An Analytic Inequality and Higher Multifractal Moments

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A General Approach

Calculations in Fractal Geometry often fall into two parts: a geometric part and an analytic part.

The geometric part may involve expressing geometric or metric aspects of a problem in mathematical terms.

The analytic part may involve estimating the integrals, sums, etc. so obtained.

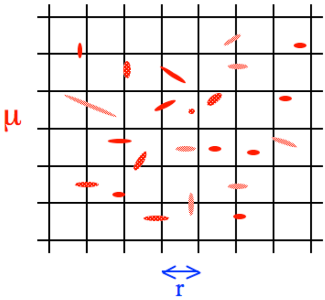
For the analytic part, there are methods may be applied to a range of apparently different fractal geometric problems - e.g. covering or potential theoretic methods for estimating dimensions.

We will look at an analytic technique which extends the potential theoretic method and give several applications.

Moment sums and L^q-dimensions

Let $\mathcal{M}_r$ be the mesh of side r. Define the q-th power moment sum of a measure μ on $\mathbb{R}^n$ by

$$M_r(q) = \sum_{C \in \mathcal{M}_r} \mu(C)^q. \quad (1)$$



Then the L^q-dimension or generalised q dimension of μ is given by

$$D_q(\mu) = \frac{1}{q-1} \lim_{r \to 0} \frac{\log M_r(q)}{\log r} \qquad (q > 0).$$

(or $\liminf, \limsup$). Equivalently we may replace (1) by a moment integral

$$M_r(q) = \int \mu(B(x,r))^{q-1} d\mu(x) \qquad (q > 0).$$

Images of measures

Now let x_ω : Metric space $\to \mathbb{R}^n$ for a parameterised family of mappings x_ω $(\omega \in \Omega)$ [e.g. projections, random functions, etc.]

Let μ be a measure on the Metric Space and let μ_ω be its image measure on $\mathbb{R}^n$ under x_ω, i.e.

$$\mu_\omega(A) = \mu(x_\omega^{-1}(A)) \quad \text{or} \quad \int f(x) d\mu_\omega(x) = \int f(x_\omega(t)) d\mu(t).$$

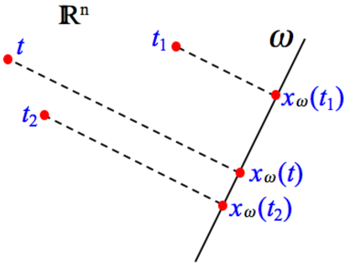
One way to get lower estimates for L^q-dimensions of μ_ω for a.a. ω is to bound the average moment integrals over ω. For $q \geq 2$ an integer:

$$\begin{aligned}
&\mathbb{E} \int \mu_\omega(B(x,r))^{q-1} d\mu_\omega(x) \\
&= \mathbb{E} \int \mu_\omega\{y_1 : |x - y_1| \leq r\} \ldots \mu_\omega\{y_{q-1} : |x - y_{q-1}| \leq r\} d\mu_\omega(x) \\
&= \mathbb{E} \int \mu\{t_1 : |x_\omega(t) - x_\omega(t_1)| \leq r\} \ldots \mu\{t_{q-1} : |x_\omega(t) - x_\omega(t_{q-1})| \leq r\} d\mu(t) \\
&= \int \cdots \int \mathbb{P}\{|x_\omega(t) - x_\omega(t_j)| \leq r \text{ for all } j\} d\mu(t_1) \ldots d\mu(t_{q-1}) d\mu(t).
\end{aligned}$$

$$\mathbb{E}\int \mu_\omega(B(x,r))^{q-1} d\mu_\omega(x)$$

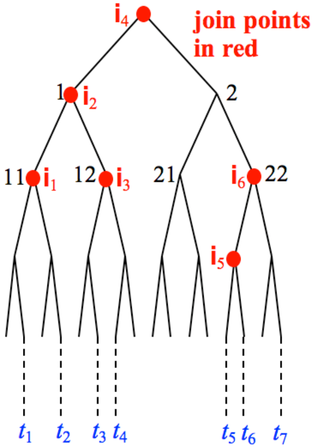
$$= \int \cdots \int \mathbb{P}\{|x_\omega(t) - x_\omega(t_j)| \leq r \text{ for all } j\} d\mu(t_1)\ldots d\mu(t_{q-1}) d\mu(t) \quad (\ddagger)$$

