(x) = B^{-1}(x + d_i),\ 1 \leq i \leq m, \quad x \in \mathbb{R}^n$, which define the corresponding ISF ("Iterated function system") and we have $K = \bigcup_{i=1}^{m} f_i(K)$.

(b) We say that the IFS $\{f_i\}_{i=1}^{m}$ satisfies the open set condition if there exists a non-empty bounded open set V such that

$$\bigcup_{i=1}^{m} f_i(V) \subset V \text{ and } f_i(V) \bigcap f_j(V) = \emptyset \text{ for } i \neq j.$$

Definition

Given B and $\mathcal{D}$ as above, we define for $k \geq 1$,

$$\mathcal{D}_k := \Big\{ \sum_{j=0}^{k-1} B^j d_j : d_j \in \mathcal{D}, j \geqslant 0 \Big\}, \quad \text{and} \quad \mathcal{D}_\infty := \bigcup_{k=1}^{\infty} \mathcal{D}_k.$$

- For ex., if $n = 1$, $B = 2$, $\mathcal{D} = \{0, 1\}$, $\mathcal{D}_\infty = \{0, 1, 2, 3, 4, \dots\}$.
- If $n = 1$, $B = 3$, $\mathcal{D} = \{0, 2\}$,
  $\mathcal{D}_\infty = \{0, 2, 6, 8, 4, 18, 20, 24, 26, \dots\}$

Theorem (He-Lau)

The IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition if and only if $\mathcal{D}_\infty$ is a uniformly discrete set and the m^k expansions in $\mathcal{D}_k$ are distinct for all $k \geq 1$.

Definition

If the matrix B above has the form $B = \rho R$, where $\rho > 1$ and R is an orthogonal matrix, then B is called a similarity with scaling factor ρ and the corresponding set K is called a self-similar set.

- Our main goal in this talk is to exhibit a relationship between the Lebesgue measure $|K|$ of K or a certain Hausdorff measure $\mathcal{H}^s(K)$, where $0 < s \leq n$, and an appropriate notion of density for the (discrete) measure μ defined by

$$\mu = \lim_{k\to\infty} \sum_{d_0,\ldots,d_{k-1}\in\mathcal{D}} \delta_{d_0+Bd_1+\cdots+B^{k-1}d_{k-1}},$$

- Note that $\mu = \sum_{a\in\mathcal{D}_\infty} \delta_a$ if the expansions defining $\mathcal{D}_k$ are all distinct for any $k \geq 1$.

The relationship will hold in the following two situations:

- The case where B is a general expansive matrix and $m = |\det(B)|$.
- The case where B is called a similarity with scaling factor $\rho > 1$ and $m \leq |\det(B)|$.

The case $m = |\det(B)|$ with B expansive

In this case, Lagarias and Wang proved the following result:

Theorem (Lagarias & Wang)

The following four conditions are equivalent.

(i) $K(B, \mathcal{D})$ *has positive Lebesgue measure.*

(ii) $K(B, \mathcal{D})$ *has non-empty interior.*

(iii) $K(B, \mathcal{D})$ *is the closure of its interior* K°, *and its boundary has zero Lebesgue measure.*

(iv) *For each* $k \geq 1$, *all* m^k *expansions in* $\mathcal{D}_k$ *are distinct, and* $\mathcal{D}_\infty$ *is a uniformly discrete set.*

Definition

Let μ be a Borel measure in $\mathbb{R}^n$. The upper Beurling density of the measure μ is defined by

$$D^+(\mu) = \limsup_{N\to\infty} \sup_{z\in\mathbb{R}^n} \frac{\mu(I_N(z))}{N^n},$$

and the lower Beurling density of the measure μ is defined by

$$D^-(\mu) = \liminf_{N\to\infty} \inf_{z\in\mathbb{R}^n} \frac{\mu(I_N(z))}{N^n},$$

where $I_N(z) = \left\{ y \in \mathbb{R}^n, |y_i - z_i| \leq \frac{N}{2}, i = 1, \ldots, n \right\}$.
If $D^+(\mu) = D^-(\mu)$, we say that the Beurling density of the measure μ exists and we denote it as $D(\mu)$.

- If $\Lambda \subset \mathbb{R}^n$ is a discrete set, we also define $D^+(\Lambda)$ and $D^-(\Lambda)$ as $D^+(\mu)$ and $D^-(\mu)$, where $\mu = \sum_{\lambda\in\Lambda} \delta_\lambda$ respectively.

