Bernoulli Convolutions and Branching Dynamical Systems

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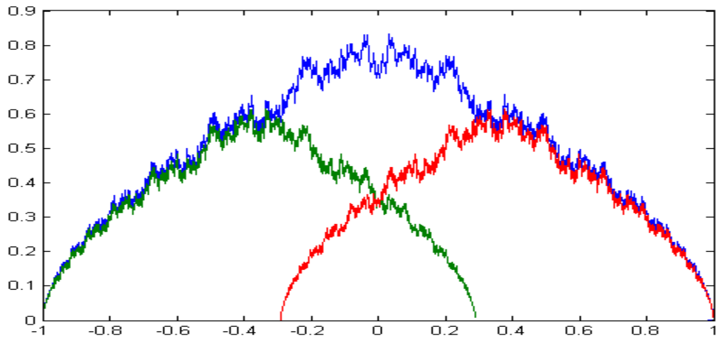
Christoph Bandt

1. Pisot and non-Pisot case
2. Branching Dynamical Systems
3. Computer experiments
4. Smooth cases

many questions – few answers



I apologize for not mentioning many beautiful results of the organizers and of the audience.

1 Pisot and non-Pisot case

- the measure ν
- the Erdös problem
- growth rate of inverse functions
- distribution of values for inverse iteration

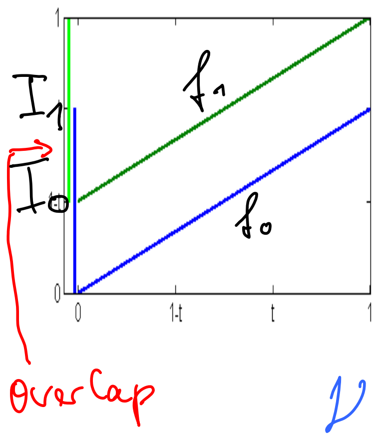
Def (Bernoulli convolution ν)

Take $t \in (1/2, 1)$ and similarity maps

$$f_0(x) = tx \qquad \text{on } I = [0,1]$$
$$f_1(x) = tx + 1 - t$$

The self-similar set for $\{f_0, f_1\}$ is

$$I = f_0(I) \cup f_1(I) = I_0 \cup I_1$$



ν is the self-similar measure on I given by f_0, f_1 and weights $p_0 = p_1 = 1/2$.

$$\nu = \frac{1}{2}\,\nu \circ f_0^{-1} + \frac{1}{2} \cdot \nu \circ f_1^{-1}$$

Old Problem: for which $t \geq 1/2$ does ν have a density function?

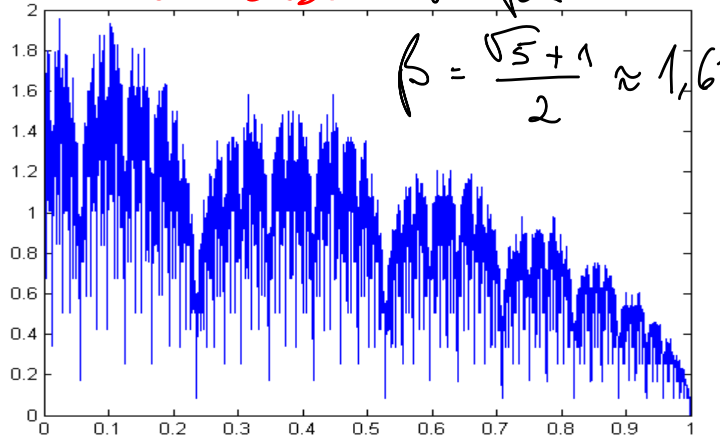
Erdős 1939: if $\beta = \frac{1}{t}$ is a Pisot number (algebraic integer, all conjugate roots β_i fulfil $|\beta_i| < 1$) there is no density function.

Wintner 1935: for $\beta = \sqrt[k]{2}$, $k = 0, 1, 2, \dots$ there is a density

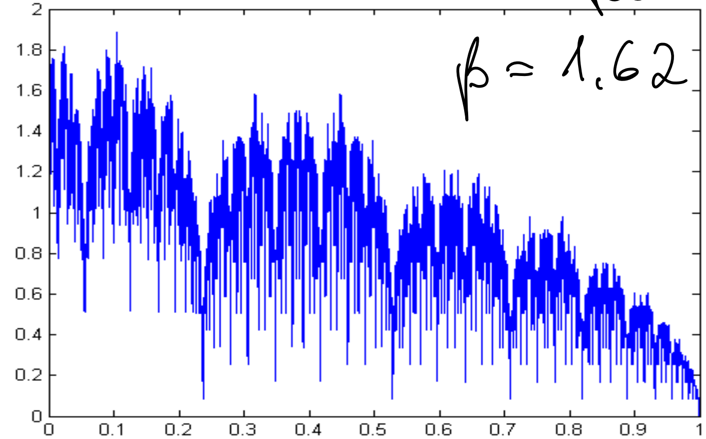
Garsia 1962: also for some algebraic integers β

Solomyak 1995: for Lebesgue almost all $\beta \in (1,2)$ there is a density.