Typically
$\mathbb{P}\{|x_\omega(t) - x_\omega(t_j)| \leq r$ for all $j\}$ depends on the relative closeness of $t_1, \ldots, t_{q-1}, t$ to each other in the metric space.



e.g. $x_\omega : \mathbb{R}^n \to \mathbb{R}$ might be projection onto a line parameterised by ω.

We may use the geometry of the situation to estimate $\mathbb{P}\{|x_\omega(t) - x_\omega(t_j)| \leq r$ for all $j\}$ and then use analytic methods to estimate the resulting integral ($\ddagger$).
In particular, bounding ($\ddagger$) by const.$r^{s(q-1)}$ will give an a.s lower bound of s for the L^q-dimensions of μ_ω.



We can often regard $t_1, t_2, \dots, t_q$ as points on an ultrametric space, say as points of $\{1,2\}^{\mathbb{N}}$, which we can identify with a binary tree.

Let $\mathbf{i}_1, \dots, \mathbf{i}_{q-1}$ be the $q-1$ join points of $t_1, \dots, t_q$.

A generalised transversality argument may lead to an estimate

$$\mathbb{P}\{|x_\omega(t_q) - x_\omega(t_j)| \le r \text{ for all } j\} \le F(t_1, t_2, \dots, t_q)$$

where F is a product over the join points

$$F(t_1, t_2, \dots, t_q) = f(\mathbf{i}_1) f(\mathbf{i}_2) \dots f(\mathbf{i}_{q-1})$$

for some f : vertices of the tree $\to \mathbb{R}^+$

So (‡) becomes

$$\mathbb{E} \int \mu_\omega(B(x,r))^{q-1} d\mu_\omega(x) \le \int \cdots \int F(t_1, t_2, \dots, t_q) d\mu(t_1) \dots d\mu(t_{q-1}) d\mu(t_q).$$

Estimation of the integrals

Special case: $q = 3$

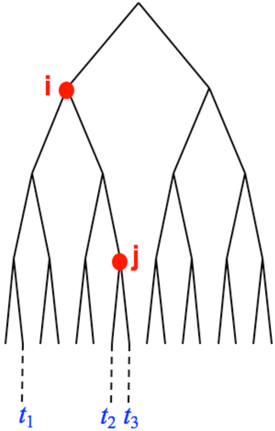
$$\iiint F(t_1, t_2, t_3)d\mu(t_1)d\mu(t_2)d\mu(t_3) \leq \left(\sum_{k=0}^{\infty} \left[\sum_{|i|=k} f(\mathbf{i})^2 \mu(C_{\mathbf{i}})^3 \right]^{1/2} \right)^2$$

Sketch of proof:

Splitting this integral into a sum over possible pairs of join points:

$$\iiint F(t_1, t_2, t_3)d\mu(t_1)d\mu(t_2)d\mu(t_3)$$
$$\leq \sum_{\mathbf{i}\in T} \sum_{\mathbf{j}\in T, \mathbf{j}\succ\mathbf{i}} f(\mathbf{i})f(\mathbf{j})\mu(C_{\mathbf{i}})\mu(C_{\mathbf{j}})^2$$

where $C_{\mathbf{i}}$ denotes the cylinder consisting of points with address starting with $\mathbf{i}$.
We first estimate this sum over vertices T of the tree at levels $|\mathbf{i}| = k$ and $|\mathbf{j}| = l > k$.