Definition

A positive Borel measure μ on $\mathbb{R}^n$ is called translation-bounded if, for every compact set $K \subset \mathbb{R}^n$, there exists a constant $C_\mu(K) \geq 0$ such that $\mu(K+z) \leq C_\mu(K)$, $z \in \mathbb{R}^n$.

Lemma

A positive Borel measure μ on $\mathbb{R}^n$ is translation-bounded if and only if $D^+(\mu) < \infty$.

Theorem (G.)

Let $\mathcal{P}(\mathbb{R}^n) = \{f \in L^1(\mathbb{R}^n),\ f \geq 0, \int f\,dx = 1\}$ and let μ be a positive Borel measure on $\mathbb{R}^n$. Then,

$$D^+(\mu) = \inf\{C \geq 0,\ \mu * f \leq C \text{ a.e. for some } f \in \mathcal{P}(\mathbb{R}^n)\}.$$

- This last result implies in particular that if μ is a positive Borel measure μ on $\mathbb{R}^n$, if $F \geq 0$ is integrable and $\mu * F \leq C$ where $C \geq 0$, then $D^+(\mu) \int F\,dx \leq C$.

- **Theorem**

  Let $B \in M_n(\mathbb{R})$ be an expansive matrix with $|\det B| = m \in \mathbb{Z}$ and let $\mathcal{D}$ be a finite subset of $\mathbb{R}^n$ with card$(\mathcal{D}) = m$. Then, $|K(B, \mathcal{D})| = (D^+(\mu))^{-1}$, where

  $$\mu = \lim_{k\to\infty} \sum_{d_0,\ldots,d_{k-1}\in\mathcal{D}} \delta_{d_0+Bd_1+\cdots+B^{k-1}d_{k-1}},$$

  with the convention that $|K(B, \mathcal{D})| = 0$ if $D^+(\mu) = \infty$.

Idea of the proof in the case when $|K(B,\mathcal{D})| > 0$:

- We have $\mu = \sum_{\lambda\in\mathcal{D}_\infty} \delta_\lambda$ by the result of Lagarias and Wang.
- Since $B^k K = \cup_{d\in\mathcal{D}_k} K + d$, we have

$$\mu * \chi_K = \lim_{k\to\infty} \sum_{d\in\mathcal{D}_k} \chi_{K+d} = \lim_{k\to\infty} \chi_{B^k K} = \chi_{\cup_k B^k K} \le 1$$

  which implies that $D^+(\mu)\,|K| \le 1$.
- On the other hand, using that same result, K contains an open ball and thus $\cup_k B^k K$ contains balls of arbitrarily large radii since B is expansive.
- This implies that $D^+(\mu * \chi_K) \ge 1$ and thus that $D^+(\mu)\,|K| \ge 1$.
- Hence, $D^+(\mu)\,|K| = 1$.

Using the previous result as well as the results of He-Lau, Lagarias-Wang, we obtain:

Theorem

Let $B \in M_n(\mathbb{R})$ be an expansive matrix with $|\det B| = m \in \mathbb{Z}$ and let $\mathcal{D}$ be a finite subset of $\mathbb{R}^n$ with card$(\mathcal{D}) = m$.

(i) *The IFS $\{f_i\}_{i=1}^m$ satisfies the open set condition.*

(ii) *The m^k expansions in $\mathcal{D}_k$ are distinct for all $k \geq 1$ and $\mathcal{D}_\infty$ is a uniformly discrete set.*

(iii) $|K(B, \mathcal{D})| > 0$

(iv) $0 < D^+(\mu) < \infty$.

(v) μ *is translation-bounded.*

Information about the structure of K can also be extracted from the lower Beurling density of μ.

Theorem

Under the previous condition, suppose that $|K(B,\mathcal{D})| > 0$. Then, then we have the following alternative:

(a) *either K contains a neighborhood of 0 and $D^+(\mathcal{D}_\infty) = D^-(\mathcal{D}_\infty) = \frac{1}{|K|}$.*

(b) *or, K does not contain a neighborhood of 0 and $D^+(\mathcal{D}_\infty) = \frac{1}{|K|}$ and $D^-(\mathcal{D}_\infty) = 0$.*

The case $m < |\det(B)|$, B a similarity with factor $\rho > 1$.

Recall the definition of Hausdorff measure:

Definition

Let E be a subset of $\mathbb{R}^n$ and let $s \geq 0$. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf\left\{ \sum_{i=1}^{\infty} [\mathrm{diam}(U_i)]^s : E \subseteq \bigcup_{i=1}^{\infty} U_i, \mathrm{diam}(U_i) < \delta \right\}.$$

Then, the s-dimensional Hausdorff measure of E is defined by

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

- Note that this definition $\mathcal{H}^n(E) = c_n |E|$ if E is Borel, where $c_n \neq 1$ if $n \geq 2$.

