The dimension of the full nonuniformly hyperbolic horseshoe

Cao Yongluo

Email: ylcao@suda.edu.cn

Soochow University

Contents

Motivation

To consider the Hausdorff dimension and the dynamics of the full non-uniformly hyperbolic horseshoe. These systems was constructed by Rios for studying the bifurcation of homoclinic tangency inside horseshoe and it appears naturally in the Henon family.

Hausdorff dimension

Definition 1.1

For any $\delta > 0$ we define

$$\mathcal{H}^s_\delta(F) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \ \delta \ \text{ cover of } \ F\}$$

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

This limit exists for any subset, though the limiting value can be 0 or ∞. We call $\mathcal{H}^s(F)$ the s-dimensional Hausdorff measure of F.

$$Dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

.

Topological pressure

Recall the notation of topological pressure.

Let $T:\Lambda\to\Lambda$ be a continuous transformation and $\phi:\Lambda\to R$ continuous function.

F is a (n,ϵ) separate set of T.

$P_n(\sigma,\phi,\epsilon)=\sup\{\sum_{x\in F}e^{S_n(\phi(x))}|F \text{ is}(n,\epsilon) \text{ separate set}\}$

Where $S_n(\phi(x))=\phi(x)+\cdots+\phi(\sigma^{n-1}x)$.

$P(\sigma,\phi)=\lim\limits_{\epsilon\to 0}\limsup\limits_{n\to\infty}\frac{1}{n}\log P_n(\sigma,\phi,\epsilon)$

Theorem 1.2

Bowen: $P(\sigma,\phi)$ topological pressure, variational principal

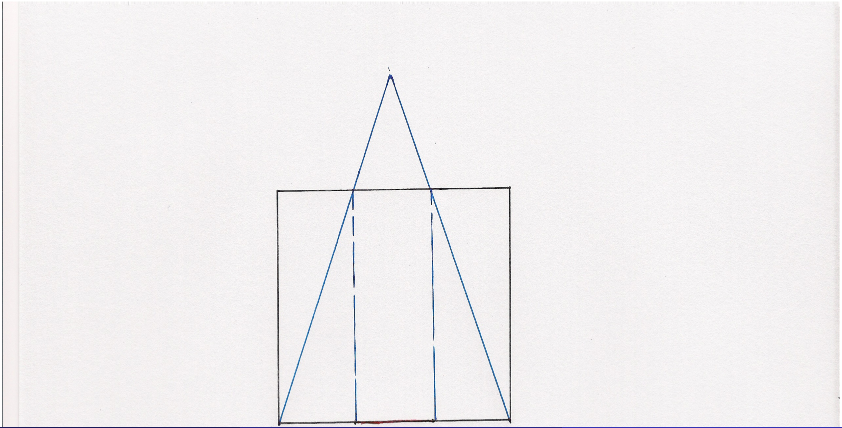
$$P(\sigma,\phi)=\sup_{\mu\in\mathcal{M}(\sigma)}\{h_\mu+\int\phi d\mu\}$$

Topological pressure

Hausdorff dimension of Standard Cantor set.

1. It is well known that the root of equation $2 \times (\frac{1}{3})^t = 1$ is it Hausdorff dimension.

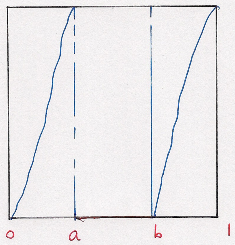
It is equivalent to $P(T, -t \log |DT|) = 0$.



2.One dimension nonlinear expanding map

Hausdorff dimension of Cantor set is the root of equation

$P(T, -t \log |DT|) = 0.$



4. $f : M \to M$ be a C^1 map, Λ is a repeller of f, Conformal.

Theorem 1.3

Let Λ be a C^1 Conformal repeller of f. Then Hausdorff dimension of repeller is the root of equation $P(T, -t\log|DT|) = 0$.

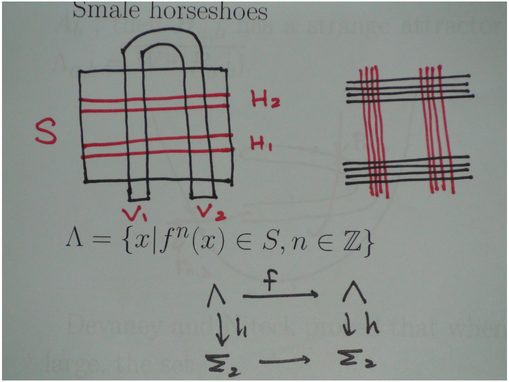
Ruelle considered the $C^{1+\alpha}$ for Hausdorff.

Falconer considered for Hausdorff and Box dimension

Gatzouras and Peres considered the C^1 case.

