Analytic Continuation of Analytic (Fractal) Functions

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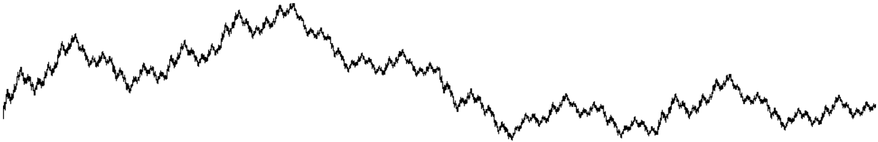
Analytic continuations of fractals

generalises analytic continuation of analytic functions



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- Given a (piece of) an analytic fractal function can we continue it?
- Analytic continuation is a cornerstone notion in mathematics and applications

Analytic iterated function systems (IFS)

$\mathcal{W} = \{X; w_1, w_2, .., w_N\}$ where $w_n : X \to X$ real analytic

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- an **attractor** is a compact set $A \in K$ s.t. $\lim_{k\to\infty} W^k(S) = A$ for all bounded nonempty $S \subset X$

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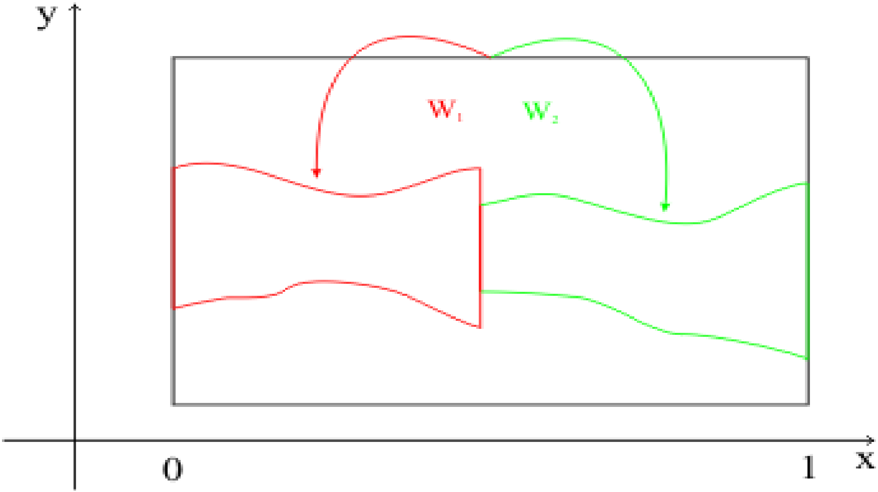
- examples include analytic functions, arcs of wavelets, fractal interpolation functions, Wieirstrass nowhere differentiable functions, and functions defined by Julia sets.

Analytic fractal functions



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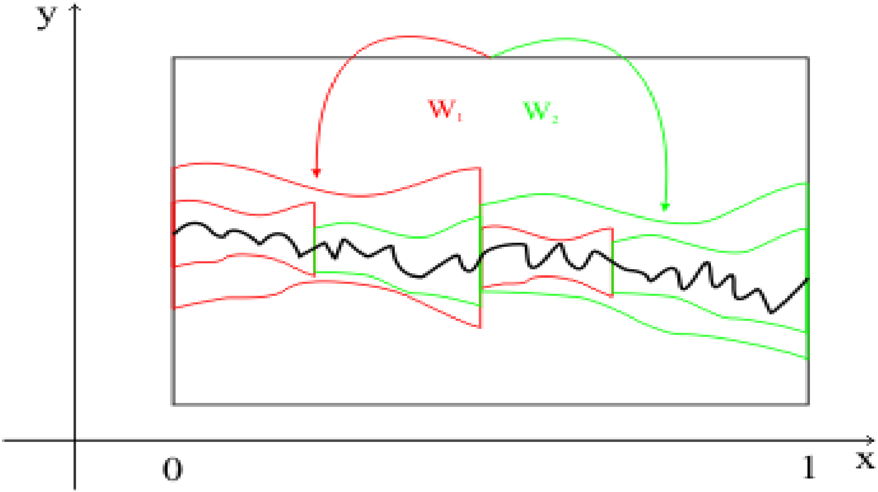