$$
\begin{aligned}
&\sum_{|\mathbf{i}|=k}\ \sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}} f(\mathbf{i})f(\mathbf{j})\mu(C_{\mathbf{i}})\mu(C_{\mathbf{j}})^2\\
&\quad\le \sum_{|\mathbf{i}|=k}\Big[f(\mathbf{i})\mu(C_{\mathbf{i}})\Big]\Big[\sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}}\big(f(\mathbf{j})\mu(C_{\mathbf{j}})^{3/2}\big)\mu(C_{\mathbf{j}})^{1/2}\Big]\\
&\quad\le \sum_{|\mathbf{i}|=k}\Big[f(\mathbf{i})\mu(C_{\mathbf{i}})\Big]\Big[\Big(\sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}} f(\mathbf{j})^2\mu(C_{\mathbf{j}})^3\Big)^{1/2}\Big(\sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}}\mu(C_{\mathbf{j}})\Big)^{1/2}\Big] \quad \text{(C-S)}\\
&\quad\le \sum_{|\mathbf{i}|=k}\Big[f(\mathbf{i})\mu(C_{\mathbf{i}})^{3/2}\Big]\Big[\sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}} f(\mathbf{j})^2\mu(C_{\mathbf{j}})^3\Big]^{1/2}\\
&\quad\le \Big[\sum_{|\mathbf{i}|=k} f(\mathbf{i})^2\mu(C_{\mathbf{i}})^3\Big]^{1/2}\Big[\sum_{|\mathbf{i}|=k}\ \sum_{|\mathbf{j}|=l,\,\mathbf{j}\succ\mathbf{i}} f(\mathbf{j})^2\mu(C_{\mathbf{j}})^3\Big]^{1/2} \quad \text{(C-S)}\\
&\quad\le \Big[\sum_{|\mathbf{i}|=k} f(\mathbf{i})^2\mu(C_{\mathbf{i}})^3\Big]^{1/2}\Big[\sum_{|\mathbf{j}|=l} f(\mathbf{j})^2\mu(C_{\mathbf{j}})^3\Big]^{1/2}
\end{aligned}
$$

Summing over levels $k \geq 0, l > k$ gives the desired inequality.

Thus

$$\iiint F(t_1,t_2,t_3)d\mu(t_1)d\mu(t_2)d\mu(t_3) \leq \left(\sum_{k=0}^{\infty}\Big[\sum_{|\mathbf{i}|=k} f(\mathbf{i})^2\mu(C_{\mathbf{i}})^3\Big]^{1/2}\right)^2.$$

More generally, for integers $q \geq 2$,

$$\int\cdot\cdot\int F(t_1,\ldots,t_q)d\mu(t_1)\ldots d\mu(t_q) \leq \left(\sum_{k=0}^{\infty} p(k)\Big[\sum_{|\mathbf{i}|=k} f(\mathbf{i})^{q-1}\mu(C_{\mathbf{i}})^q\Big]^{\frac{1}{q-1}}\right)^{q-1} \quad (\star)$$

where p is a polynomial.

Notes:

- when $q = 2$ this is close to the usual potential theoretic estimate
- the tree can be m-ary rather than just binary
- such estimates can be extended to non-integral $q > 1$.

In applications $f(\mathbf{i}) \equiv f_s(\mathbf{i})$ typically depends on a parameter s such that

$$\sum_{|\mathbf{i}|=k} f_s(\mathbf{i})^{q-1}\mu(C_{\mathbf{i}})^q \asymp (\lambda_s)^k$$

where $\lambda_s > 0$. Then:

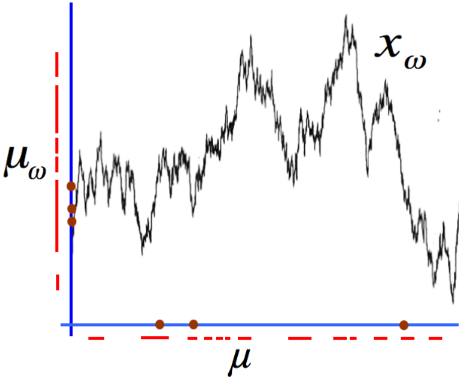
$$\mathbb{E}\int \mu_\omega(B(x,r))^{q-1}d\mu_\omega(x) \leq c\left(\sum_{k=0}^{\infty} p(k)(\lambda_s)^{k/(q-1)}\right)^{q-1}.$$

The value of s such that $\lambda_s = 1$ is critical for convergence.