Pisot case: ν for $\beta = \frac{\sqrt{5}+1}{2} \approx 1{,}618$



non-Pisot case: ν for $\beta = 1{,}62$

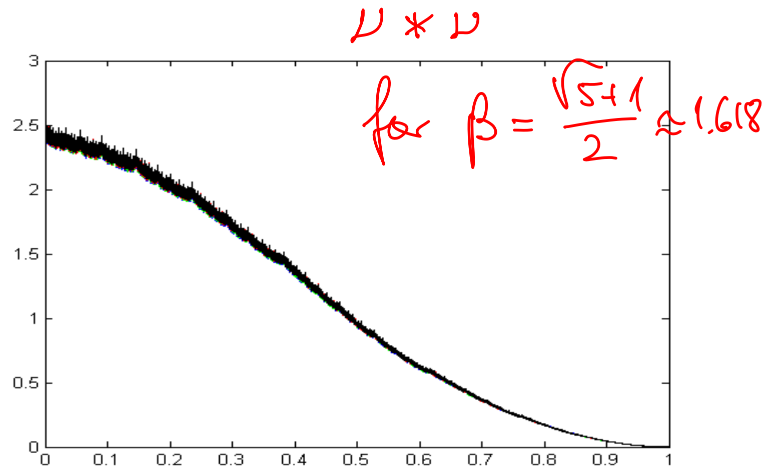


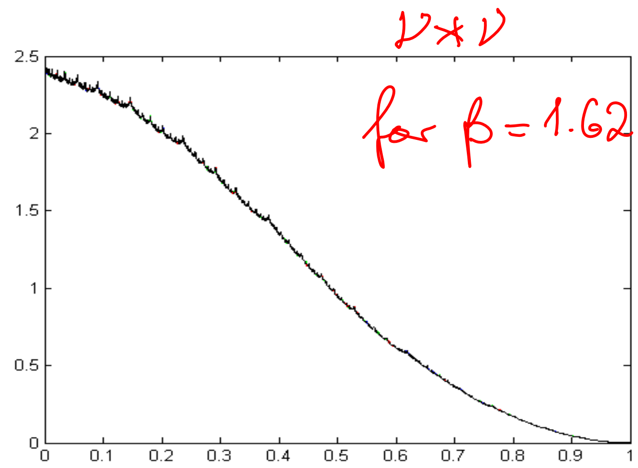
Convention: since measures are symmetric, we show only their right half, and we normalize so that the support is again $[0,1]$.

Construction: histogram of orbit of random iteration of f_0, f_1 (IFS).

We used a more accurate method described below.

For the correlation measure $\nu * \nu$, the difference is easier to recognize.

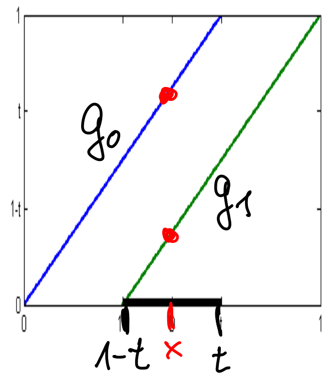




To see a real difference, we consider the inverse functions of f_i

$$g_0(x) = \beta x \ , \quad x \in I_0 = [0, t]$$
$$g_1(x) = \beta x + 1 - \beta \ , \quad x \in I_1 = [1-t, 1]$$

where $\beta = \frac{1}{t} \in (1, 2)$.



$G = \{g_0, g_1\}$ is a multivalued map.

For $x \in D = I_0 \cap I_1$, $G(x)$ consists of two points.

How many values do we obtain if we n times apply G?

How does $|G^n(x)|$ depend on x?

Actually, $|G^n(x)|$ for large n is a proxy for a density $d(x)$ of ν.

Theorem (Feng + Sidorov, Kempton)

$$\lim_{n\to\infty} \sqrt[n]{|G^n(x)|} = \gamma$$

for Lebesgue almost all x.