All real analytic functions are analytic fractal functions

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- **PROOF**: If f is strictly increasing with $f'(x)$ not varying too much, IFS is

$$\{X; w_1 = (\frac{x}{2}, f(\frac{f^{-1}(y)}{2})), w_2 = (x/2 + 1/2, f(\frac{f^{-1}(y)+1}{2}))\}.$$

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- Otherwise, first conjugate by $T(x,y) = (x, y + Cx)$ with C sufficiently large. ■

Examples: QUADRATIC, EXPONENTIAL

- The graph of $f : [0,1] \to \mathbb{R}, f(x) = x^2$ is the attractor of the analytic IFS

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- $\widehat{B}(\theta) := \cup_{k\in\mathbb{N}}\widehat{B}(\theta_1\theta_2...\theta_k)$ is a **continuation** of A.

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- Let A be attractor of an analytic IFS $\mathcal{W} = \{X; w_1, w_2\}$.
- **Theorem 2:** *If A is an arc of the graph $G(f)$ of a real analytic function $f : D \subset \mathbb{R} \to \mathbb{R}$, then $\widehat{B}(\theta)$ is the graph of an analytic continuation of f.*

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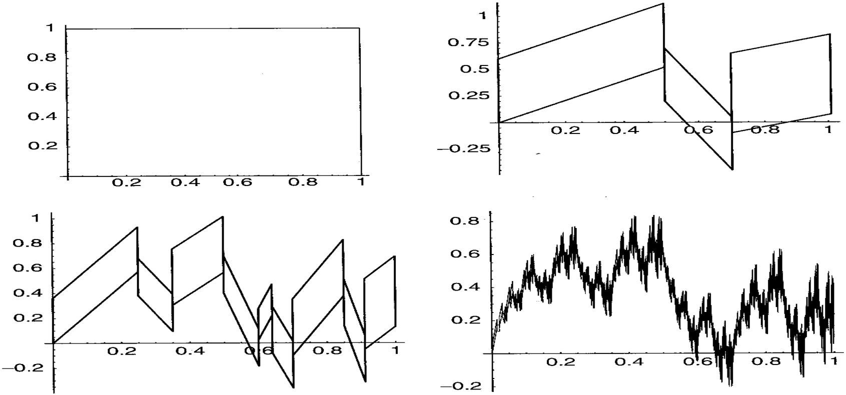
- If the w_is are real analytic then f is called an **analytic FIF**.

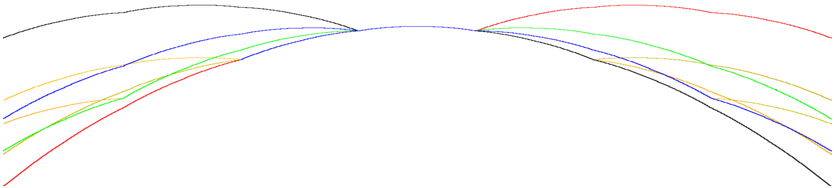
Analytic FIF may be rough, or rectifiable, or smooth, or analytic



Continuation of real analytic attractors

- **Theorem 2:** *If* A *is an arc of the graph of an analytic FIF* $f : D \subset \mathbb{R} \to \mathbb{R}$*, then* $\widehat{B}(\theta)$ *is the graph of* $f_\theta : I_\theta \to \mathbb{R}$*, which coincides with* f on D. Moreover, $\dim_H(G(f)) = \dim_H(\widehat{B}(\theta_1\theta_2...\theta_k)$ for all k.

