Billiard orbits in self-similar Sierpinski carpets

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Joint work with Robert Niemeyer (University of New Mexico)

Sierpinski carpet $S_{\mathbf{a}}$

$$\mathbf{a} = \{a_1, a_2, \cdots\} \in \{3, 5, 7, \cdots\}^{\mathbb{N}}.$$

Divide the unit square into a_1^2 squares of side a_1^{-1}, and remove the middle square. Then for each of the remaining $a_1^2 - 1$ squares, remove from it a center square of side $a_1^{-1} \cdot a_2^{-1}$, etc.
Let $S_{\mathbf{a},n}$ be the nth-level approximation of $S_{\mathbf{a}}$.

Example: $S_{\mathbf{3}}$, where $\mathbf{a} = \{3, 3, \cdots\}$.

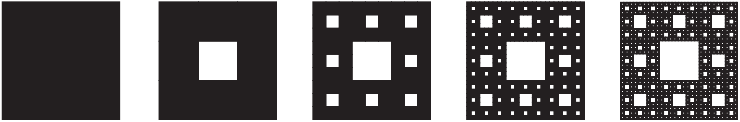


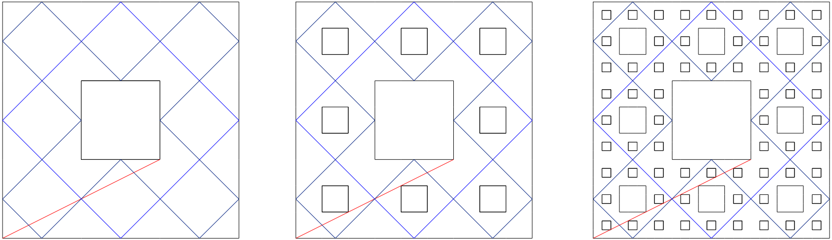
Figure : $S_{\mathbf{3},0} = [0,1]^2$, $S_{\mathbf{3},1}$, $S_{\mathbf{3},2}$, $S_{\mathbf{3},3}$, $S_{\mathbf{3},4}$

Task: Identify the nontrivial rectifiable curves / line segments in $S_{\mathbf{a}}$.

- For $S_{\mathbf{3}}$ this was answered by C. Bandt & M. Mubarak (2004).
- For more general self-similar $S_{\mathbf{a}}$, the result is due to E. Durand-Cartagena & J. Tyson (2011).

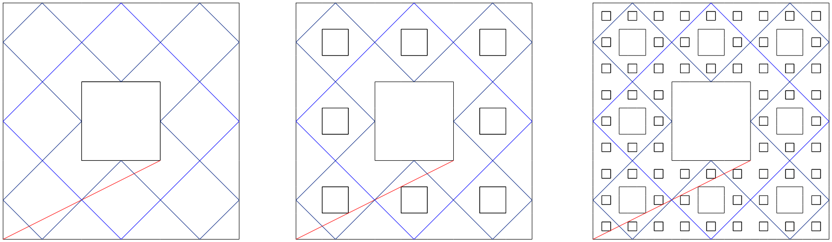
Our goal: Understand the nature of billiard trajectories in $S_{\mathbf{a}}$.

Illustrative example: Standard Sierpinski carpet (S_3)



- The boundary of an open square removed in the construction of $S_{\mathbf{a}}$ is called a peripheral square of $S_{\mathbf{a}}$. (Note: the boundary of the unit square is *not* a peripheral square.)
- A nontrivial line segment of $S_{\mathbf{a}}$ is a (straight-line) segment of the plane contained in $S_{\mathbf{a}}$ that has nonzero length.
- A line segment in $S_{\mathbf{a}}$ is said to be maximal if it connects two points on the boundary of the unit square.

