On the dynamics of strongly tridiagonal competitive-cooperative system

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Setting

Consider the nonautonomous tridiagonal system

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, x_2), \\ \dot{x}_i &= f_i(t, x_{i-1}, x_i, x_{i+1}), \quad 2 \leq i \leq n-1 \\ \dot{x}_n &= f_n(t, x_{n-1}, x_n)\end{aligned} \tag{1}$$

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(A2) There are $\varepsilon_0 > 0$ such that for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$,

$$\frac{\partial f_i}{\partial x_{i+1}}(t, x) \ge \varepsilon_0, \ \frac{\partial f_{i+1}}{\partial x_i}(t, x) \ge \varepsilon_0 \quad 1 \le i \le n-1.$$

for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$.

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$H(f) = \overline{\{g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \mid \exists \tau \in \mathbb{R},\ \text{s.t. } g(t, x) = f(\tau + t, x)\}}$

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For convenience, we assume the hull $H(f)$ is minimal under the shift action defined by $f \cdot \tau \triangleq f(\tau + \cdot, \cdot)$. It is reasonable to characterize the structure of $\omega(x_0, f)$ in terms of $H(f)$.

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1. Biologically, it describes the relationship between the environment and the variety of number of species;
2. Mathematically, it covert the complexity of $\omega(x_0, f)$ to the one of $H(f)$.

(1) can generates a skew-product semiflow π on $\mathbb{R}^n \times H(f)$, by

$$\pi(t, x_0, g) \triangleq (x_0, g) \cdot t \triangleq (x(t, x_0, g), g \cdot t), \tag{2}$$

where $x(t, x_0, g)$ is the solution of initial value problem

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An omega limit set of system (1) corresponds to an omega limit set of system (2) (see George R. Sell [1]).

Hyperbolic Case

Theorem (F.-Gyllenberg-Wang)

Let $\pi(t, x_0, g_0)$ *be a positively bounded motion of system* (2)*. If* $\omega(x_0, g_0)$ *is hyperbolic, then* $\omega(x_0, g_0)$ *is 1-cover of* $H(f)$*.*

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Definition: A set $Y \subset \mathbb{R}^n \times H(f)$ is said to be an *1-cover* of $H(f)$ if $\#(Y \cap P^{-1}(g)) = 1$ for all $g \in H(f)$, where $P : \mathbb{R}^n \times H(f) \to H(f), P(x, g) = g$ is the natural projection.

Hyperbolic Case

Definition: Let $Y \subset \mathbb{R}^n \times H(f)$ be a compact invariant set of (2). For each $y = (x, g) \in Y$, the linearized equation of (2) along $y \cdot t = (x, g) \cdot t$ reads:

$$\dot{x} = A(y \cdot t)x. \tag{3}$$

Y is *hyperbolic* if the system (3) admits an *exponential dichotomy* over Y, i.e. there is a projector $Q : \mathbb{R}^n \times H(f) \to \mathbb{R}^n \times H(f)$ and positive constants K and α such that

(i) $\Phi(t, y)Q(y) = Q(y \cdot t)\Phi(t, y), \quad t \in \mathbb{R},$

(ii) $|\Phi(t, y)(1 - Q(y))| \leq Ke^{-\alpha t}, \quad t \in \mathbb{R}^+,$
$|\Phi(t, y)Q(y)| \leq Ke^{\alpha t}, \quad t \in \mathbb{R}^-,$

for all $y \in H(f)$.

Hyperbolic Case - Perturbation Theory

Consider a perturbed system of (1)

$$\dot{x} = f(t, x) + h(t, x).$$

Assume the perturbation item h and its Jacobi matrix with respect to x are uniformly continuous and there exist $0 < \delta < 1$ such that

$$|h(t, x)| < \delta, \quad |\partial h(t, x)/\partial x| < \delta \tag{4}$$

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Theorem (F.-Gyllenberg-Wang)

Suppose the skew-product flow (2) generated by $\dot{x} = f(t, x)$ admits a hyperbolic ω-limit set $\omega(x_0, f)$. Then there exist a C^1 neighborhood $\mathcal{F}$ of f in the sense of (4) and a neighborhood U of $\omega(x_0, f)$ such that for any $g \in \mathcal{F}$, there exist an ω-limit set $\omega(x_0', g) \subset \mathbb{R}^n \times H(g) \cap U$, moreover, it is 1-cover of $H(g)$.

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Lemma (F.-Gyllenberg-Wang)

For any π-invariant set $Y \subset \mathbb{R}^n \times H(f)$ of (2), the linearized system (3): $\dot{x} = A(y \cdot t)x$ admits an $(1, \cdots, 1)$-dominated splitting.

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Definition: System (3) is said to admit a (n_1, n_2)-*dominated splitting* if there exists π-invariant splitting $X_1(Y) \oplus X_2(Y)$ of $\mathbb{R}^n \times Y$ such that there exist positive numbers K and ν satisfies

$$\frac{|\Phi(t, y)x_2|}{|\Phi(t, y)x_1|} \leq Ke^{-\nu t}, \quad t \geq 0,$$

for all $y \in Y$ and $x_1 \in X_1(y)$, $x_2 \in X_2(y)$ with $|x_1| = |x_2| = 1$.

General Case

Next theorem shows that after a suitable functional distortion, dominated splitting becomes to hyperbolicity.

Theorem (F.-Gyllenberg-Liu)

Let $f : M \to M$ be a diffeomorphism on a closed manifold M and $\Lambda \subset M$ be any compact f-invariant set. A splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ of tangent bundle over Λ is $(n_1, \cdots, n_k)$-dominated if and only if there exist continuous real functions $p_i : \Lambda \to \mathbb{R}^+$, $i = 1, \cdots, k$, with $\log p_1, \cdots, \log p_k$ are summably separated with respect to f, such that for $i = 1, \cdots, k$ the linear cocycle $(f, p_i Df)$ admits a hyperbolicity over Λ with stable subspace of dimension $n_1 + \cdots + n_i$.

General Case - Central Dimension One

For any given $\lambda \in \mathbb{R}$, consider

$$\dot{x} = (A(y \cdot t) - \lambda \mathrm{Id})x, \quad y \in Y, \tag{5}$$

Definition: $\sum(Y) = \{\lambda \in \mathbb{R}^1 \mid (5) \text{ has no exponential dichotomy on } Y\}$ is called the *Sacker-Sell spectrum of* (3).

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Remark

1. $\sum(Y) = \cup_{i=1}^{k} I_i$, where $I_i = [a_i, b_i]$ and $\{I_i\}$ is ordered from right to left. Denote the invariant subbundle associated with I_i is $X_i(Y)$, then $X_1(Y) \oplus \cdots \oplus X_k(Y) = \mathbb{R}^n \times Y$.
2. If Y is hyperbolic, $0 \notin \sum(Y)$.

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Suppose $Y \subset \mathbb{R}^n \times H(f)$ is a minimal set and is of central dimension one. Then the flow $(Y, \cdot)$ is topologically conjugated to a scalar skew-product subflow of $(\mathbb{R}^1 \times H(f), \cdot)$.

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Corollary

Suppose f is almost periodic in t and let $Y \subset \mathbb{R}^n \times H(f)$ be a minimal set and unique ergodic. Then $(Y, \cdot)$ is topologically conjugated to a scalar skew-product subflow of $(\mathbb{R}^1 \times H(f), \cdot)$.

Future Research

Question

How about the case that the central dimension is bigger than one, for example two?

Thank you for your attention!

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