Spectra on fractal measure

Xinrong Dai
Sun Yat-sen University
daixr@mail.sysu.edu.cn

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Abstract

In this talk, we will discuss some spectra properties associated with some fractal measures. In particular, we will consider spectral property of the Bernoulli convolution and the Cantor type measure, the tree structure of the maximal orthogonal set and spectrum, and the sparsity of the spectrum.

I. Self-similar set and measure: Cantor type

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- Cantor sets

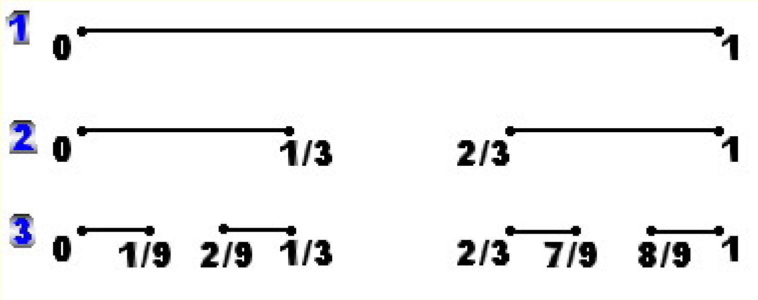

Figure 1: $E = \tau(E) \cup \tau_1(E)$

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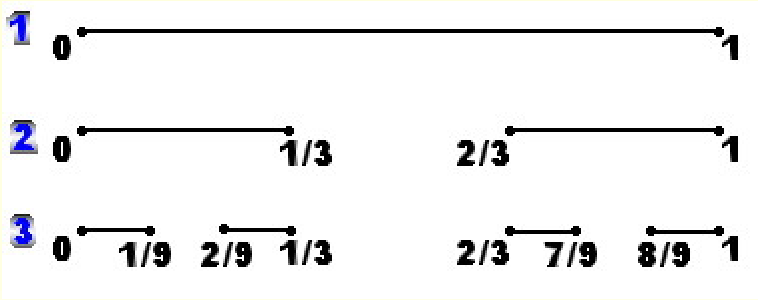

Figure 1: $E = \tau(E) \cup \tau_1(E)$

- 1/4-Cantor measure:

$$\mu_{1/4} = \frac{1}{2}\mu_{1/4}(4\cdot) + \frac{1}{2}\mu_{1/4}(4\cdot -2).$$

- Cantor type measure:

$$\mu_{\rho,n} = \frac{1}{n} \sum_{k=0}^{n-1} \mu_{\rho,n} \left(\rho^{-1} \cdot -k \right)$$

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- Bernoulli convolution: density function μ for the random walk $\sum_{j=0}^{\infty} \pm\rho^j, \rho \in (0,1)$.

$$\mu_\rho = \frac{1}{2}\mu_\rho\left(\rho^{-1}\cdot\right) + \frac{1}{2}\mu_\rho\left(\rho^{-1}\cdot - 1\right)$$

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J.-P. Kahane, Mem. Soc. Math. France 1969, 1971;
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  False:
  (1) 5 and higher dimension, (T. Tao, 2004);
  (2) 3 and higher dimension, (M. N. Kolountzakis and M. Matolcsi, 2006).

- Fourier basis and frame:

  J. Ortega-Cerdà and K. Seip, (2002 Ann. Math.)

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- Singular measure: $\mu_{1/4}$ is spectral measures, and

$$\begin{aligned}\Lambda &= \left\{\sum_{j=0}^{n}\epsilon_j 4^j : n \geq 0, \epsilon_j \in \{0,1\}\right\} \\ &= \{0,1,4,5,16,17,20,21,32,33,36,37,48,47,52,53,\cdots\}\end{aligned}$$

  is a spectrum of $\mu_{1/4}$.

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- The Fourier transform of μ_ρ:

$$\hat{\mu}_\rho(\xi) = \prod_{j=1}^{\infty} \cos\left(2\pi\rho^j\xi\right).$$

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- How about $\rho = \frac{3}{8}$? (2003 or earlier.)

