Modified singular value functions and self-affine carpets

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Self-affine sets

Given an iterated function system (IFS) consisting of contracting affine maps, $\{A_i + t_i\}_{i=1}^m$, where the A_i are linear contractions and the t_i are translation vectors, it is well-known that there exists a unique non-empty compact set F satisfying

$$F = \bigcup_{i=1}^{m} S_i(F)$$

which is termed the *self-affine* attractor of the IFS.

Self-affine sets

The singular value function

The *singular values* of a linear map, $A : \mathbb{R}^n \to \mathbb{R}^n$, are the positive square roots of the eigenvalues of $A^T A$. For $s \in [0, n]$ define the *singular value function* $\phi^s(A)$ by

$$\phi^s(A) = \alpha_1 \alpha_2 \dots \alpha_{\lceil s \rceil - 1} \alpha_{\lceil s \rceil}^{s - \lceil s \rceil + 1}$$

where $\alpha_1 \geqslant \dots \geqslant \alpha_n$ are the singular values of A.

Returning to our IFS, let $\mathcal{I}^k$ denote the set of all sequences $(i_1, \dots, i_k)$, where each $i_j \in \{1, \dots, m\}$, and let $d(A_1, \dots, A_m) = s$ be the solution of

$$\lim_{k \to \infty} \left(\sum_{\mathcal{I}^k} \phi^s(A_{i_1} \circ \dots \circ A_{i_k}) \right)^{1/k} = 1.$$

This number is called the *affinity dimension* of the attractor, F.

Falconer's theorem

Theorem

Let $A_1, \dots, A_m$ be contracting linear self-maps on $\mathbb{R}^n$ with Lipschitz constants strictly less than $1/2$. Then, for $\left(\prod_{i=1}^m \mathcal{L}^n\right)$-almost all $(t_1, \dots, t_m) \in \times_{i=1}^m \mathbb{R}^n$, the unique non-empty compact set F satisfying

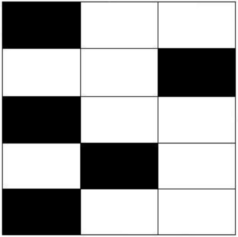
$$F = \bigcup_{i=1}^{m} (A_i + t_i)(F)$$

has

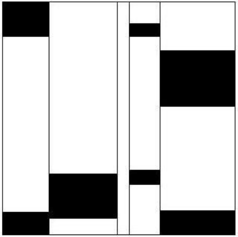
$$\dim_{\mathrm{B}} F = \dim_{\mathrm{P}} F = \dim_{\mathrm{H}} F = \min\left\{n, d\left(A_1, \dots, A_m\right)\right\}.$$

In fact, the initial proof required that the Lipschitz constants be strictly less than $1/3$ but this was relaxed to $1/2$ by Solomyak who also observed that $1/2$ is the optimal constant.

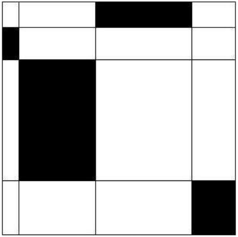
Exceptional constructions



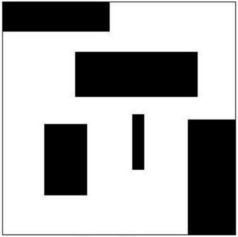
Bedford-McMullen



Gatzouras-Lalley

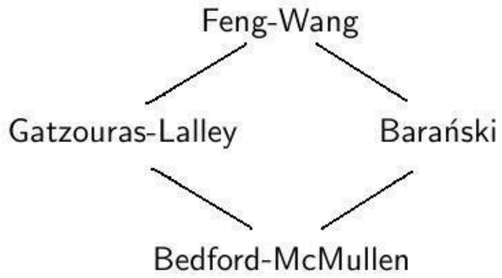


Barański



Feng-Wang

Exceptional constructions



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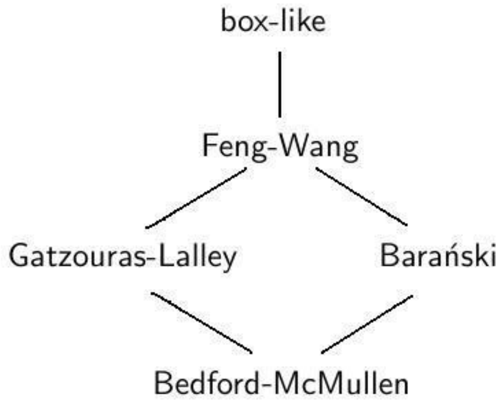

