A new approach to Gaussian heat kernel upper bounds on doubling metric measure spaces

Thierry Coulhon, Australian National University

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Setting

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Two models : Riemannian manifolds, fractals.
Fractal manifolds.

Heat kernel

Let p_t be the heat kernel of M, that is the smallest positive fundamental solution of the heat equation:

$$\frac{\partial u}{\partial t} + \Delta u = 0,$$

or the kernel of the heat semigroup $e^{-t\Delta}$:

$$e^{-t\Delta} f(x) = \int_M p_t(x,y) f(y) d\mu(y), \ \ f \in L^2(M,\mu), \ \mu - a.e. \ x \in M.$$

Measurable, non-negative.

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In a general metric space setting, continuity is an issue.

On-diagonal bounds: the uniform case

Want to estimate

$$\sup_{x,y \in M} p_t(x,y) = \sup_{x \in M} p_t(x,x)$$

as a function of $t \to +\infty$.

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$p = 2$ (Coulhon-Grigor'yan): L^2 isoperimetric profile, $\sup_{x\in M} p_t(x,x) \simeq m(t)$, where

$$t = \int_0^{1/m(t)} [\varphi(v)]^2 \frac{dv}{v}. \tag{1}$$

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Go down in the scale: Pseudo-Poincaré inequalities:

$$\|f - f_r\|_p \le Cr\||\nabla f|\|_p, \quad \forall\, f \in \mathcal{C}_0^\infty(M),\ r > 0,$$

where $f_r(x) = \frac{1}{V(x,r)} \int_{B(x,r)} f(y)\, d\mu(y)$. Groups, covering manifolds

Examples

Polynomial volume growth

- $$V(x,r) \geq cr^D$$
- $$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{|\Omega|^{1/D}}$$
- $$\lambda_1(\Omega) \geq \frac{c}{|\Omega|^{2/D}} \Leftrightarrow \sup_{x\in M} p_t(x,x) \leq Ct^{-D/2}$$

Exponential volume growth

- $$V(x,r) \geq c\exp(cr)$$
- $$\frac{|\partial\Omega|}{|\Omega|} \geq \frac{c}{\log|\Omega|}$$
- $$\lambda_1(\Omega) \geq \frac{c}{(\log|\Omega|)^2} \Leftrightarrow \sup_{x\in M} p_t(x,x) \leq C\exp(-ct^{1/3})$$

Off-diagonal bounds

There is a nice connection between the geometry of a metric measure space and the on-diagonal estimates of the heat kernel, but to do analysis, one needs much more, namely **pointwise** estimates of the heat kernel, that is estimates of $p_t(x, y)$ depending on x, y.

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Gaussian:

$$p_t(x, y) \simeq \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{t}\right), \text{ for } \mu\text{-a.e. } x, y \in M, \ \forall\, t > 0.$$

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Sub-Gaussian, for $\omega \geq 2$ (fractals!):

$$p_t(x,y) \simeq \frac{1}{V(x,t^{1/\omega})} \exp\left(-\left(\frac{d^\omega(x,y)}{t}\right)^{\frac{1}{\omega-1}}\right), \text{ for } \mu\text{-a.e. } x,y \in M, \ \forall\, t > 0.$$

Conditions on the volume growth of balls

$B(x, r)$ open ball of center $x \in M$ and radius $r > 0$.

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Polynomial volume growth of exponent $D > 0$:
$\exists c, C > 0$ such that

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Volume doubling condition :
$\exists C > 0$ such that

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Examples: manifolds with non-negative Ricci curvature, but also...

Consequences of the volume doubling condition

$\exists\, C, \nu > 0$ such that

$$V(x,r) \le C \left(\frac{r}{s}\right)^{\nu} V(x,s), \quad \forall\, r \ge s > 0,\ x \in M. \tag{D_ν}$$

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Less well-known: if M is connected and non-compact, **reverse doubling**, that is $\exists\, c, \nu' > 0$ such that

$$c \left(\frac{r}{s}\right)^{\nu'} \le \frac{V(x,r)}{V(x,s)}, \quad \forall r \ge s > 0,\ x \in M. \qquad (RD_{\nu'})$$

Heat kernel estimates 1

Assume doubling.
On-diagonal upper estimate:

$$(DUE) \qquad p_t(x,x) \leq \frac{C}{V(x,\sqrt{t})}, \; \forall \, x \in M, \; t > 0.$$

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Heat kernel estimates 2

Gradient upper estimate

$$(G) \qquad |\nabla_x p_t(x,y)| \leq \frac{C}{\sqrt{t}V(y,\sqrt{t})}, \forall x,y \in M,\ t>0.$$

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Connection with the L^p-boundedness of the Riesz transform

Theorem

Let M be a complete non-compact Riemannian manifold satisfying (D) and (G). Then the equivalence

$$(R_p) \qquad \| \, |\nabla f| \, \|_p \simeq \|\Delta^{1/2} f\|_p, \; \forall\, f \in \mathcal{C}_0^\infty(M),$$

holds for $1 < p < \infty$.

