Recent advances in Mandelbrot martingales theory

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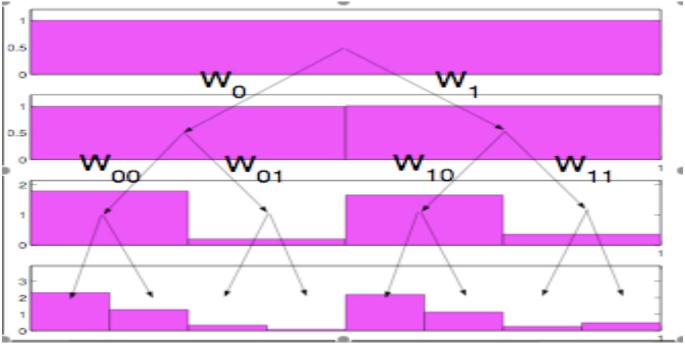
Mandelbrot martingales

Let $\Sigma = \{0,1\}^{\mathbb{N}_+}$ and for $n \geq 1$, $\Sigma_n = \{0,1\}^n$.
W: positive rv. with $\mathbb{E}\, W = 1/2$. $\{W(w)\}_{\sigma \in \bigcup_{n\geq 1} \Sigma_n}$ independent rv's equidistributed with W.



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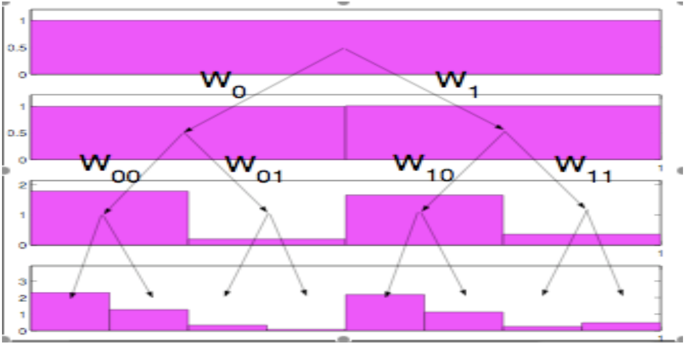

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Let $Y = \lim_{n\to\infty} Y_n$. Writing

$$Y_{n+1} = \sum_{j\in\{0,1\}} W(j) \times \sum_{\sigma\in\Sigma_n} W(j\cdot\sigma|_1)W(j\cdot\sigma|_2)\cdots W(j\cdot\sigma|_n)$$
$$= W(0)\,Y_n(0) + W(1)\,Y_n(1)$$

yields

$$Y = W(0)\,Y(0) + W(1)\,Y(1),$$

where $\{W(j), Y(j)\}_{j\in\{0,1\}}$ are independent, $W(j) \sim W$, $Y(j) \sim Y$. Moreover, $\mathbb{P}(Y > 0) \in \{0, 1\}$.

Using this recursively yields the Mandelbrot random measure on $[0, 1]$

$$\mu(I_\sigma) = W(\sigma|_1)\,W(\sigma|_2)\cdots W(\sigma|_{|\sigma|})\,Y(\sigma).$$

Theorem (Kahane (1976))

The following assertions are equivalent: (1) $\mathbb{P}(Y > 0) = 1$*; (2)* $(Y_k)_{k\geq 1}$ *is uniformly integrable; (3)* $\mathbb{E}\,W \log W < 0$.

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Mandelbrot martingales. Normalization and related equation

Natural questions arise:

1. (Mandelbrot, 1974) When $\mathbb{E}\, W \log W \geq 0$, does there exist $A_n > 0$ such that (Y_n/A_n) converges to a non-trivial limit Z, at least in distribution?

   If so A_n/A_{n+1} converges to A, $0 < A < \infty$, and the limit satisfies

   $$Z \stackrel{d}{=} A\, W(0)\, Z(0) + 1\, W(1)\, Z(1)$$

2. (Durrett and Liggett, 1983) In general, what are the non-trivial solutions to

   $$(E) \quad Z \stackrel{d}{=} W(0)\, Z(0) + W(1)\, Z(1) \quad ?$$

3. Are there natural multifractal measures associated with solutions of (E)?

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The derivative martingale in the critical case.