Box-like self-affine sets

We call a self-affine set *box-like* if it is the attractor of an IFS consisting of contracting affine self-maps on $[0,1]^2$, each of which maps $[0,1]^2$ to a rectangle with sides parallel to the axes.

The affine maps which make up such an IFS are necessarily of the form $S = T \circ L + t$, where T is a contracting linear map of the form

$$T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for some $a, b \in (0,1)$; L is an isometry of $[0,1]^2$ (i.e., a member of D_4); and $t \in \mathbb{R}^2$ is a translation vector.

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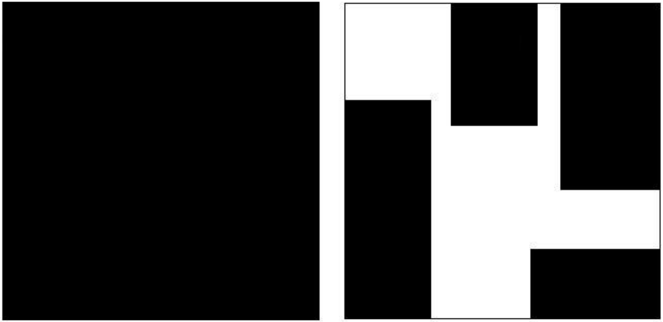

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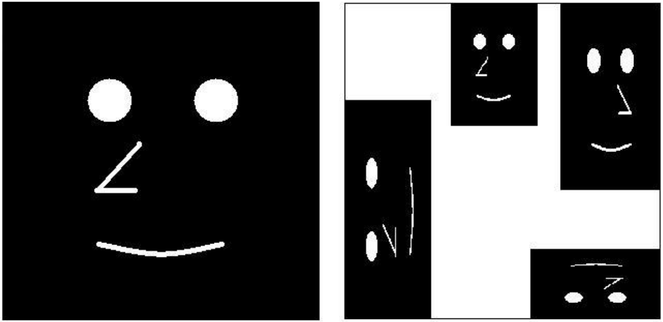

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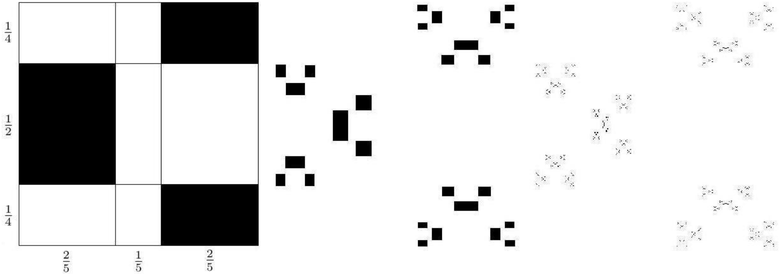

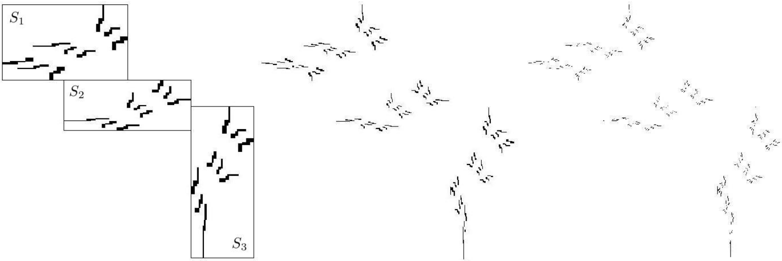

Box-like self-affine sets

Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

(1) $\pi_1(F)$ and $\pi_2(F)$ are either self-similar sets or they are a pair of graph-directed self-similar sets. This shows that the box dimensions of $\pi_1(F)$ and $\pi_2(F)$ always exist and are equal in the graph-directed case.