- Given B a similarity with factor $\rho > 1$, consider the contractions f_i, $1 \le i \le m$. By a classical result of Hutchinson, there is a unique Borel probability measure σ supported on the set $K(B, \mathcal{D})$ satisfying

$$\int f \, d\sigma = \frac{1}{m} \sum_{i=1}^{m} \int f \circ f_i \, d\sigma, \quad f \in C_c(\mathbb{R}^n).$$

- The number $s = \log_\rho(m)$ is called the similarity dimension of the set $K(B, \mathcal{D})$.

Theorem (Falconer)

Suppose that the open set condition holds for the similarities f_i, $1 \le i \le m$ on $\mathbb{R}^n$ with ratio $\rho > 1$. Then the Hausdorff dimension of $K(B, \mathcal{D})$ is given by the formula $s := \log_\rho(m)$. Moreover, for this value of s, the corresponding Hausdorff measure of $K(B, \mathcal{D})$ is positive and finite, i.e. $0 < \mathcal{H}^s(K) < \infty$.

- Falconer proved that the probablity measure σ in the result of Hutchinson's is the restriction of $\mathcal{H}^s$ to K normalized so as to give $\sigma(K) = 1$.

- **Definition**

If μ is a positive Borel measure on $\mathbb{R}^n$, we define the upper s-density of μ to be the quantity

$$\mathcal{E}_s^+(\mu) = \limsup_{r\to\infty} \sup_{\mathrm{diam}(U)\geq r>0} \frac{\mu(U)}{[\mathrm{diam}(U)]^s},$$

where the supremum is over all compact convex sets U with $\mathrm{diam}(U) \geq r > 0$.

Lemma

Let μ be a positive Borel measure on $\mathbb{R}^n$ and σ be a Borel probability measure. Then, $\mathcal{E}_s^+(\mu * \sigma) = \mathcal{E}_s^+(\mu)$.

Definition

(a) A subset $E \subset \mathbb{R}^n$ is called an s-set ($0 \leq s \leq n$) if E is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$.

(b) If E is an s-set E and $x \in \mathbb{R}^n$, we define the upper convex density of E at x, to be the quantity

$$D_c^s(E, x) = \overline{\lim_{r \to 0}} \sup_{0 < \text{diam}(U) \leq r} \frac{\mathcal{H}^s(E \bigcap U)}{[\text{diam}(U)]^s},$$

where the supremum is over all convex sets U with $x \in U$ and $0 < \text{diam}(U) \leq r$.

Theorem (Falconer)

If E is an s-set in $\mathbb{R}^n$, then $D_c^s(E, x) = 1$ at $\mathcal{H}^s$-almost all $x \in E$ and $D_c^s(E, x) = 0$ at $\mathcal{H}^s$-almost all $x \in E^c$.

Corollary

Let K be a self-similar set and contractions f_i, $1 \le i \le m$ satisfy the open set condition. Then

$$\overline{\lim_{r\to 0}} \sup_{0<diam(U)\le r} \frac{\sigma(U)}{[diam(U)]^s} = (\mathcal{H}^s(K))^{-1},$$

where s is the Hausdorff dimension of the set K, σ is the Hutchinson probability measure and the supremum is taken over all convex sets U with $U \bigcap K \ne \emptyset$ and $0 < diam(U) \le r$.

Lemma

Let σ and K be as above.and define

$$\mu_N = \sum_{d_0,\ldots,d_{N-1}\in\mathcal{D}} \delta_{d_0+Bd_1+\cdots+B^{N-1}d_{N-1}}.$$

Then, for any Borel measurable set $W \subset \mathbb{R}^n$, we have $\sigma(B^{-N}W) = \frac{1}{m^N}\mu_N * \sigma(W)$.

Theorem

Let K be a self-similar set and $s := \log_\rho(m) \leq n$ be the similarity dimension of K. Then

$$\mathcal{H}^s(K) = (\mathcal{E}_s^+(\mu))^{-1},$$

where $\mu = \lim_{N\to\infty} \mu_N$, with the convention that $\mathcal{E}_s^+(\mu) = \infty$ if $\mathcal{H}^s(K) = 0$.

Corollary

Under the same conditions, we have

$$D^+(\mu) = \infty \iff \mathcal{E}_s^+(\mu) = \infty.$$