Example of continuation of FIFs



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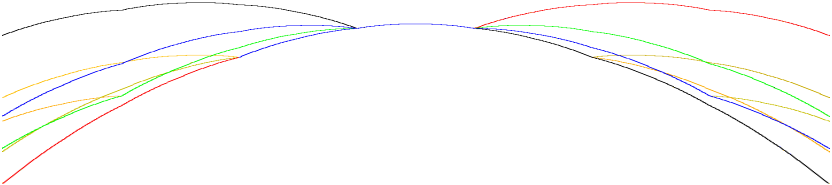

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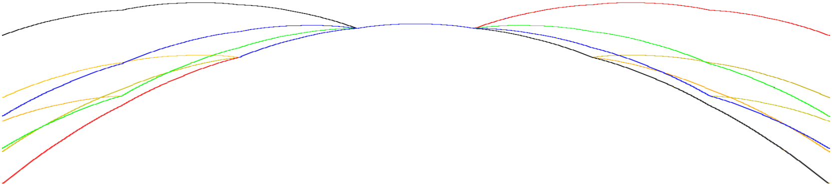

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- here $p = 0.3$ and the graph is of finite length

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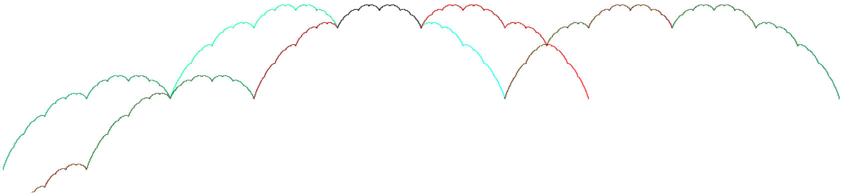

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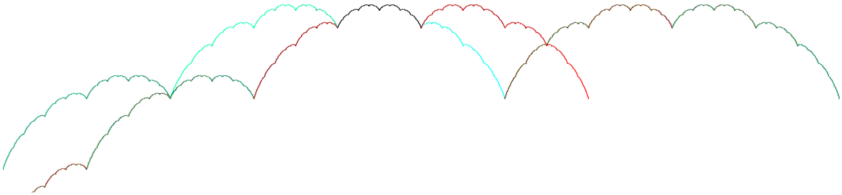
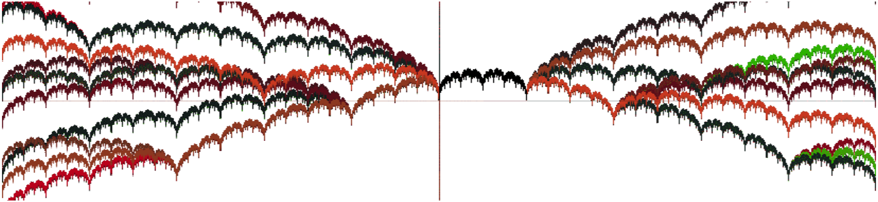


- same as the previous example, but now $p = 0.5$
- once differentiable except at countably many points

Example: fractal continuations



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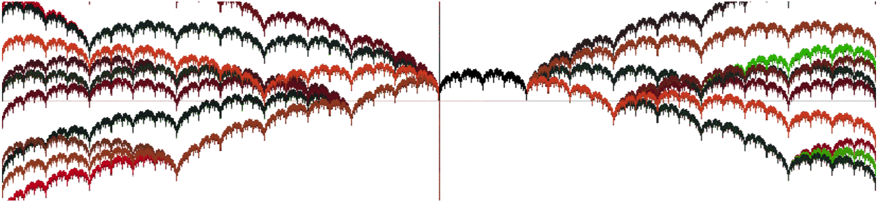