Images of measures under Gaussian processes

Let $\{x_\omega : [0,1] \to \mathbb{R}, \omega \in \Omega\}$ be index-α fractional Brownian motion on a probability space Ω. Let μ be a (finite) measure on $[0,1]$ and let μ_ω be the measure induced by x_ω on $\mathbb{R}$.

What is the relationship between the L^q-dimensions $D_q(\mu_\omega)$ and $D_q(\mu)$ (assumed to exist)?



Theorem (with Yimin Xiao)

For $q > 1$,

$$D_q(\mu_\omega) = \min\left\{1, \frac{D_q(\mu)}{\alpha}\right\}$$

almost surely, where α is the index of the fractional Brownian motion x_ω.

Proof '$\leq$': Follows since index-α fBm is a.s. $\alpha - \epsilon$ Hölder.

'$\geq$': Using local non-determinism of fBm (roughly that the variance of $x_\omega(t_1)$ conditional on $x_\omega(t_2), \ldots, x_\omega(t_q)$ is controlled by the variance of $x_\omega(t_1) - x_\omega(t_j)$ such that $|t_1 - t_j|$ is least) we get

$$\mathbb{E} \int \mu_\omega(B(x,r))^{q-1} d\mu_\omega(x)$$
$$\leq c r^{s(q-1)} \int \cdots \int m^{-|\mathbf{i}_1|\alpha s} m^{-|\mathbf{i}_m|\alpha s} \cdots m^{-|\mathbf{i}_{q-1}|\alpha s} d\mu(t_1) \ldots d\mu(t_q)$$

where Euclidean distance on $[0,1]$ has been replaced by an m-ary ultrametric $d(t_1, t_2) = m^{-|t_1 \wedge t_2|}$ and $\mathbf{i}_1, \ldots, \mathbf{i}_{q-1}$ are the $q-1$ join points of $t_1, \ldots, t_q$. Taking $f(\mathbf{i}) = m^{-|\mathbf{i}|\alpha s}$ in $(\star)$,

$$\leq c r^{s(q-1)} \Big(\sum_{k=0}^{\infty} p(k) \Big[\sum_{|\mathbf{i}|=k} \lambda_{s,k} \Big]^{1/(q-1)} \Big)^{q-1} \qquad (\star\star)$$

where

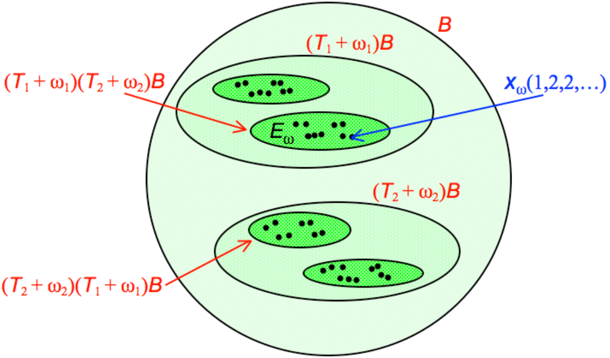
$$\lambda_{s,k} \equiv \sum_{|\mathbf{i}|=k} f(\mathbf{i})^{q-1} \mu(C_\mathbf{i})^q = m^{-|\mathbf{i}|\alpha s(q-1)} \sum_{|\mathbf{i}|=k} \mu(C_\mathbf{i})^q.$$