Suppose that $\mathbb{E}\, W \log W = 0$. For all $\beta \in [0,1)$, set

$$W_\beta = \frac{W^\beta}{2\,\mathbb{E}\, W^\beta}. \text{ It satisfies } \begin{cases} \mathbb{E}\, W_\beta = 1/2, \\ \mathbb{E}\, W_\beta \log W_\beta < 0 \end{cases}.$$

Define

$$Y_n(\beta) = \sum_{\sigma \in \Sigma_n} W_\beta(\sigma|_1) W_\beta(\sigma|_2) \cdots W_\beta(\sigma|_n) \text{ and } Y_n' = -\frac{\mathrm{d}}{\mathrm{d}\beta} Y_n(\beta).$$

Theorem (Biggins-Kyprianou (1997), Liu (2000))

If $\mathbb{E}\, W^{1+\epsilon} < \infty$ for some $\epsilon > 0$, then (Y_n') converges almost surely to Y', $Y' = W(0)\, Y'(0) + W(1)\, Y'(1)$, $\mathbb{E}\, Y' = \infty$.

This yields a.s. on $[0,1]$ the "critical" measure

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Other solutions to (E): $Z \overset{d}{=} W(0)\,Z(0) + W(1)\,Z(1)$

Suppose that $\mathbb{E}\,W^{1+\epsilon} < \infty$ for some $\epsilon > 0$.

- If $\mathbb{E}\,W \log W < 0$ (resp. $\mathbb{E}\,W \log W = 0$), Durrett and Liggett prove that up to multiplicative positive constant the unique solution to (E) is $Y = \|\mu\|$ (resp. $Y' = \|\mu'\|$).
- If the distribution of $\log(W)$ is non-lattice and $\mathbb{E}\,W \log W > 0$, let β be the unique solution of $\mathbb{E}\,W^\beta = 1/2$ in $(0,1)$.
  Setting $\widetilde{W} = W^\beta$, we have $\mathbb{E}\,\widetilde{W} \log \widetilde{W} > 0$. This yields a non-degenerate Mandelbrot measure $\widetilde{\mu}$. Durrett and Liggett prove that the unique solutions to the functional equation (E) : are, up to a positive constant, of the form $L_\beta(\|\widetilde{\mu}\|)$, where L_β is a stable Lévy subordinator of index β.
- If the distribution of $\log(W)$ is non-lattice and $\mathbb{E}\,W \log W > 0$, but $\mathbb{E}\,W \neq 1/2$, then other kind of solutions appear, all reducible to the form $L_\beta(\|\widetilde{\mu}'\|)$, where $\|\widetilde{\mu}'\|$ is a critical Mandelbrot measure independent of L_β.

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Random measures associated with (E): $Z \stackrel{d}{=} W(0)\,Z(0) + W(1)\,Z(1)$

There are 4 kind of natural measures associated with (E), each providing a nice candidate to illustrate the multifractal formalism. Each satisfies for all $n \geq 1$

$$(\nu(I_\sigma))_{\sigma\in\Sigma_n} \stackrel{d}{=} \big(W(\sigma|_1)W(\sigma|_2)\cdots W(\sigma|_n)\,Z(\sigma)\big)_{\sigma\in\Sigma_n}.$$

- Mandelbrot measures μ (studied by many authors: Holley-Waymire (1992), Falconer (1996), Molchan (1996), B. (2000))
- Critical Mandelbrot measures μ' (studied by B. (2000)), with the question of existence of atoms left opened.
- $L'_{\beta,\mu}$: the derivative of the Lévy process L_β in multifractal Mandelbrot time $\mu([0,t])$ (studied by Jaffard (1999) when μ is the Lebesgue mesure, and in general by B.-Seuret (2007)).
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Normalization: the critical case

Theorem (Aidekon and Shi, Webb (log-gaussian case), (2011))

Suppose that $\mathbb{E}\, W \log W = 0$ *and* $\mathbb{E}\, W^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$. *Then, there exists* $c > 0$ *such that*

$$c\, n^{1/2} Y_n \xrightarrow[n\to\infty]{\mathbb{P}} Y'.$$

Normalization of Mandelbrot measures: the supercritical case

Suppose that $\mathbb{E}\, W \log W > 0$. The normalization problem is closely related to the critical case. Once again for all $\beta \in [0,1]$, set

$$W_\beta = \frac{W^\beta}{2\,\mathbb{E}\, W^\beta}. \text{ It satisfies } \begin{cases} \mathbb{E}\, W_\beta = 1/2, \\ f(\beta) = \mathbb{E}\, W_\beta \log W_\beta \text{ is increasing}, \\ f(0) = -\log(2)/2 < 0,\ f(1) = \mathbb{E}\, W \log W > 0 \end{cases}.$$

There is a unique $\beta_0 \in (0,1)$ such that $\mathbb{E}\, W_{\beta_0} \log W_{\beta_0} = 0$.