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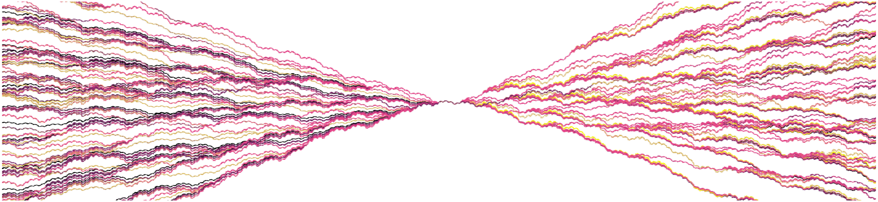
- nowhere differentiable and $dim(\widehat{B}(\theta)) = dim(A) > 1$

Fractal continuations



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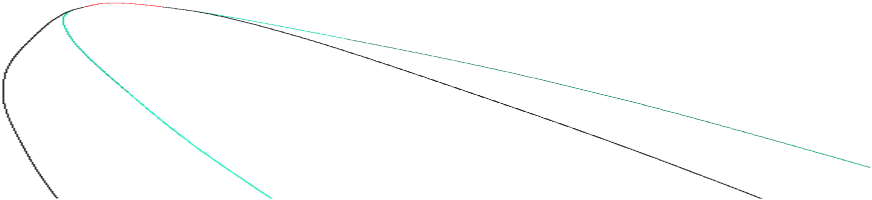
Examples: differentiable functions



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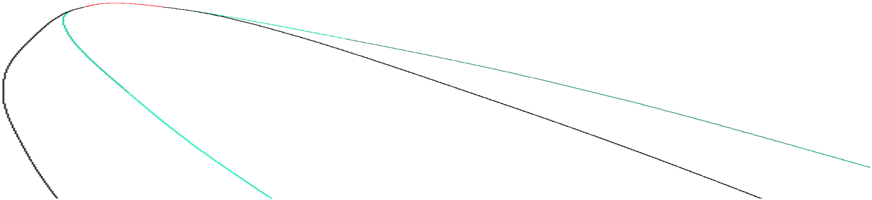

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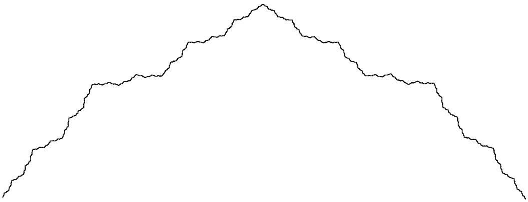
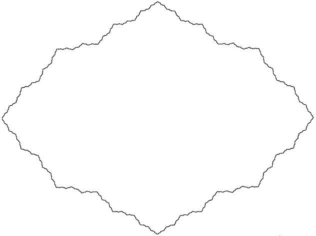
- Related to the Kigami triangle.

Examples: julia sets



- EXAMPLE (vi): Part of the Julia set J for $z^2 - 0.3$. Because of symmetries, also the attractor of $\{\mathbb{D} \subset \mathbb{C}; \sqrt{z+0.3}, -\sqrt{-z-0.3}\}$.

Examples: julia sets continued



- Continuation of EXAMPLE: in this example $\widehat{B}$ is a quasi-circle. Continuation is unique, equals the full Julia set.

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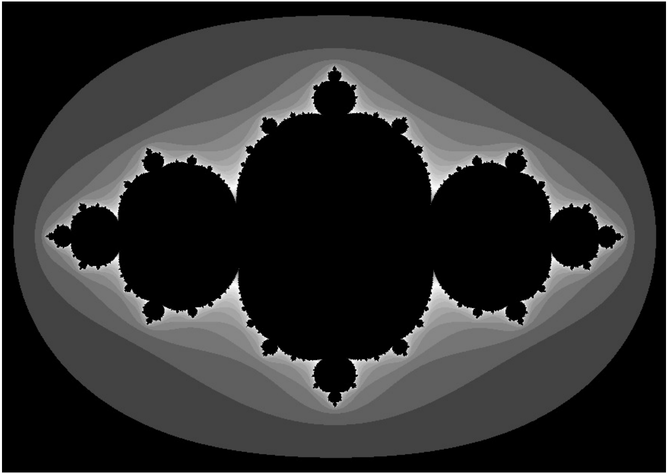
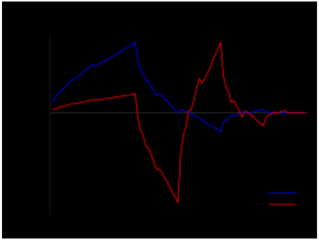