Illustrative example: Standard Sierpinski carpet (S_3)



The set of slopes of nontrivial line segments in S_3 is

$$\left\{0, \pm\frac{1}{2}, \pm 1, \pm 2, \infty\right\}$$

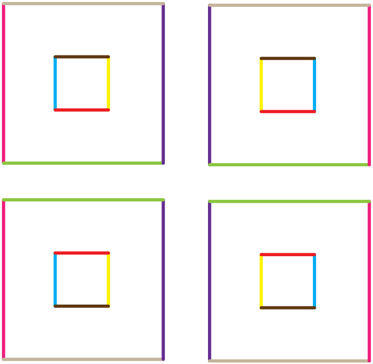
Some terminology from mathematical billiards

Let D be a polygonal domain, possibly with obstacles.

- $\Omega(\partial D)$: billiard table determined by the boundary of D.
- $\mathscr{O}(x^0, \alpha)$: billiard orbit in $\Omega(\partial D)$ with initial base point x^0 and slope α (pointing inward).
- $f^m(x^0, \alpha)$: mth collision point of the orbit $\mathscr{O}(x^0, \alpha)$ with ∂D.

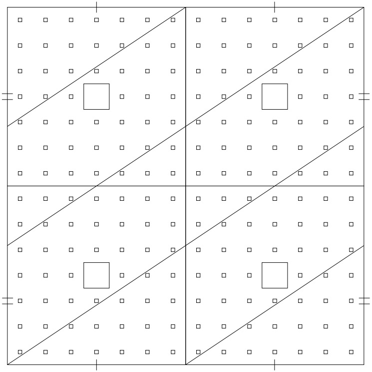
- An orbit $\mathscr{O}(x^0, \alpha)$ of $\Omega(\partial D)$ is said to be closed if the orbit consists of finitely many line segments.
- Let $\mathscr{O}(x^0, \alpha)$ be a closed orbit of $\Omega(\partial D)$. If there exists a least integer $m \geq 1$ such that $f^m(x^0, \alpha) = (x^0, \alpha)$, then we say the orbit $\mathscr{O}(x^0, \alpha)$ is a periodic orbit.
- If under the billiard flow a billiard ball intersects a point of the boundary for which reflection cannot be determined in a well-defined manner, then the billiard ball trajectory terminates at that point. Such an orbit is then called singular. Note that a singular orbit is a closed orbit, because the number of line segments in the orbit is finite.

Translation surface of $S_{\mathbf{a},n}$



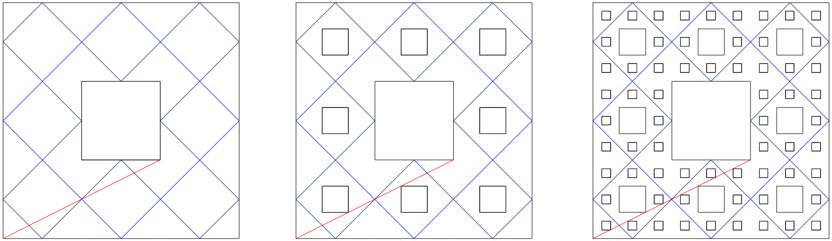
- Billiard flow on a polygonal domain is equivalent to geodesic flow on the associated translation surface.
- The translation surface of $S_{\mathbf{a},n}$ is obtained by taking 4 copies of $S_{\mathbf{a},n}$, then making proper identifications on both the outer edges (of the unit square) and the inner edges (of peripheral squares).
- The genus of the translation surface of $S_{\mathbf{a},n}$ grows to infinity as $n \to \infty$.

Law of reflection at corners



- Each of the 4 corners of the original square has conic angle $2\pi(= 4 \times \pi/2)$, and the law of reflection is well-defined there.
- However, a corner of an (inner) peripheral square has conic angle $6\pi(= 4 \times 3\pi/2)$, and reflection is ill-defined there. Any orbit which hits such a corner is considered a singular orbit, and terminates there.

Billiard orbits in S_3



$\mathscr{O}_n(x^0, \alpha)$: billiard orbit in $\Omega(\partial S_{\mathbf{a},n})$ with base point x^0 and slope α.