De-Jun Feng + Sidorov 2009

Pisot case: $\gamma < \frac{2}{\beta}$

T. Kempton 2012

when density exists: $\gamma = \frac{2}{\beta}$

Remark. For Lebesgue measure λ on I,

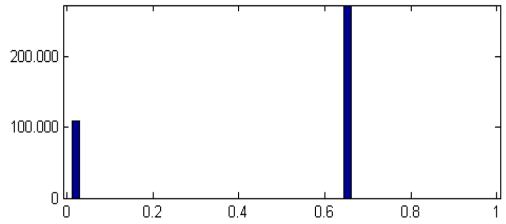
$$\lambda \circ G^{-1} = \lambda g_0^{-1} + \lambda g_1^{-1} = \frac{2}{\beta} \cdot \lambda$$

Remark. Even for $\beta = \frac{\sqrt{5}+1}{2}$, the difference of γ and $\frac{2}{\beta}$ is small:

$$\gamma \approx 0{,}996 \cdot \frac{2}{\beta}$$

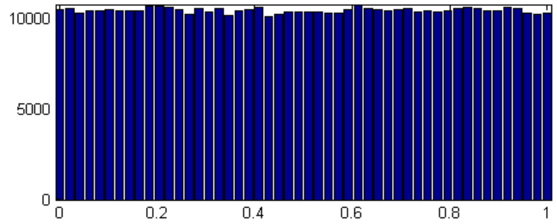
In the theorem, each $y \in G^n(x)$ is counted with its multiplicity.

Remark. A much more obvious difference between Pisot and non-Pisot case is the distribution of the values in $G^n(x)$ for fixed n.



Histogram of the values $G^n(x)$, $n=60$ for the Pisot case $\beta = \frac{\sqrt{5}+1}{2}$.

Two values with large multiplicity.



Histogram of the values $G^n(x)$, $n=60$ for the non-Pisot case $\beta = 1{,}62$.

Values seem to be equidistributed.

Prop. For Pisot β there is a C such that for all n and x, $G^n(x)$ has at most C different points.

Prop. ν has a density if and only if for Lebesgue almost all x, the distribution of the points of $G^n(x)$ tends to the equidistribution on $[0,1]$ for $n \to \infty$.

② Branching Dynamical Systems

Let $J_1, \dots, J_m$ be subsets of a set J, and $g_i : J_i \to J$ mappings.

Then $G = \{g_1, \dots, g_m\}$ is called BDS.

We put $G(x) = \{g_i(x) \mid x \in J_i\}$

and $G(A) = \bigcup_{a \in A} G(x)$ for sets A.

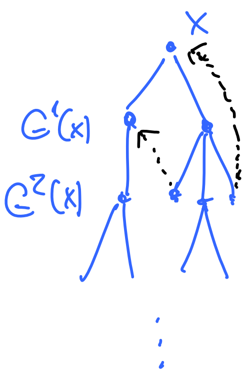
Points in $G^n(x) = \underbrace{G(G \dots (G(x)) \dots)}_{n}$ are called successors of x in generation n.

The forward orbit of x,

$$G^\infty(x) = \bigcup_{n=0}^{\infty} G^n(x)$$

has a tree-like structure.

Cycles can appear.



Question: Determine the periodic orbits of G – the x for which $G^\infty(x)$ consists of a finite number of points.

Pisot case: many / else: very few

Jordan, Shmerkin, Solomyak 2011, Th. 1.5

Talk of R. Zeller

Let $N^n(x) = |G^n(x)|$ be the number of n-th generation successors of x, counted with multiplicity.

If $\gamma(x) = \lim_{n\to\infty} \sqrt[n]{N^n(x)}$

exists, it is called growth factor of successors of x.

A finite measure μ on J is invariant for G, with growth factor γ, if

$$\gamma \cdot \mu(B) = \sum_{i=1}^{m} \mu(g_i^{-1}(B))$$

for measurable sets $B \subseteq J$.

γ is a kind of average of all $\gamma(x)$.

Questions. For which G does an invariant measure μ exist?

Under which conditions will $\gamma(x)$ exist and $\gamma(x) = \gamma$ for μ-almost all x?

Study the ergodic and mixing properties of tree-like orbits.

When $G^n(x)$ is considered as a probability measure $\mu_n(x)$, will there be some limit for $n \to \infty$?

Does the limit depend on x?

How is it related with μ?