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(3) Compositions of maps in our IFS also map $[0, 1]^2$ to a rectangle, and the singular values are just the lengths of the sides of the rectangle.

Modified singular value functions

For $\mathbf{i} \in \mathcal{I}^k$ let $s(\mathbf{i})$ be the box dimension of the projection of $S_{\mathbf{i}}(F)$ onto the longest side of the rectangle $S_{\mathbf{i}}\big([0,1]^2\big)$ and note that this is always equal to either s_1 or s_2.

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For $s \geqslant 0$ and $\mathbf{i} \in \mathcal{I}^*$, we define the *modified singular value function*, ψ^s, of $S_{\mathbf{i}}$ by

$$\psi^s\big(S_{\mathbf{i}}\big) = \alpha_1(\mathbf{i})^{s(\mathbf{i})}\ \alpha_2(\mathbf{i})^{s-s(\mathbf{i})},$$

and for $s \geqslant 0$ and $k \in \mathbb{N}$, we define a number Ψ_k^s by

$$\Psi_k^s = \sum_{\mathbf{i} \in \mathcal{I}^k} \psi^s(S_{\mathbf{i}})$$

Properites of ψ^s and Ψ_k^s

For $s \geqslant 0$ and $\mathbf{i}, \mathbf{j} \in \mathcal{I}^*$ we have

(1) If $s < s_1 + s_2$, then $\psi^s(S_{\mathbf{i}} \circ S_{\mathbf{j}}) \leqslant \psi^s(S_{\mathbf{i}})\,\psi^s(S_{\mathbf{j}})$

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For $s \geqslant 0$ and $k, l \in \mathbb{N}$ we have

(4) If $s < s_1 + s_2$, then $\Psi_{k+l}^s \leqslant \Psi_k^s\,\Psi_l^s$

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Properites of ψ^s and Ψ^s_k

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It follows by standard properties of sub- and super-multiplicative sequences that we may define a function $P : [0, \infty) \to [0, \infty)$ by:

$$P(s) = \lim_{k \to \infty} (\Psi^s_k)^{1/k}$$

Properties of our ‘pressure’ function P

P is the exponential of the function

$$P^*(s) = \lim_{k \to \infty} \tfrac{1}{k} \log \Psi_k^s$$

which one might call the *topological pressure* of the system.

(1) For all $s, t \geqslant 0$ we have

$$\alpha_{\min}^s P(t) \leqslant P(s+t) \leqslant \alpha_{\max}^s P(t)$$

(2) P is continuous and strictly decreasing on $[0, \infty)$

(3) There is a unique value $s \geqslant 0$ for which $P(s) = 1$

Definition

An IFS $\{S_i\}_{i=1}^m$ satisfies the rectangular open set condition (ROSC) if there exists a non-empty open rectangle, $R = (a, b) \times (c, d) \subset \mathbb{R}^2$, such that $\{S_i(R)\}_{i=1}^m$ are pairwise disjoint subsets of R.

Dimension result

Definition

An IFS $\{S_i\}_{i=1}^m$ *satisfies the rectangular open set condition (ROSC) if there exists a non-empty open rectangle,* $R = (a, b) \times (c, d) \subset \mathbb{R}^2$*, such that* $\{S_i(R)\}_{i=1}^m$ *are pairwise disjoint subsets of* R*.*

Theorem

Let F *be a box-like self-affine set. Then* $\dim_P F = \overline{\dim}_{\mathrm{B}} F \leqslant s$ *where* $s \geqslant 0$ *is the unique solution of* $P(s) = 1$*. Furthermore, if the ROSC is satisfied, then* $\dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = s$*.*

Some discussion

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(4) Hausdorff dimension for box-like sets. In the Gatzouras-Lalley and Barański cases, the Hausdorff dimension is equal to the supremum of the Hausdorff dimensions of the Bernoulli measures supported on the attractor. Perhaps the same is true for box-like sets? Or perhaps one can compute the Hausdorff dimension via a function based on singular values analogous to our P?

Thank you!