- $\left\{\mathscr{O}_n((0,0), \frac{1}{2})\right\}_{n=0}^{\infty}$ forms a sequence of compatible singular orbits, whose lengths tend to 0 as $n \to \infty$.
- $\left\{\mathscr{O}_n((\frac{1}{2}, 0), 1)\right\}_{n=0}^{\infty}$ forms an (eventually) constant sequence of compatible periodic orbits.
- In fact, for any $m \in \mathbb{N}$ and any positive odd integer $p < 2 \cdot 3^m$, $\left\{\mathscr{O}_n((\frac{p}{2 \cdot 3^m}, 0), 1)\right\}_{n=0}^{\infty}$ forms an eventually constant sequence of compatible periodic orbits.
(Use reflected-unfolding + self-similarity)

The slope sets of nontrivial line segments in $S_{\mathbf{a}}$

Let Slope($S_{\mathbf{a}}$) be the set of slopes, with values in $[0,1]$, of nontrivial line segments in $S_{\mathbf{a}}$.
(By isometry, one can obtain from slope $\alpha \in \text{Slope}(S_{\mathbf{a}})$ slopes of $-\alpha$, α^{-1}, and $-\alpha^{-1}$.)

Theorem 4.1 in [Durand-Cartagena & Tyson '11]

Let $\mathbf{a} = (a, a, \ldots)$ be a constant sequence. Then the set of slopes Slope($S_{\mathbf{a}}$) is the union of the following two sets:

$$A = \left\{ \frac{p}{q} \ : \ p+q \le a,\ 0 \le p < q \le a-1,\ p,q \in \mathbb{N} \cup \{0\},\ p+q \text{ is odd} \right\}, \tag{1}$$

$$B = \left\{ \frac{p}{q} \ : \ p+q \le a-1,\ 0 \le p \le q \le a-2,\ p,q \in \mathbb{N},\ p \text{ and } q \text{ are odd} \right\}. \tag{2}$$

Moreover, (*) if $\alpha \in A$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α touches vertices of peripheral squares, while if $\alpha \in B$, then each nontrivial line segment in $S_{\mathbf{a}}$ with slope α is disjoint from all peripheral squares. For each $\alpha \in A \cup B$, there exist maximal line segments in $S_{\mathbf{a}}$ with slope α. Finally, if $b < a$, then any maximal nontrivial line segment in $S_{\mathbf{b}}$ is also contained in $S_{\mathbf{a}}$. In particular, Slope($S_{\mathbf{b}}$) $\subseteq$ Slope($S_{\mathbf{a}}$).

- Any line segment in $S_{\mathbf{a}}$ with slope $\alpha \notin \text{Slope}(S_{\mathbf{a}})$ has zero length.
- In their proof [Du-CaTy] gave existence of nontrivial line segments with slope $\in \text{Slope}(S_{\mathbf{a}})$, but did not discuss in detail the set of base points from which these segments emanate.
- Furthermore, (*) turns out to be incorrect.

The slope set Slope($S_{\mathbf{a}}$)

Slope($S_{\mathbf{a}}$) is the disjoint union of $A_{\mathbf{a}}$ and $B_{\mathbf{a}}$, where

$$\begin{aligned} A_{\mathbf{a}} &= \left\{ \frac{p}{q} \;:\; p+q \leq a,\, 0 \leq p < q \leq a-1,\, p,q \in \mathbb{N}_0,\, p+q \text{ is odd} \right\}, \\ B_{\mathbf{a}} &= \left\{ \frac{p}{q} \;:\; p+q \leq a-1,\, 0 \leq p \leq q \leq a-2,\, p,q \in \mathbb{N},\, p \text{ and } q \text{ are odd} \right\}. \end{aligned}$$