The BDS G acts as an operator on the piecewise constant functions $S^n(x) = N^n(x) \cdot \gamma^{-n}$ (standardized number of successors).

$$S^{n+1}(x) = \tilde{G}(S^n(x)) = \frac{1}{\gamma} \cdot \sum_{x \in J_i} S^n(g_i(x))$$

This action can be extended to functions $h: J \to \mathbb{R}$.

$$\tilde{G}h(x) = \frac{1}{\gamma} \sum_{x \in J_i} h(g_i(x))$$

What properties does this operator have on function spaces $\mathcal{L}_1, \mathcal{C}, \mathcal{C}^k$?

Prop. If μ is invariant, then $\tilde{G}$ is a positive continuous operator on $\mathcal{L}_1(\mu)$, and $\|\tilde{G}h\|_1 = \|h\|_1$ for $h \geq 0$.

(straightforward, cf. Kempton)

Bernoulli convolutions are a nice testbed for all these questions.

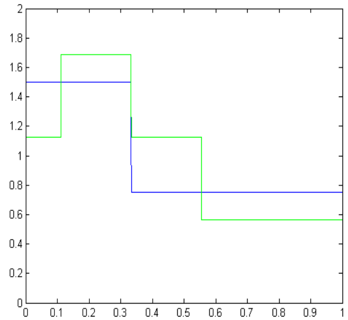
③ Computer experiments

For $\beta = 3/2$, we determine the standardized number of successors for 10^5 equally spaced x.

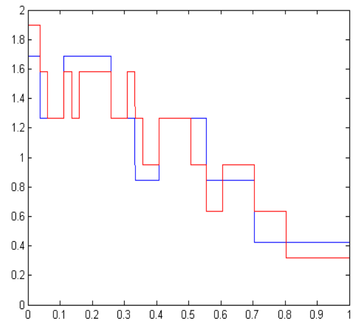
Starting with $h_0 = 1$, we iterate $\tilde{G}$:

$$h_n = \tilde{G} h_{n-1} \qquad n = 1, 2, \dots, 30$$

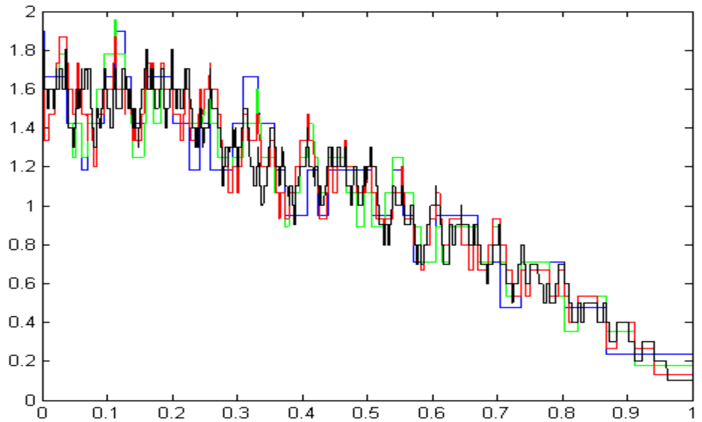
$h_n = S^n(x)$ offspring in generation n