Smale Hoesrshoes

Next we consider two dimension case.



$\forall x \in \Lambda$, $T_x M = E^s + E^u$

$\|Df^n(v_1)\| \leq e^{\lambda n}\|v_1\|, \ v \in E^s$

$\|Df^n(v_2)\| \geq e^{-\lambda n}\|v_2\|, \ v \in E^u \ (\lambda < 0).$

The Hausdorff dimension of hyperbolic horseshoe

MaCluskey, H. and Manning, A., 1983(ETDS) prove that for $C^{1+\alpha}$, the Hausdorff dimension of Λ

$$Dim_H \Lambda = t^s + t^u$$

Where t^u is the root of $P(-t \log |df|_{E^u}|) = 0$ and t^s is the root of $P(t \log |Df|_{E^s}|) = 0$.

J.Palis and Viana prove that the formula as above holds for C^1 diffeomeorphism.

Lyapunove Exponents

Lyapunov exponents:

$f : M \to M$ M is a Riemann manifold .

$\forall x \in M, v \in T_x M$ if

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\|Df^n(x)v\|}{\|v\|}$$

exists, it is called Lyapunov exponent, and denote it by $\lambda(x, v)$.

μ is f invariant measure, $\mathcal{M}(f)$.

$\mu(A) = \mu(f^{-1}(A))$ for every measurable set.

Ergodic invariant measure $\mathcal{E}(f)$. Ergodic means that for every invariant set A, $\mu(A) = 0, or 1$.

Oseledec Theorem

$A \subset M$ $\mu(A) = 1$ for every $\mu \in \mathcal{M}(f)$

$x \in A$

$1. \lambda_1(x) \leq \cdots \leq \lambda_s(x)$

$2. V_0(x) \subset V_1(x) \subset \cdots \subset V_s(x) = T_x M$

$$\lim_{n\to\infty} \frac{1}{n} \log |Df_x^n(v)| = \lambda_i(x), v \in V_i \setminus V_{i-1}$$

$\lambda_i(x)$ is defined on A

μ is ergodic, $\lambda_i(x)$ constant $a.e\mu$ and denote it by $\lambda_i(\mu)$.

The dynamics of $a = a^*$

Henon map

$$H_{a,b}(x, y) = (1 - ax^2 + y, bx)$$

Theorem 3.1

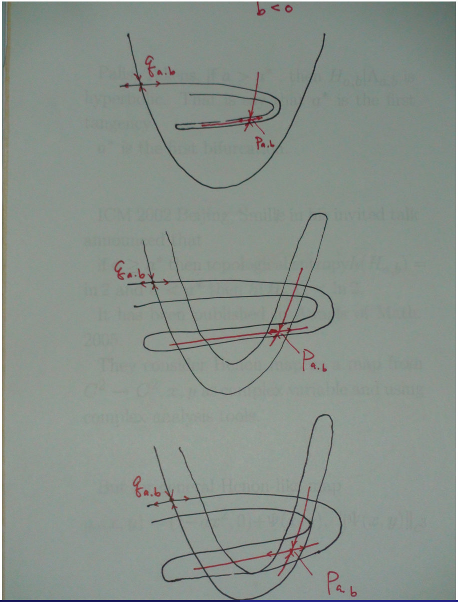
If b is small, then there is an $a = a^*$, the corresponding map $H_{a^*,b}$, $\exists B \subset \Lambda$ with $\mu(B) = 1$ for $\forall \mu \in \mathcal{M}(H, \Lambda)$ and $\forall x \in B$

$\lambda_1(x) < c_1 < 0 < c_2 < \lambda_2(x)$

μ is hyperbolic measure.

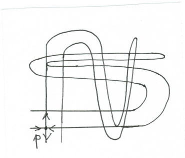
What is Hausdorff dimension ?

Horseshoe with infinite branches



The example of Rios's for homoclinic tangency inside

I.Rios gave an example of systems with homoclinic tangeny inside of invariant set in 2001, Nonlinearity. Luzzatto, Rios and Cao prove that all the Lyapunov exponents of all invariant measures are uniformly bounded away from 0(2006, DCDS). Now it is called the full nonuniformly hyperbolic