Examples (Color codes: $A_{\mathbf{a}-2}$, $A_{\mathbf{a}} \setminus A_{\mathbf{a}-2}$, $B_{\mathbf{a}-2}$, $B_{\mathbf{a}} \setminus B_{\mathbf{a}-2}$)

$$\text{Slope}(S_3) = \left\{0, \frac{1}{2}, 1\right\}.$$

$$\text{Slope}(S_5) = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}.$$

$$\text{Slope}(S_7) = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}.$$

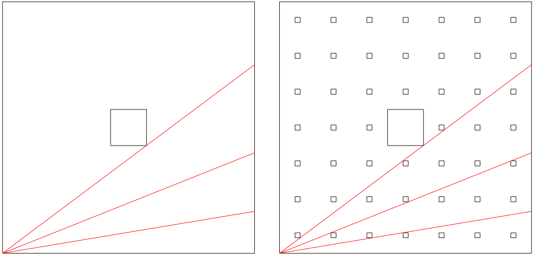
$$\text{Slope}(S_9) = \left\{0, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, 1\right\}.$$

- Relation to Farey sequences: Let F_m be the set of reduced fractions with denominator $\leq m$. Then $F_{(a+1)/2} \subset \text{Slope}(S_{\mathbf{a}}) \subset F_{a-1}$.

Line segments with slope drawn from $A_{\mathbf{a}} \setminus A_{\mathbf{a-2}}$ in $S_{\mathbf{a}}$

$$A_{\mathbf{a}} \setminus A_{\mathbf{a-2}} = \left\{ \frac{p}{q} : p + q = a,\ 1 \leq p < q \leq a-1,\ p, q \text{ are coprime} \right\}$$

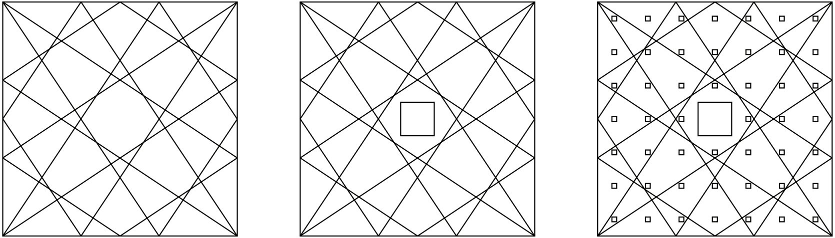
Example: $S_{\mathbf{7}}$, initial condition $((0,0), \alpha \in A_{\mathbf{7}} \setminus A_{\mathbf{5}})$.



Any line segment starting at $(0,0)$ with slope $\alpha \in A_{\mathbf{a}} \setminus A_{\mathbf{a-2}}$ must hit a corner of a peripheral square in $S_{\mathbf{a}}$. By self-similar scaling, we can prove that $\{\mathscr{O}_n((0,0),\alpha)\}_{n=0}^{\infty}$ forms a sequence of compatible singular orbits whose lengths tend to 0.

A counterexample to the result of Durand-Cartagena & Tyson

Example: $S_{\mathbf{7}}$, initial condition $\left((0,0), \frac{2}{3} \in A_{\mathbf{5}}\right)$.



- The orbit hits none of the peripheral squares of $S_{\mathbf{7}}$.
- The sequence of orbits is an (eventually) constant sequence of compatible periodic orbits.
- Using a simple algebraic argument + self-similarity, we can prove that any orbit in $S_{\mathbf{a}}$ with initial condition $((0,0), \alpha \in A_{\mathbf{a-2}})$ avoids all peripheral squares of $S_{\mathbf{a}}$.