[Auscher, Coulhon, Duong, Hofmann, Ann. Sc. E.N.S. 2004]

Relations

Theorem

$$(DUE) \Leftrightarrow (UE) \Rightarrow (DLE) \not\Rightarrow (LE)$$

$$(G) \Rightarrow (LE) \Rightarrow (DUE)$$

$$(LE) \not\Rightarrow (G)$$

Explain: Davies-Gaffney [Coulhon-Sikora, Proc. London Math. Soc. 2008 and Colloq. Math. 2010]
[Grigory'an-Hu-Lau, CPAM, 2008, Boutayeb, Tbilissi Math. J. 2009]
Three levels: (G), (LY), (UE)

Davies-Gaffney

Heuristics of

$$(DUE) \Leftrightarrow (UE)$$

(Coulhon-Sikora's approach). For simplicity, consider the polynomial case

$$p_t(x,x) \leq C\, t^{-D/2},\ \forall\, t > 0$$

can be reformulated as

$$|<\exp(-zL)f_1, f_2>| \leq K(\mathrm{Re} z)^{-D/2}\|f_1\|_1\|f_2\|_1,\ \forall\, z \in \mathbb{C}_+, f_1, f_2 \in L^1(M, d\mu).$$

Interpolate with the Davies-Gaffney estimate, namely

$$|\langle \exp(-tL)f_1, f_2\rangle| \leq \exp\left(-\frac{r^2}{4t}\right)\|f_1\|_2\|f_2\|_2$$

for all $t > 0$, $f_1, f_2 \in L^2(M, d\mu)$, supported respectively in U_1, U_2, with $r = d(U_1, U_2)$.
Finite propagation speed for the wave equation.
Not on fractals !!

Upper bounds and Faber-Krahn inequality

A fundamental characterization of (UE) or (DUE) was found by Grigor'yan. One says that M admits the relative Faber-Krahn inequality if there exists $c > 0$ such that, for any ball $B(x, r)$ in M and any open set $\Omega \subset B(x, r)$:

$$\lambda_1(\Omega) \geq \frac{c}{r^2} \left(\frac{V(x,r)}{|\Omega|} \right)^{\alpha}, \qquad (FK)$$

where c and α are some positive constants and $\lambda_1(\Omega)$ is the smallest Dirichlet eigenvalue of Δ in Ω. Grigor'yan proves that (FK) is equivalent to the upper bound (DUE) together with (D). The proof of this fact is difficult (Moser iteration).

Upper bounds: back to the uniform case

Assume $V(x,r) \simeq r^D$. Then (DUE) reads

$$(*)\ p_t(x,x) \leq Ct^{-D/2}, \ \ \forall\, t > 0, \ x \in M,$$

$(*)$ is equivalent to:

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- the Sobolev inequality:

$$\|f\|_{\alpha D/(D-\alpha p)} \le C\|\Delta^{\alpha/2} f\|_p, \quad \forall f\in C_0^\infty(M),$$

for $p>1$ and $0<\alpha p<D$ [Varopoulos 1984, Coulhon 1990].

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- the Nash inequality:

$$\|f\|_2^{2+(4/D)} \leq C\|f\|_1^{4/D}\mathcal{E}(f), \quad \forall f \in C_0^\infty(M).$$

[Carlen-Kusuoka-Stroock 1987]

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[Carlen-Kusuoka-Stroock 1987]
-the Gagliardo-Nirenberg type inequalities, for instance

$$\|f\|_q^2 \leq C\|f\|_2^{2-\frac{q-2}{q}D}\mathcal{E}(f)^{\frac{q-2}{2q}D}, \quad \forall f \in C_0^\infty(M),$$

for $q > 2$ such that $\frac{q-2}{2q}D < 1$ [Coulhon 1992].

One-parameter weighted Sobolev inequalities 1

Denote

$$V_r(x) := V(x,r),\ r > 0,\ x \in M.$$

Introduce

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Introduce

$$\|f\|_2^2 \leq C(\|fV_r^{-1/2}\|_1^2 + r^2\mathcal{E}(f)), \quad \forall\, r > 0, \quad \forall f \in \mathcal{F}. \tag{N}$$

(equivalent to Nash if $V(x, r) \simeq r^D$) and

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(equivalent to Nash if $V(x,r) \simeq r^D$) and for $q > 2$,

$$\|fV_r^{\frac{1}{2}-\frac{1}{q}}\|_q^2 \leq C(\|f\|_2^2 + r^2\mathcal{E}(f)), \quad \forall\, r > 0, \quad \forall f \in \mathcal{F}, \tag{GN_q}$$

(equivalent to Gagliardo-Nirenberg if $V(x,r) \simeq r^D$)
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(equivalent to Gagliardo-Nirenberg if $V(x, r) \simeq r^D$)
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Theorem

Assume that M satisfies (D) and Davies-Gaffney. Then (DUE) is equivalent to (N), and to (GN_q) if ν is as in (D_ν) and $q > 2$ is such that $\frac{q-2}{2q}\nu < 1$.