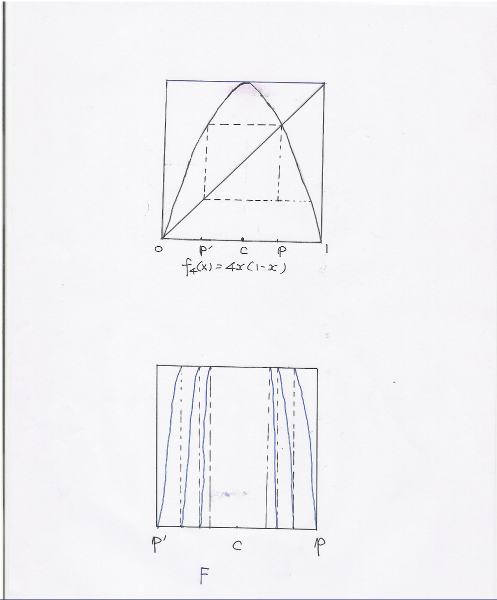


Horseshoe with infinite branches

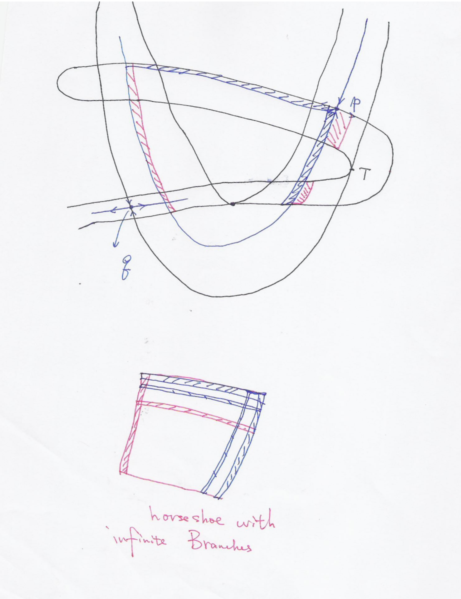
Furthermore, we will consider the ergodicity of this map. We will construct a inducing map (Horseshoe with infinite branches.)

One dimension $f_4(x) = 4x(1-x)$.

Horseshoe with infinite branches



Horseshoe with infinite branches



$\tilde{\Lambda} = \cup_{i=2}^{\infty} \Lambda_i$ and $F(x) = f^{\tau}(x)(x)$ and $\tau(x) = i$ for $x \in \Lambda_i$.

$(\tilde{\Lambda}, F)$, the first return map to $\tilde{\Lambda}$. There hyperbolic product structure.

Stable foliation γ^s and unstable foliation γ^u.

For $y \in \gamma^s(x)$,

$$\log \Pi_{i=n}^{\infty} \frac{DF^u(F^i(x)}{DF^u(F^i(y))} \leq c\lambda^n.$$

For $y \in \gamma^u(x)$, and they are in the same n cylinder,

$$\log \Pi_{i=0}^{n} \frac{DF^u(F^i(x)}{DF^u(F^i(y))} \leq c\lambda^n.$$

Consider the one-side full shift of countable type $(S^{\mathbb{N}}, \sigma)$, where S countable set.

Let $\Phi : S^{\mathbb{N}} \to R$ be a some real function. The variations of Φ are defined as

$$Var_n(\Phi) = \sup\{|\Phi(x) - \Phi(y)| : x, y \text{ in the same n-cylinder}\}$$

If there are constants $C > 0$ and $\theta \in (0, 1)$ such that $Var_n(\Phi) < C\theta^n$ for all $n \geq 2$, then we call Φ the weakly Holder continuous.

A Gibbs measure

$$\frac{1}{B} \leq \frac{m[a_0, \cdots, a_{n-1}]}{e^{\Phi_n(x) - nP}} \leq B \text{ for all } x \in [a_0, \cdots, a_{n-1}].$$

$\Phi_n = \sum_{i=0}^{n-1} \Phi(\sigma^i(x))$.

The Gurevich pressure of Φ

$$P_G(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma^n(x) = x} e^{\Phi_n(x)} 1_{[a]}(x)$$

Consider the full shift of countable type $(S^{\mathbb{Z}}, \sigma)$, where S countable set.

Let $\psi : S^{\mathbb{Z}} \to R$ be a some real function. The variations of Φ are defined as

$$Var_n(\psi) = \sup\{|\psi(x) - \psi(y)| : x_i = y_i (i = -(n-1), \cdots, n-1)\}$$

A function $\psi : S^{\mathbb{Z}} \to \mathbb{R}$ is called one-side, if $\psi(\underline{x}) = \psi(\underline{y})$ for every $\underline{x}, \underline{y} \in S^{\mathbb{Z}}$ with $x_i = y_i$ for all $i \geq 0$. It has

Proposition 1

If $\psi : S^{\mathbb{Z}} \to R$ is weakly Holder continuous and $var_1 \phi < \infty$, then there exists a bound Holder continuous function φ such that $\phi = \psi + \varphi - \varphi \circ \sigma$ is weakly Holder continuous and one-side.

Sarig, Mauldin & Urbanski

Theorem 1

For full shift map as above, if $\sum_{k\geq 1} Var_k(\Phi) < \infty$*, then* Φ *has an invariant Gibbs measure iff* Φ *has finite Gurevich pressure.*

If $\phi : \Lambda \to R$ is Holder continuous function, then we can define an induced potential on $\tilde{\Lambda}$ as follow: $\Phi = \sum_{i=0}^{\tau(x)-1} \phi(f^i(x))$.