The sum in $(\star\star)$ is finite if $\limsup_{k\to\infty} \lambda_{s,k} < 1$, that is if $\alpha s > D_q(\mu)$.$\square$

Measures on almost self-affine sets

For $j = 1, \ldots, m$ let T_j be linear contractions on $\mathbb{R}^n$ and let ω_j be translation vectors. The iterated function system $\{T_j(x) + \omega_j\}$ has an attractor E satisfying $E = \cup_{j=1}^m (T_j(E) + \omega_j)$ which is a self-affine set. The attractor E may be characterised in terms of m-ary sequences: $E_\omega = \bigcup_t x_\omega(t)$ where $x_\omega : \{1, \ldots, m\}^{\mathbb{N}} \to \mathbb{R}^n$ is given by

$$x_\omega(t) \equiv x_\omega(t_1, t_2, \ldots) = \bigcap_{k=1}^{\infty} (T_{t_1} + \omega_{t_1})(T_{t_2} + \omega_{t_2}) \cdots (T_{t_k} + \omega_{t_k})(B)$$

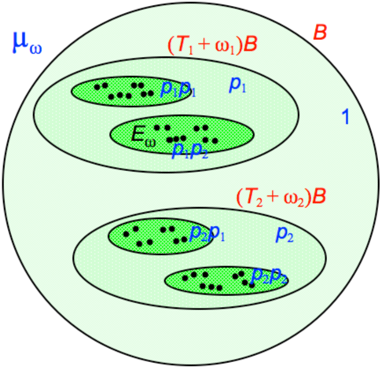


Let $p_1, \ldots, p_m$ be probabilities (so $0 < p_i < 1$ and $\sum p_i = 1$). Let μ be the Bernoulli probability measure on $\{1, \ldots, m\}^{\mathbb{N}}$ defined by

$$\mu(C_{\mathbf{i}}) = p_{i_1} p_{i_2} \ldots p_{i_k}$$

where $\mathbf{i} = (i_1, \ldots, i_k)$ and $C_{\mathbf{i}}$ is the corresponding cylinder.

Let μ_ω be the image measure of μ under x_ω, which is supported by E_ω.



Thus $\mu_\omega\big((T_{t_1} + \omega_{t_1}) \cdots (T_{t_k} + \omega_{t_k})(B)\big) = p_{i_1} p_{i_2} \ldots p_{i_k}$.

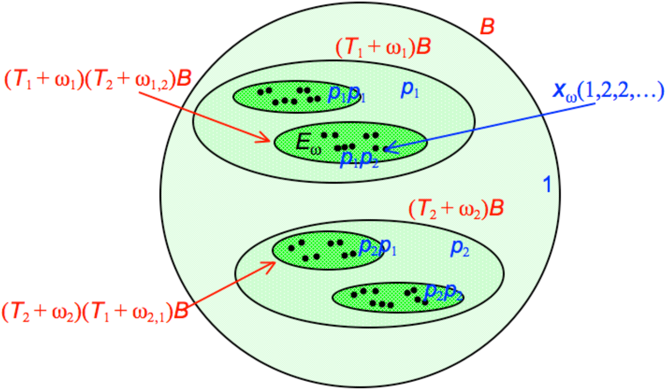
We would like to find $D_q(\mu_\omega)$, at least for a.a. translation vectors $\omega = (\omega_1, \ldots, \omega_m)$. This can be done for $1 < q \leq 2$, but for $q > 2$ there is 'not enough transversality' for the required estimates.