A refinement of the result of Durand-Cartagena & Tyson

Theorem (C. & Niemeyer '12)

Let $\mathbf{a} = (a, a, a, ...)$ *be a constant sequence. Then the set of slopes* $Slope(S_{\mathbf{a}})$ *is the union of the following two sets:*

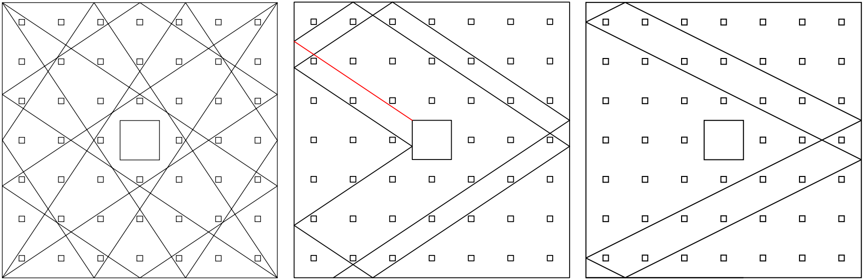
$$A_{\mathbf{a}} = \left\{ \frac{p}{q} \; : \; p+q \leq a, \, 0 \leq p < q \leq a-1, \, p, q \in \mathbb{N} \cup \{0\}, \, p+q \text{ is odd} \right\},$$

$$B_{\mathbf{a}} = \left\{ \frac{p}{q} \; : \; p+q \leq a-1, \, 0 \leq p \leq q \leq a-2, \, p, q \in \mathbb{N}, \, p \text{ and } q \text{ are odd} \right\}.$$

Moreover, if $\alpha \in A_{\mathbf{a}} \setminus A_{\mathbf{a-2}}$*, then each nontrivial line segment in* $S_{\mathbf{a}}$ *with slope* α *beginning from* $(0,0)$ *touches vertices of peripheral squares, while if* $\alpha \in A_{\mathbf{a-2}}$*, then each nontrivial line segment in* $S_{\mathbf{a}}$ *with slope* α *beginning from* $(0,0)$ *is disjoint from all peripheral squares. If* $\alpha \in B_{\mathbf{a}}$*, then each nontrivial line segment in* $S_{\mathbf{a}}$ *with slope* α *beginning at* $(\frac{p}{2a^n})$*,* $n \geq 1$ *and* $p < 2a^n$ *being a positive odd integer, is disjoint from all peripheral squares with side* $< \frac{1}{a^n}$*.*
In addition, For each $\alpha \in A_{\mathbf{a}} \cup B_{\mathbf{a}}$*, there exist maximal line segments in* $S_{\mathbf{a}}$ *with slope* α*. Finally, if* $b < a$*, then any maximal nontrivial line segment in* $S_{\mathbf{b}}$ *is also contained in* $S_{\mathbf{a}}$*. In particular,* $Slope(S_{\mathbf{b}}) \subseteq Slope(S_{\mathbf{a}})$*.*

Orbits with slope drawn from $A_{\mathbf{a}-2}$ in $S_{\mathbf{a}}$

Example: $\mathbf{a} = 7$.



(a) $\mathscr{O}\left((0,0), \frac{2}{3}\right)$ (b) $\mathscr{O}\left((\frac{1}{7},0), \frac{2}{3}\right)$ (c) $\mathscr{O}\left((\frac{1}{7},0), \frac{1}{2}\right)$

- Starts at $(0,0)$, slope $\frac{2}{3}$: The orbit avoids all peripheral squares of $S_{7,n}$ for any $n \in \mathbb{N}$. $\rightarrow$ A periodic orbit.
- Starts at $(\frac{1}{7},0)$, slope $\frac{2}{3}$: The orbit avoids all peripheral squares of side 7^{-m}, $m \geq 2$, but hits a corner of the central peripheral square of side 7^{-1} (red). $\rightarrow$ A singular orbit.
- Starts at $(\frac{1}{7},0)$, slope $\frac{1}{2}$: The orbit avoids all peripheral squares of S_7, and forms a periodic orbit.

Periodic orbits in self-similar Sierpinski carpets

Theorem (C. & Niemeyer '12)

Let $\Omega(\partial S_{\mathbf{a}})$ be a self-similar Sierpinski carpet billiard. Let $\alpha \in Slope(S_{\mathbf{a}})$.