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2. Weak-regularity of Bernoulli convolution.

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(2) A spectral measure has different many spectrums.

- Can we characterizing the structure of the spectrums of these $\mu_{\rho,n}$?

- **Example** (Jorgensen and Pedersen, 1998):

$$\begin{aligned}\Lambda &= \left\{ \sum_{j=0}^{n} \epsilon_j 4^j : n \geq 0, \epsilon_j \in \{0,1\} \right\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 32, 33, 36, 37, 48, 47, 52, 53, \cdots\}\end{aligned}$$

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$$\begin{gathered} \{0,1\} \\ 4+\{0,1\} \\ 16+\{0,1\}+4\{0,1\} \\ 32+\{0,1\}+(4+\{0,1\})+(16+\{0,1\}+4\{0,1\}) \\ \vdots \end{gathered}$$

- (Dutkay, Han and Sun, Adv. Math. 2009)
  $\Lambda \subset \mathbb{R}$ with $0 \in \Lambda$ to be a maximal orthogonal set of $\mu_{1/4}$ iff there exists a spectral labeling L of the binary tree such that $\Lambda = \Lambda(L)$.
  (1) One edge labeled by 0 or 2, and another by 1 or 3;
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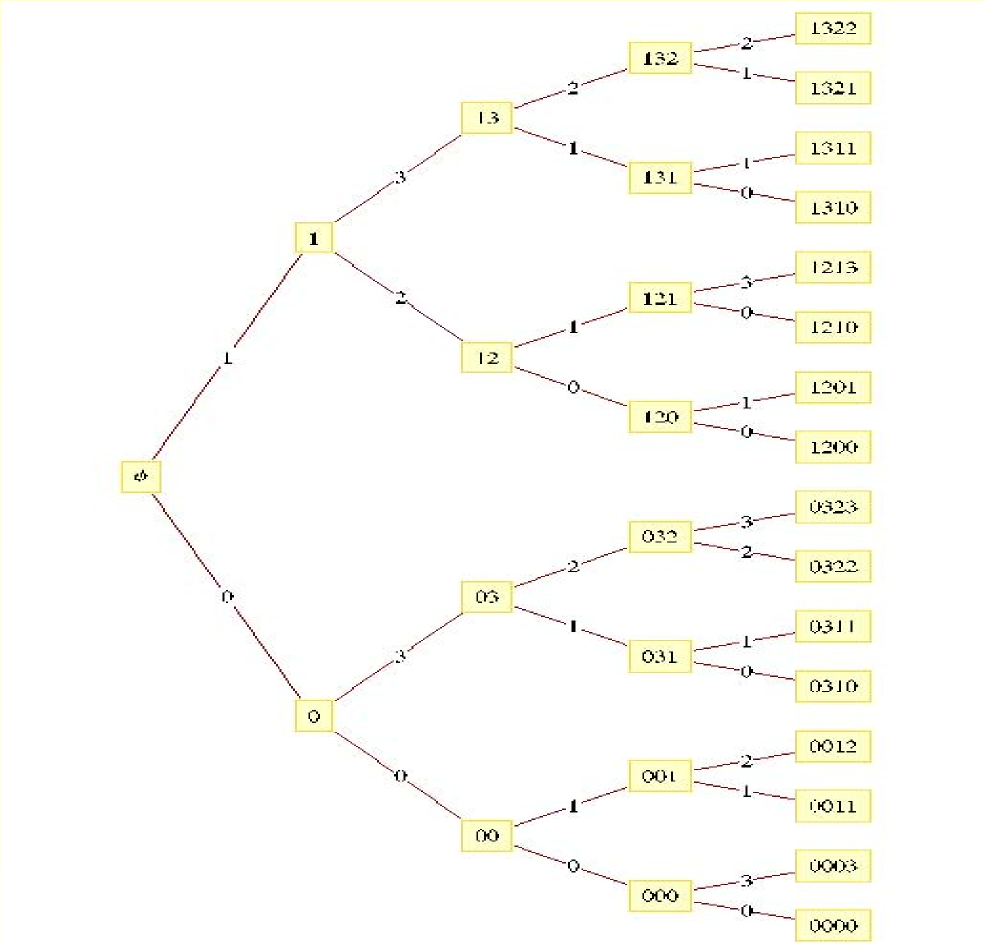
- Tree structure of maximal orthogonal set of $\mu_{1/4}$: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Theorem 2. (Dai, He and Lai, 2012)
$\Lambda \subset \mathbb{R}$ with $0 \in \Lambda$ to be a maximal orthogonal set of $\mu_{q^{-1},n}$ iff there exists a spectral labeling L of the n-adic tree such that $\Lambda = \Lambda(L)$.
(1) Each edge labeled by digit syatem $\{-1, 0, 1, \ldots, q-2\}$;
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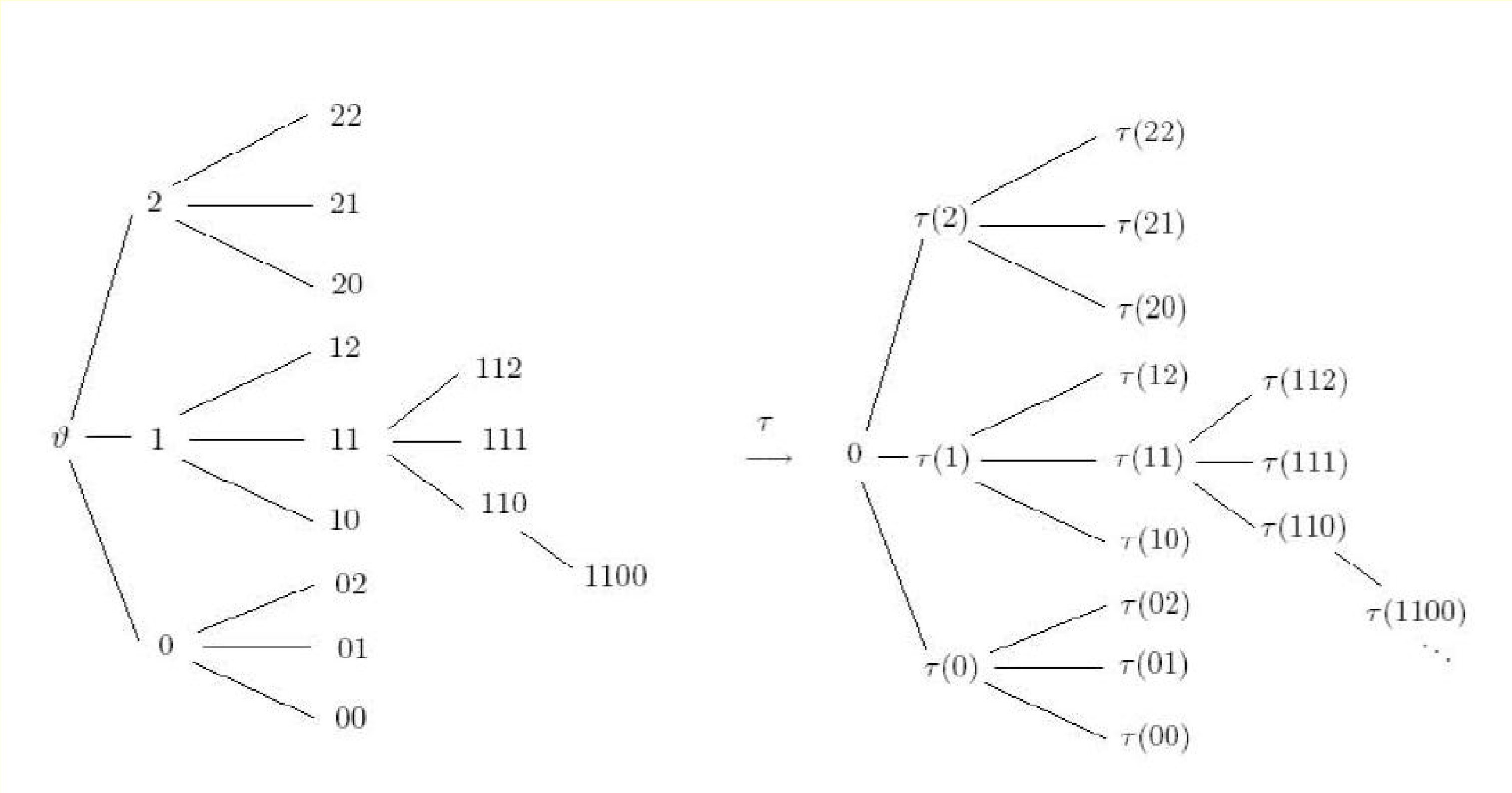
- Tree structure for a spectrum of $\mu_{\rho,n}$:



- $\tau(i_1 i_2 i \dots i_k) \in \{-1, 0, 1, \dots, q-2\}$ and $\tau(i_1 i_2 i \dots i_k) \equiv i_k \mod n$

Theorem 3. (Dai, He and Lai, 2012)
Suppose Λ is a regular,
(1) If there exists M such that $\mathfrak{N}_K^* < M$ for all $K \in \mathbb{N}$, then Λ is a spectrum of $L^2(\mu_{\rho,n})$;
(2) If $\sum_{K=1}^{\infty} C_0^{\mathfrak{N}_K^*} < \infty$, then Λ is not a spectrum of $L^2(\mu)$.

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Regular: The bottom path of each vertex end by $\dot{0}$.

$\mathfrak{N}_K^*$**:** Cardinality of non-zero elements in the bottom path of vertex τ_K.

V. Sparsity of the spectrum: Beurling density

Example The spectrum of $\mu_{1/4}$:

$$\begin{aligned}\Lambda &= \{\sum_{j=0}^{n} \epsilon_j 4^j : n \geq 0, \epsilon_j \in \{0,1\}\} \\ &= \{0, 1, 4, 5, 16, 17, 20, 21, 32, 33, 36, 37, 48, 47, 52, 53, \cdots\}\end{aligned}$$

Elements of spectrum Λ contained in the interval $[0, 4^n)$ are $\{\sum_{j=0}^{n-1} \epsilon_j 4^j; \epsilon \in \{0,1\}\}$. Hence the cardinality is 2^n. Therefore

$$\lim_{n\to\infty} \frac{\#(E \cap [0, 4^n))}{(4^n)^{\log_4 2}} = 1.$$

In fact,

$$0 < \liminf_{N\to\infty} \frac{\#(E \cap [0, N))}{N^{\log_4 2}} \le \lim_{N\to\infty} \frac{\#(E \cap [0, N))}{N^{\log_4 2}} < \infty$$

Observe that $\log_4 2$ is the fractal dimension of the Cantor set.

Beurling density :

$$D^-(\Lambda) = \liminf_{h\to\infty} \frac{\#(\Lambda \cap [x-h, x+h))}{2h}$$

and

$$D^+(\Lambda) = \limsup_{h\to\infty} \frac{\#(\Lambda \cap [x-h, x+h))}{2h}$$

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Dutkey, Han, Sun and Weber, 2011, Adv. math.

K.Seip, etc. 2002, JAMS, Ann. MAth.

R.Strichartz, 2006, J. Anal. Math

Theorem 4. (Dai, HE and Lai, 2012) Let $\mu = \mu_{q,b}$ be a Cantor type self-similar measure (especially, Bernoulli convolution). Then given any increasing non-negative function g on $[0, \infty)$, there exists a spectrum Λ of $L^2(\mu)$ such that

$$\lim_{h\to\infty} \sup_{x\in\mathbb{R}} \frac{\#(\Lambda \cap [x-h, x+h))}{g(h)} = 0.$$

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THANK YOU !