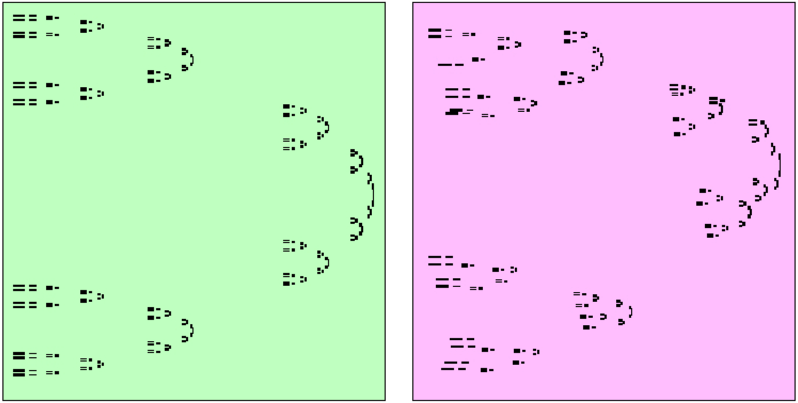
So we introduce more randomness by allowing the translation components to vary at each stage of the construction, by taking:

$$x_\omega(t) = \bigcap_{k=1}^{\infty} (T_{t_1} + \omega_{t_1})(T_{t_2} + \omega_{t_1,t_2})(T_{t_3} + \omega_{t_1,t_2,t_3}) \cdots (T_{t_k} + \omega_{t_1,t_2,\ldots t_k})(B)$$

for $t = (t_1, t_2, \ldots)$, where $\omega = \{\omega_{t_1,t_2,\ldots,t_k}\}$ is a family of i.i.d random variables. We call $E_\omega = \bigcup_t x_\omega(t)$ an almost self-affine set (Jordan, Pollicott & Simon 2007).

Again let μ_ω be the image of the Bernoulli measure μ under x_ω.



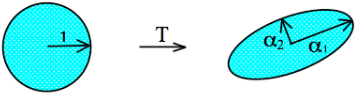


A self-affine set and an almost self-affine set with the same linear components in the defining mappings.

Write $\phi^s(T)$ for the singular value function of a linear mapping T (e.g. for $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$\phi^s(T) = \begin{cases} \alpha_1^s & (0 \le s \le 1) \\ \alpha_1 \alpha_2^{s-1} & (1 \le s \le 2) \end{cases}$$

where α_1, α_2 are the semi-axis lengths of T(unit ball):



[if T is a similarity then $\phi^s(T)$ is just the (scaling ratio of $T)^s$]. Let

$$\Phi_q^s = \lim_{k\to\infty} \Big(\sum_{i_1 \dots i_k} \phi^s(T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k})^{1-q} (p_{i_1} p_{i_2} \dots p_{i_k})^q \Big)^{1/k}.$$

Theorem

For $q > 1$ let s_q satisfy $\Phi_q^{s_q} = 1$. Then for almost all $\omega = \{\omega_{t_1,t_2,\dots,t_k}\}$ the L^q-dimensions of the image measure μ_ω on the almost self-affine set E_ω are given by

$$D_q(\mu_\omega) = \min\{s_q, n\}.$$

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Proof

'$\leq$': Covering argument.
'$\geq$': (Case of $q \geq 2$ an integer) Using the geometry and randomness

$$\mathbb{E}\int \mu_\omega(B(x,r))^{q-1} d\mu_\omega(x)$$
$$\leq cr^{s(q-1)} \int \cdots \int \phi^s(T_{\mathbf{i}_1})^{-1}\phi^s(T_{\mathbf{i}_2})^{-1}\ldots\phi^s(T_{\mathbf{i}_{q-1}})^{-1} d\mu(t_1)\ldots d\mu(t_q)$$

where $\mathbf{i}_1, \ldots, \mathbf{i}_{q-1}$ are the join points of $t_1, \ldots, t_q$.
Then taking $f(\mathbf{i}) = \phi^s(T_{\mathbf{i}})^{-1}$ in inequality $(\star)$, and using the definition of Φ_q^s, this is finite if $\Phi_q^s < 1$. $\square$