Theorem (Madaule, Webb (log-gaussian case) (2011))

Suppose that $\mathbb{E}\, W \log W > 0$, $\mathbb{E}\, W^{1+\epsilon} < \infty$ for some $\epsilon > 0$, and the distribution of $\log W$ is non-lattice. Then

$$n^{\frac{3}{2\beta_0}}\, c^n\, Y_n \xrightarrow[n\to\infty]{d} Z > 0,$$

where $c = (2\,\mathbb{E}\, W^{\beta_0})^{-1/\beta_0}$ and $Z \overset{d}{=} c\, W(0)\, Z(0) + c\, W(1)\, Z(1)$.

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$$n^{\frac{3}{2\beta_0}}\, c^n\, Y_n \xrightarrow[n\to\infty]{d} Z > 0,$$

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Normalization of Mandelbrot measures: the supercritical case

Suppose that $\mathbb{E}\, W \log W > 0$. The normalization problem is closely related to the critical case. Once again for all $\beta \in [0,1]$, set

$$W_\beta = \frac{W^\beta}{2\,\mathbb{E}\, W^\beta}. \text{ It satisfies } \begin{cases} \mathbb{E}\, W_\beta = 1/2, \\ f(\beta) = \mathbb{E}\, W_\beta \log W_\beta \text{ is increasing}, \\ f(0) = -\log(2)/2 < 0, \ f(1) = \mathbb{E}\, W \log W > 0 \end{cases}.$$

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Normalization of Y_n

We summarize.

Theorem (Aidekon and Shi (2011), Webb (2011))

Suppose that $\mathbb{E}\, W \log W = 0$ *and* $\mathbb{E}\, W^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$. *Then, there exists a* $c_W > 0$ *such that*

$$c\, n^{1/2} Y_n \xrightarrow[n\to\infty]{\mathbb{P}} Y'.$$

Theorem (Webb (log-gaussian case), Madaule (2011))

Suppose that $\mathbb{E}\, W \log W > 0$, $\mathbb{E}\, W^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$, *and the distribution of* $\log W$ *is non-lattice. Then there exists a unique* $\beta \in (0,1)$ *such that*

$$n^{\frac{3}{2\beta}}\, c^n\, Y_n \xrightarrow[n\to\infty]{d} Z > 0,$$

where $c = (2\,\mathbb{E}\, W^{\beta})^{-1/\beta}$ *and* $Z \overset{d}{=} c\, W(0)\, Z(0) + c\, W(1)\, Z(1)$ *(recall that* β *is the unique solution of* $\mathbb{E}\, W_\beta \log W_\beta = 0$ *in* $(0,1)$*).*

Identification of the limit for the associated measures

Recall: let

$$\mu_n(I_\sigma) = W(\sigma|_1)\, W(\sigma|_2) \cdots W(\sigma|_{|\sigma|}) \quad \text{for } \sigma \in \Sigma_n.$$

If $\mathbb{E}\, W \log W \geq 0$ then almost surely the martingale $\mu_n \stackrel{weakly}{\longrightarrow} \mu = 0$ as $n \to \infty$.

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Suppose that $\mathbb{E}\, W \log W \geq 0$, $\mathbb{E}\, W^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$, *and the distribution of* $\log W$ *is non-lattice.*

1. *(Johnson and Waymire, 2011) If* $\mathbb{E}\, W \log W = 0$ *then,*

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$$n^{\frac{3}{2\beta}}\, c^n\, \mu_n \xrightarrow[n\to\infty]{\text{weakly in } d} L'_{\beta,\mu'_\beta}.$$

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Another natural normalization

$$\mu_n(I_\sigma) = W(\sigma|_1)\, W(\sigma|_2) \cdots W(\sigma|_{|\sigma|}) \quad \text{for } \sigma \in \Sigma_n.$$

Corollary

Suppose that $\mathbb{E}\, W \log W \geq 0$, $\mathbb{E}\, W^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$.