1. *If $\alpha \in A_{\mathbf{a-2}}$, then the sequence of compatible orbits $\{\mathscr{O}_n((0,0),\alpha)\}_{n=0}^{\infty}$ is a sequence of compatible periodic orbits.*
2. *If $\alpha \in B_{\mathbf{a}}$ and $p < 2a^n$ is an positive odd integer, then the sequence of compatible orbits $\{\mathscr{O}_n((\frac{p}{2a^n},0),\alpha)\}_{n=0}^{\infty}$ is a sequence of compatible periodic orbits.*

Furthermore, in each case, the sequence of compatible periodic orbits is an eventually constant sequence of compatible orbits. The trivial limit of an eventually constant sequence of periodic orbits constitutes a periodic orbit of a self-similar Sierpinski carpet billiard.

More (sequence of compatible) closed orbits in $S_{\mathbf{a}}$

Proposition

Given $\alpha \in A_{\mathbf{a-2}}$, $m \in \mathbb{N}$, and $k \in \{1, 2, \cdots, 2^m - 1\}$, there exists an $N \in \mathbb{N}$ such that $\{\mathscr{O}_n((\frac{k}{a^m}, 0), \alpha)\}_{n=N}^{\infty}$ is an eventually constant sequence of compatible closed orbits, whose limit $\mathscr{O}\left((\frac{k}{a^m}, 0), \alpha\right)$ has nonzero length.

Remark. Such an orbit is either periodic or singular, depending on whether it hits corners of peripheral squares of side $\geq a^{-m}$. Which type it is seems to depend sensitively on the choice of α, m, and k.

Proposition

Given $\alpha \in A_{\mathbf{a}} \setminus A_{\mathbf{a-2}}$, $m \in \mathbb{N}$, and $k \in \{0, 1, 2, \cdots, 2^m - 1\}$, there exists an $N \in \mathbb{N}$ such that $\{\mathscr{O}_n((\frac{k}{a^m}, 0), \alpha)\}_{n=N}^{\infty}$ forms a sequence of compatible singular orbits, whose lengths tend to zero as $n \to \infty$.

This is a case where we are certain that reflected-unfolding an eventually trivial line segment onto $\Omega(\partial S_{\mathbf{a}})$ leads to an eventually trivial billiard orbit.

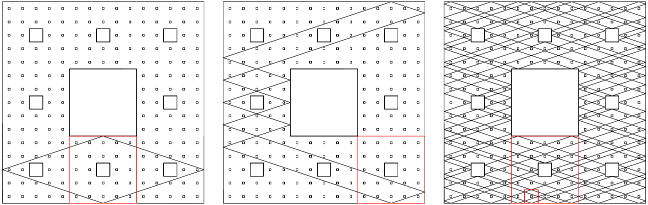
Question: Does reflected-unfolding *some* eventually trivial line segment lead to an "interesting" billiard orbit in $\Omega(\partial S_{\mathbf{a}})$? Specific examples to be explored:

- $((\frac{k}{a^n}, 0), \alpha \in B_{\mathbf{a}})$, where $k \in \{0, 1, \cdots, 2^n - 1\}$.
- $((\frac{p}{2a^n}, 0), \alpha \in A_{\mathbf{a}})$, where p is a positive odd integer $< 2a^n$.

Future work

Extend the analysis to other billiard tables of infinite genus and positive area. One natural candidate consists of the non-self-similar Sierpinski carpets.

Example: $S_{\mathbf{a}}$ with $\mathbf{a} = \{3, 5, 7, \cdots\}$ and $\sum_{j=1}^{\infty} a_j^{-2} < \infty$.



In this setting, it seems plausible that one could define a dynamical system for the corresponding billiard flow.

(Big) open question: Is there a connection between properties of periodic orbits in $\Omega(\partial S_{\mathbf{a}})$ and properties of Laplacian eigenfunctions on $S_{\mathbf{a}}$? (Microlocal analysis)

Preprint available upon request next week, will go on arXiv soon.