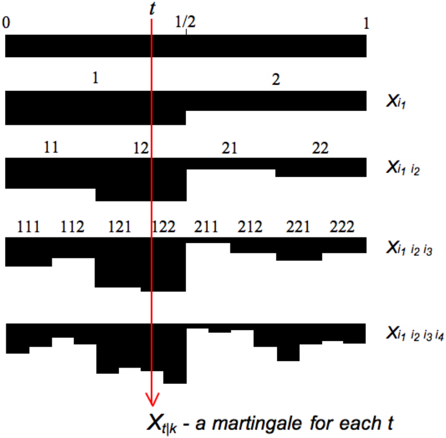
Random multiplicative cascade measures

Let $W_{\mathbf{i}}$ be independent positive random variables indexed by $\mathbf{i} \in \cup_{k=0}^{\infty}\{1,2\}^k \equiv T$, which may be identified with a binary subdivision of $[0,1]$.

Let $X_{\mathbf{i}} = W_{i_1} W_{i_1 i_2} \cdots W_{i_1 i_2 \dots i_k}$ where $\mathbf{i} = (i_1, i_2, \dots, i_k)$.

Assume that $\mathbb{E}(W_{\mathbf{i}}) = 1$ for all $\mathbf{i} \in T$.

Then $X_{t|k}$ is a martingale for each $t \in \{1,2\}^{\mathbb{N}}$.



These martingales were introduced and studied in the 1970s by Mandelbrot, Kahane, Peyrière, in particular for self-similar random multiplicative measures, i.e. when the $W_{\mathbf{i}}$ are identically distributed.

Let μ be a probability measure on $\{1,2\}^{\mathbb{N}}$, and let $q > 1$.

Theorem

If

$$\limsup_{k\to\infty}\Big(\sum_{|\mathbf{i}|=k}\mathbb{E}\Big(\big(X_{\mathbf{i}}\,\mu(C_{\mathbf{i}})\big)^q\Big)\Big)^{1/k} < 1$$

then

$$\limsup_{k\to\infty}\mathbb{E}\Big(\Big(\sum_{|\mathbf{i}|=k}X_{\mathbf{i}}\,\mu(C_{\mathbf{i}})\Big)^q\Big) < \infty$$

and $\int X_{t|k}\,d\mu(t)$ *converges a.s. and in* L^q.

Note that we do not require the $W_{\mathbf{i}}$ *to be identically distributed.*
Results of this type were obtained by Kahane & Peyrière in the i.i.d. case for all $q > 1$ and Barrel in the general case for $1 < q \leq 2$.

Proof A variant of inequality ($\star$) holds using the independence of the $W_{\mathbf{i}}$, taking

$$F(t_1, t_2, \ldots, t_q) \\ = \mathbb{E}(X_{\mathbf{i}_1} X_{\mathbf{i}_1} \cdots X_{\mathbf{i}_{q-1}}) \mu(C_{\mathbf{i}_1}) \mu(C_{\mathbf{i}_2}) \cdots \mu(C_{\mathbf{i}_{q-1}})$$

where $\mathbf{i}_1, \ldots, \mathbf{i}_{q-1}$ are the join points of $t_1, \ldots, t_q$. $\square$

Conclusion

We have considered a particular method of estimating higher moments of fractal measures and seen some examples. There are other situations where a similar approach is possible.

On the other hand, there are certainly other methods for addressing moment problems.

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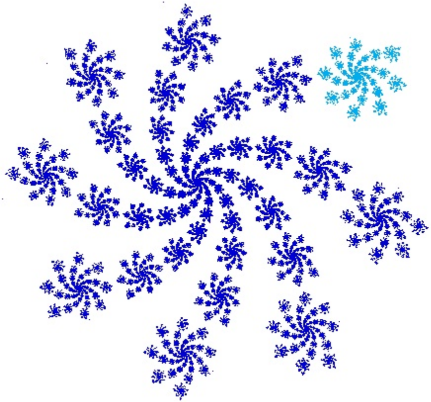
Fractal geometry has developed beyond recognition since I was first attracted to the subject in the 1980s.
As this conference shows, there is more interest, more activity and more open problems than ever, and I am sure the area has a great future.



Thank you!

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