[Boutayeb-Coulhon-Sikora, in preparation]

One-parameter weighted Sobolev inequalities 2

Kigami, local inequalities à la Saloff-Coste, Faber-Krahn: all equivalent

Nash inequality:

$$\|f\|_2^2 \le C(\|fV_r^{-1/2}\|_1^2 + r^2\mathcal{E}(f)), \quad \forall\, r > 0,\ f \in \mathcal{F}. \qquad (N)$$

Kigami-Nash inequality:

$$\|f\|_2^2 \le C\left(\frac{\|f\|_1^2}{\inf_{x\in supp(f)} V_r(x)} + r^2\mathcal{E}(f)\right), \quad \forall\, r > 0,\ f \in \mathcal{F}_0. \qquad (KN)$$

Localised Nash inequalities: there exists $\alpha, C > 0$ such that for every ball $B = B(x, r)$, for every $f \in \mathcal{F} \cap \mathcal{C}_0(B)$,

$$\|f\|_2^{\frac{1+\alpha}{2}} \le \frac{C}{V_r^{\alpha}(x)}\|f\|_1^{2\alpha}\left(\|f\|_2^2 + r^2\mathcal{E}(f)\right). \qquad (LN)$$

One-parameter weighted Sobolev inequalities 3

Sketch of the proof of $(GN_q) \Leftrightarrow (DUE)$
(GN_q) is equivalent to

$$\sup_{t>0} \|M_{V_{\sqrt{t}}^{\frac{1}{2}-\frac{1}{q}}} e^{-tL}\|_{2\to q} < +\infty \qquad (VE_{2,q})$$

(DUE) is equivalent to

$$\sup_{t>0} \|M_{V_{\sqrt{t}}^{\frac{1}{2}}} e^{-tL}\|_{2\to\infty} < +\infty \qquad (VE_{2,\infty})$$

Extrapolation; commutation: again, finite speed propagation of the associated wave equation.

One gets a characterization of (DUE) that does not use any kind of Moser iteration.

One can replace the volume $V(x,r)$ by a more general doubling function $v(x,r)$ (except in the equivalence with Faber-Krahn).

Heat kernel estimates: the sub-Gaussian case 1

Sub-Gaussian upper estimate

$$(UE^\omega) \quad p_t(x,y) \le \frac{C}{V(x,t^{1/\omega})} \exp\left(-c\left(\frac{d^\omega(x,y)}{t}\right)^{\frac{1}{\omega-1}}\right), \forall\, x,y \in M,\ t>0.$$

On-diagonal lower sub-Gaussian estimate

$$(DLE^\omega) \qquad p_t(x,x) \ge \frac{c}{V(x,t^{1/\omega})}, \forall\, x \in M,\ t>0.$$

Full sub-Gaussian lower estimate

$$(LE^\omega) \quad p_t(x,y) \ge \frac{c}{V(x,\sqrt{t})} \exp\left(-C\left(\frac{d^\omega(x,y)}{t}\right)^{\frac{1}{\omega-1}}\right), \forall\, x,y \in M,\ t>0$$

Relations remain, but one needs an exit time estimate. No more Davies-Gaffney !

The sub-Gaussian case 2

Theorem

Let $\mathcal{E}$ be a regular, local and conservative Dirichlet form on $L^2(M,\mu)$ with domain $\mathcal{F}$. Let $q>2$ such that $\frac{q-2}{q}\nu<\omega$, where $\nu>0$ is as in (D_ν). Assume the exit time estimate:

$$cr^\omega \le E_x(\tau_{B_r(x)}) \le Cr^\omega,\ \text{for a.e. } x\in M,\ \text{all } r>0,$$

Then the following conditions are equivalent: (UE^ω)

$$\|fV_r^{\frac{1}{2}-\frac{1}{q}}\|_q^2 \le C(\|f\|_2^2 + r^\omega \mathcal{E}(f)),\ \forall\, r>0,\ f\in\mathcal{F},$$

$$\|f\|_2^2 \le C(\|fV_r^{-1/2}\|_1^2 + r^\omega \mathcal{E}(f)), \quad \forall\, r>0,\ f\in\mathcal{F},$$

$$\lambda_1(\Omega) \ge \frac{c}{r^\omega}\left(\frac{V(x,r)}{|\Omega|}\right)^{\omega/\nu},$$

for every ball $B(x,r)\subset M$ and every open set $\Omega\subset B(x,r)$.

Questions

- Doubling case, Gaussian
  Get a more handy characterization of (LE), get a characterization of (G).
- Sub-Gaussian ?
  We do use Grigory'an-Telcs, Grigor'yan-Hu-Lau