Examples: Daubechies wavelets



- These wavelets are piecewise fractal interpolation functions, attractors of analytic IFSs. Pieces have unique analytic continuations $\widehat{B}(\theta)$

Uniqueness of Fractal Continuations

- **Theorem 4** *If the graph* $G(f)$ *of* $f:[0,1]\rightarrow\mathbb{R}$ *is the (FIF) attractor of two different analytic IFSs, each of the form* $\{\mathbb{R}^2; (a_n x + h_n, F_n(x,y))\ n = 1,2,...,N\}$*, then the continuation associated with* 1111.... *is same for each.*

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- **PROOF**: Commutation and Wieirstass preparation theorem.

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- arXiv:1209.6100v2 (Nov 2012)

Uniqueness of Fractal Continuations

- **Theorem 5** *If the graph* $G(f)$ *of* $f : [0,1] \to R$ *is a (FIF) attractor of an analytic IFS of the form* $\{\mathbb{R}^2, (l_n(x), F_n(x,y), N = 2\}$, *and* $f(x)$ is infinitely differentiable, or $G(f)$ *has a tangent at all except countably many points, then the set of continuations is unique.*
- **PROOF:**The main steps:
- (i) Theorem A: f is Lipshitz (bootstrap argument).
- (ii) Theorem B: if $G(f)$ is rectifiable then $f'(x)$ is discontinuous only at points with two addresses.
- (iii) If two IFSs, Theorem B implies set of points with double addresses is same.
- (iv) Theorem A implies interpolation nodes are the same. Then Wieirstrass.
- arXiv:1209.6100v2 (Nov 2012)
- **Conjecture:** *continuations of analytic FIF is unique.*

Fast basins

- the **fast-basin** is

$$\widehat{B} := \{x \in X : \exists k \in \mathbb{N} \text{ s.t.} W^k(\{x\}) \cap A \neq \emptyset\} = \cup_{\theta \in \{1,2\}^\infty} \widehat{B}(\theta)$$

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- Reverse invariance: $W^{-1}(\widehat{B}) = A \cup W^{-1}(\widehat{B})$

Picture shows fast basin for Sierpinski

Inherited properties

- **Theorem 6** (with K. Lesniak) The following properties are inherited from A to $\widehat{B}$ and $\widehat{B}(\theta)$: (i) (arc-) connected; (ii) empty interior; (iii) σ-porous (*); (iv) topological covering dimension; (v) Hausdorff dimension (*).

Inherited properties

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- (*) means : assumes $\mathcal{W}$ is bi-lipshitz.

Branched) fracta(*l-mani*)folds

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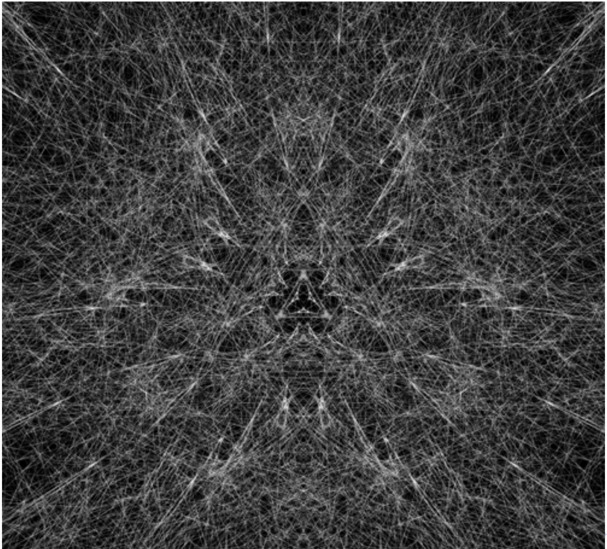
Fractal manifolds