1. *(Johnson and Waymire, 2011) If* $\mathbb{E}\, W \log W = 0$ *then,*

$$\frac{\mu_n}{\|\mu_n\|} \xrightarrow[n\to\infty]{\text{weakly in } \mathbb{P}} \frac{\mu'}{\|\mu'\|}.$$

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The random energy model point of view

Assume that W is normalized to $\mathbb{E}\, W \log W = 0$. Write

$$W(\sigma) = e^{\xi(\sigma)},\ W(\sigma|_1)W(\sigma|_2)\cdots W(\sigma_{|\sigma|}) = e^{X(\sigma)},\ X(\sigma) = \sum_{i=1}^{n} \xi(\sigma|_k).$$

Define the partition function

$$\beta \geq 0 \mapsto Z_n(\beta) = \sum_{\sigma\in\Sigma_n} e^{\beta X(\sigma)}$$

and for each $\beta \geq 0$ consider the sequence of Gibbs measures

$$\mu_{\beta,n}(I_\sigma) = \frac{e^{\beta X(\sigma)}}{Z_n(\beta)}.$$

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Suppose that $\mathbb{E}\, e^{(1+\epsilon)\xi} < \infty$ for some $\epsilon > 0$.

Theorem (Collet and Koukiou (1992), Waymire-Williams (1994), ...)

With probability 1,

$$\frac{1}{n} \log Z_n(\beta) \underset{n\to\infty}{\longrightarrow} \begin{cases} \log(2\,\mathbb{E}\, e^{\beta\xi}) & \text{if } \beta \in [0,1), \\ 0 & \text{if } \beta \geq 1 \end{cases}.$$

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Suppose that $\mathbb{E}\, e^{(1+\epsilon)\xi} < \infty$ for some $\epsilon > 0$. Let μ' be the critical Mandelbrot measure. Suppose that the law of ξ is non-lattice.

Theorem

1. *if* $\beta \in [0,1)$ *then a.s., a non trivial Mandelbrot measure* μ_β *is associated with* W_β, $\frac{Z_n(\beta)}{(2\,\mathbb{E}\, e^{\beta\xi})^n} \xrightarrow[n\to\infty]{} \|\mu_\beta\|$, $\mu_{\beta,n} \xrightarrow[n\to\infty]{weakly} \frac{\mu_\beta}{\|\mu_\beta\|}$.
2. *If* $\beta = 1$, $c\, n^{1/2} Z_n(1) \xrightarrow[n\to\infty]{\mathbb{P}} \|\mu'\|$, $\frac{\mu_{1,n}}{\|\mu_{1,n}\|} \xrightarrow[n\to\infty]{weakly\ in\ \mathbb{P}} \frac{\mu'}{\|\mu'\|}$.
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Theorem (Aidekon (2010), Webb (in the log-Gaussian case, 2011))

$$\mathbb{P}(n^{3/2} \max_{\sigma\in\Sigma_n} e^{X_\sigma} \leq z) \xrightarrow[n\to\infty]{} \mathbb{E}\exp(-c\|\mu'\|/z).$$

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Continuity of the critical Mandelbrot measure

Here we also assume that $\mathbb{E}\, W^{-\epsilon} < \infty$ for some $\epsilon > 0$.

Theorem (B., Kupiainen, Nikula, Saksman, Webb (2012))

For any $\gamma \in [0, 1/2)$ *we have*

$$n^{\gamma} \max_{\sigma \in \Sigma_n} \mu'(I_\sigma) \xrightarrow{\mathbb{P}} 0 \quad \text{as} \quad n \to \infty,$$

and for any $\gamma \in (1/2, \infty)$ *we have*

$$n^{\gamma} \max_{\sigma \in \Sigma_n} \mu'(I_\sigma) \xrightarrow{\mathbb{P}} \infty \quad \text{as} \quad n \to \infty.$$

Corollary

Almost surely the limit measure μ' *has no atoms.*

Modulus of continuity of the critical measure

Theorem (B., Kupiainen, Nikula, Saksman, Webb (2012))