We can prove that $\sum_{k \geq 1} Var_k(\Phi) < \infty$.

Lemma 3.2

If μ_F is an ergodic measure on $(\tilde{\Lambda}, F)$ with $\int \tau d\mu_F < \infty$, and μ is the projected measure on (λ, f) then

$$h_{\mu_F}(F) = (\int \tau d\mu_F) h_\mu(f)$$

$$\int \Phi d\mu_F = (\int \tau d\mu_F) \int \phi d\mu.$$

$\psi_s = \phi - s$, then induced potential $\Psi_s = \Phi - ts$.

Lemma 3.3

If $P_G(\Psi_{s*}) < \infty$, then $P_G(\Psi_s)$ is decrease and continuous on $[s^*, \infty)$.

Equilibrium state

Furthermore if $\phi = -t \log |Df^u(x)|,\ t > 0$, then we can prove $\sum_{k\geq 1} Var_k(\Phi) < \infty$. There exists an unique Gibbs measure μ_t which is equilibrium state of induced systems $(\tilde{\Lambda}, F)$.

Theorem 3.4

There is an open set U such that if $t \in U$ then

$$P_G(\tilde{\Lambda},\, \Phi_t)$$

has an unique equilibrium state of Φ_t which is Gibbs measure and the project of μ_t onto (Λ, f) is the equilibrium state of $-t \log |Df^u(x)|$.

The properties of Gibbs measure μ_t has exponential decay of correlations and satisfies the CLT for the class of functions whose induced functions are Holder continuous.

Upper semi-continuity

We also use the stable foliation and unstable foliation as above to prove that there is a surjective $\Pi : \{0,1\}^Z \to \Lambda$, finite to one and Holder continuous. Full probability set: one to one.

Measure entropy $h_\mu : (f) \to R$ upper semi-continuity. In fact it is continuous.

Rios and Leplaideur have related results for the nonhyperbolic horseshoe which was constructed by Rios.

In fact, our method for constructing infinite branches horseshoe can be applied to the nonhyperbolic horseshoe which was constructed by Rios.

Hausdorff dimension

Furthermore we study the Hausdorff dimension of Λ. We can prove that

$$Dim_H \Lambda = t^s + t^u$$

Where t^u is the root of $P_G(-t \log |\tilde{D} f^u(x)|) = 0$ and t^s is the root of $P_G(t \log |\tilde{D} f^s(x)|) = 0$.

Where $-t \log |\tilde{D} f^u(x)|$, and $t \log |\tilde{D} f^s(x)|$ are induced potential of $-t \log |D f^u(x)|$ and $t \log |D f^s(x)|$ respectively.

Hausdorff dimension

In fact, we can prove that

$$P_G(-t^u \log |\tilde{D} f^u(x)|) = p(-t^u \log |D f^u(x)|)$$
$$= \sup\{h_\mu - t^u \int \log |D f^u(x)| d\mu\} = 0,$$

$$P_G(t^s \log |\tilde{D} f^s(x)|) = p(t^s \log |D f^s(x)|)$$
$$= \sup\{h_\mu + t^s \int \log |D f^s(x)| d\mu\} = 0.$$

Multifractal structure of LE

Now we consider the level set

$$K_{\alpha,\beta} = \{x \in \Lambda : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |df(f^i(x))|_{E^s_{f^i(x)}}| = \alpha,$$
$$- \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |df^{-1}(f^{-i}(x))|_{E^u_{f^{-i}(x)}}| = \beta, \}.$$

Let

$$\mathcal{P}^+ = \{ \int \log |df|_{E^s(x)}| d\mu, \ \ \mu \in \mathcal{M} \}$$

and

$$\mathcal{P}^- = \{ \int \log |df|_{E^u(x)}| d\mu, \ \ \mu \in \mathcal{M} \}.$$

Multifractal structure of LE

We can prove that for $(\alpha, \beta) \in int\mathcal{P}^+ \times int\mathcal{P}^-$,

$$dim_H K_{\alpha,\beta} = \max\{\frac{h_\mu}{-\alpha} : \mu \in \mathcal{M} \text{ and } \int \log |df|_{E^s(x)}| d\mu = \alpha\}$$
$$+ \max\{\frac{h_\mu}{\beta} : \mu \in \mathcal{M} \text{ and } \int \log |df|_{E^u(x)}| d\mu = \beta\}.$$

The same result as above for uniformally hyperbolic horseshoe in the surface was obtained by Barreira and Valls 2006(CMP), Barreira and Doutor 2009(Nonlinearity).

THANK YOU!

谢谢!