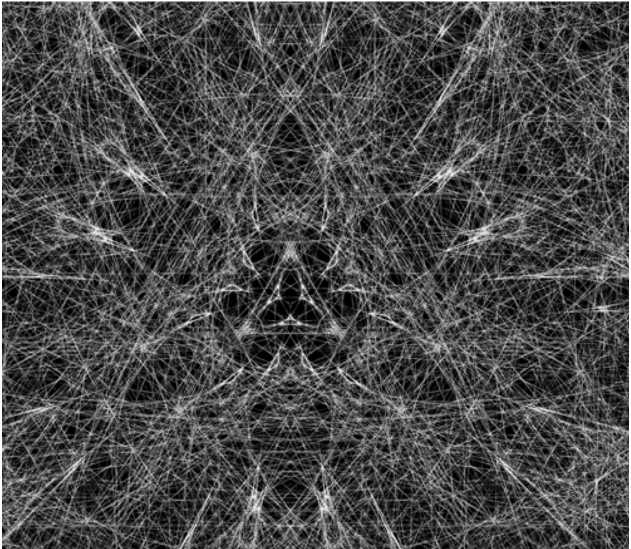
- fractal manifold using fern

Fast basins and fractal manifolds

Kigami fast-basin



Kigami fast-basin



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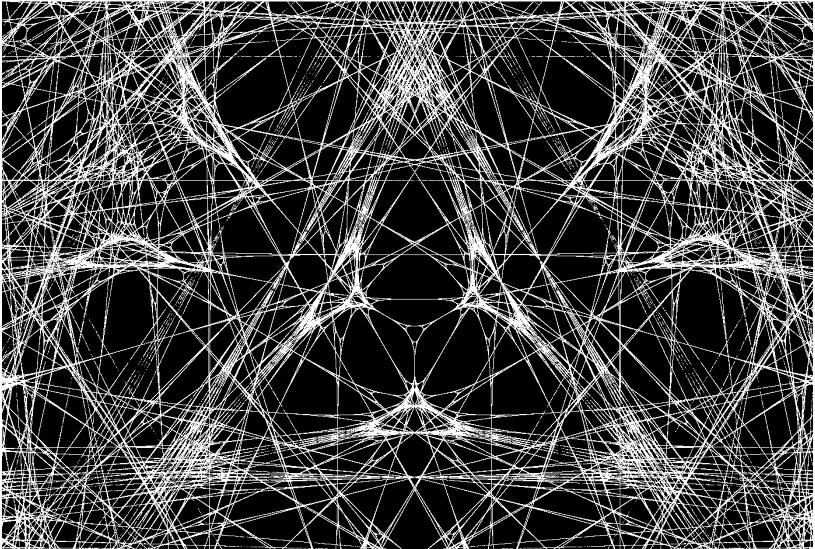

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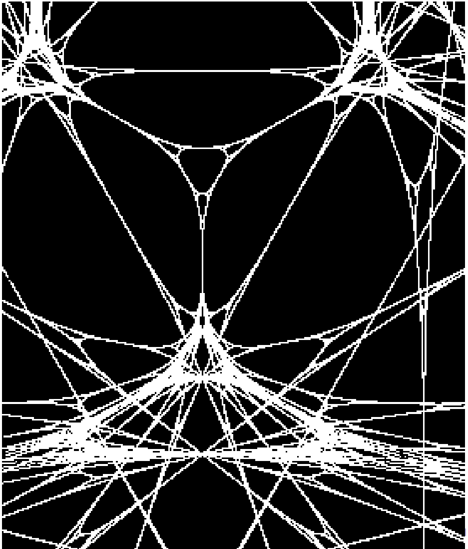

Kigami fractal manifold