For any $\gamma \in (0,1/2)$*, with probability 1, there exists* $C(\omega) \in \mathbb{R}_+^*$ *such that*

$$\mu'(I) \leq C(\omega)\left(\log\left(1+\frac{1}{|I|}\right)\right)^{-\gamma}$$

for all subintervals I *of* $[0,1]$*. Moreover, one cannot take* $\gamma > 1/2$ *in the above statement.*

Application to the modulus of continuity of the subcritical measure

Here we suppose that $\mathbb{E}\, W^q < \infty$ for all $q > 0$. Set

$$\varphi(q) = 1 + \log_2 \mathbb{E}\, W^q.$$

Notice that $0 < \varphi(q) < 1$ over $(0,1)$ and $\varphi(0) = 0$.
Recall that for $\beta \in (0,1)$, μ_β is the Mandelbrot measure defined as

$$\mu_\beta(I_\sigma) = e^{\beta X(\sigma)} Y_\beta(\sigma).$$

Theorem (B., Kupiainen, Nikula, Saksman, Webb (2012))

Let $\beta \in (0,1)$ and $\gamma \in (0,1/2)$. With probability 1, there exists $C(\omega) \in \mathbb{R}_+^*$ such that

$$\mu_\beta(I) \le C(\omega)|I|^{\varphi(\beta)} \left(\log\left(1 + \frac{1}{|I|}\right)\right)^{-\gamma\beta}$$

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L^q-spectrum of the critical measure

For $\beta \geq 0$ set

$$\widetilde{Z}_n(\beta) = \sum_{\sigma\in\Sigma_n} \mu'(I_\sigma)^\beta = \sum_{\sigma\in\Sigma_n} e^{\beta X(\sigma)} Y'(\sigma)^\beta \quad \text{(recall that } Z_n(\beta) = \sum_{\sigma\in\Sigma_n} e^{\beta X(\sigma)}\text{)}.$$

Theorem (Collet and Koukiou (1992), Waymire-Williams (1994), ...)

With probability 1,

$$\frac{1}{n}\log_2 \widetilde{Z}_n(\beta) \underset{n\to\infty}{\longrightarrow} \begin{cases} 1+\log_2(\mathbb{E}\, e^{\beta\xi}) & \text{if } \beta\in[0,1), \\ 0 & \text{if } \beta \geq 1 \end{cases}.$$

L^q-spectrum of the critical measure

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Theorem

1. *(deduced from Ossiander-Waymire (2000)) If $\beta \in [0,1)$ then a.s., a non trivial Mandelbrot measure μ_β is associated with W_β,*
$$\frac{\widetilde{Z}_n(\beta)}{(2\,\mathbb{E}\,e^{\beta\xi})^n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}(\|\mu'\|^\beta)\|\mu_\beta\|.$$
2. *(B., Kupiainen, Nikula, Saksman, Webb (2012), log-gaussian case) If $\beta = 1$, $c\,n^{1/2}\widetilde{Z}_n(1) \overset{d}{\underset{n\to\infty}{\longrightarrow}} \|\mu'\|$.*
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$$\widetilde{Z}_n(\beta) = \sum_{\sigma \in \Sigma_n} \mu'(I_\sigma)^\beta.$$

Theorem

1. *(deduced from Ossiander-Waymire (2000)) If $\beta \in [0,1)$ then a.s., a non trivial Mandelbrot measure μ_β is associated with W_β,*
   $$\frac{\widetilde{Z}_n(\beta)}{(2\,\mathbb{E}\,e^{\beta\xi})^n} \underset{n\to\infty}{\longrightarrow} \mathbb{E}(\|\mu'\|^\beta)\|\mu_\beta\|.$$
2. *(B., Kupiainen, Nikula, Saksman, Webb (2012), log-gaussian case)*
   If $\beta = 1$, $c\, n^{1/2}\widetilde{Z}_n(1) \xrightarrow[n\to\infty]{d} \|\mu'\|$.
3. *(B., Kupiainen, Nikula, Saksman, Webb (2012), log-gaussian case)*
   If $\beta > 1$, $c\, n^{\beta/2}\widetilde{Z}_n(\beta) \xrightarrow[n\to\infty]{d} L_{1/\beta}(\|\mu'\|)$.