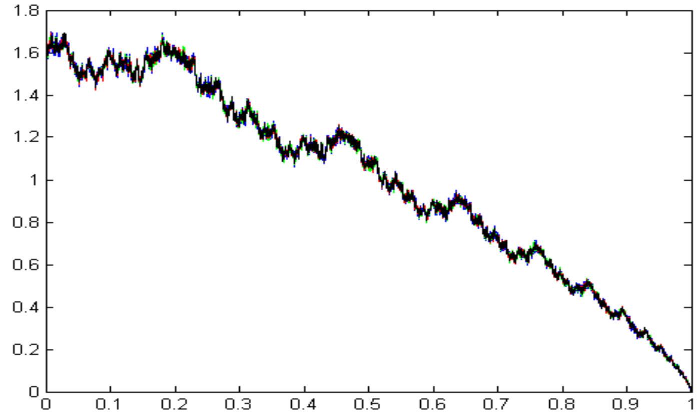
h_1, h_2



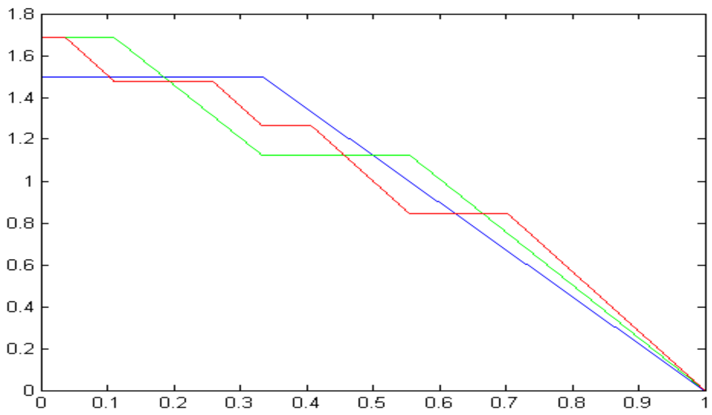
h_3, h_4



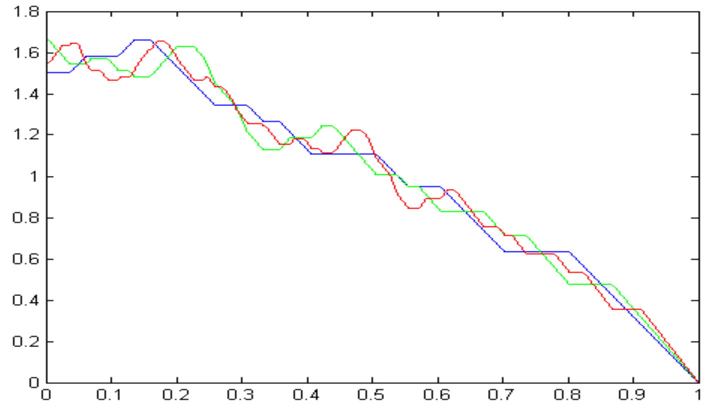
h_7 up to h_{10}



after 30 iterations: ν



Starting with a linear function h_0



Iteration remains within $C[0,1]$.

Prop. The following subspaces of $\mathcal{L}_1(\lambda)$ are invariant under $\tilde{G}$.

$$\mathcal{C}_0 = \{ h \mid \text{continuous},\ h(1)=0 \}$$

$$\mathcal{C}_0^k = \{ h \mid k \text{ times differentiable},\ h^{(j)}(1) = 0 \text{ for } j=0,..k \}$$

Prop. IFS on $[0,1]$, $f_i(x) = tx + v_i$ with inverse functions

$$g_i(x) = \beta\,(x - v_i), \quad \beta = \frac{1}{t}.$$

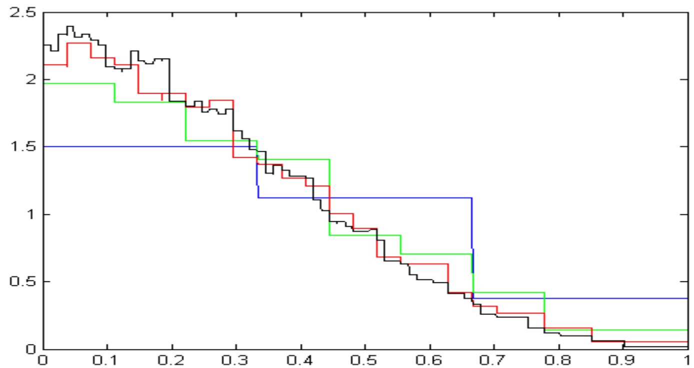
Then the operator $\tilde{G}$ is just the restriction of Hutchinson's operator to $\mathcal{L}_1(\lambda)$.

Remember: Hutchinson's operator acts on the space of finite measures κ by

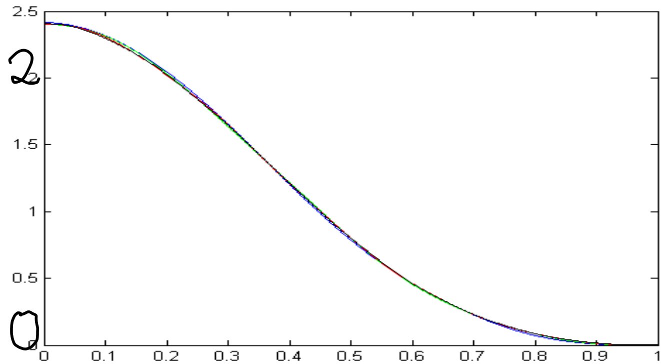
$$H\kappa = \frac{1}{m}\cdot\sum_{i=1}^{m} \kappa\circ f_i^{-1} = \frac{1}{m}\sum_{i=1}^{m} \kappa\circ g_i$$

It is contractive with respect to transport distance.

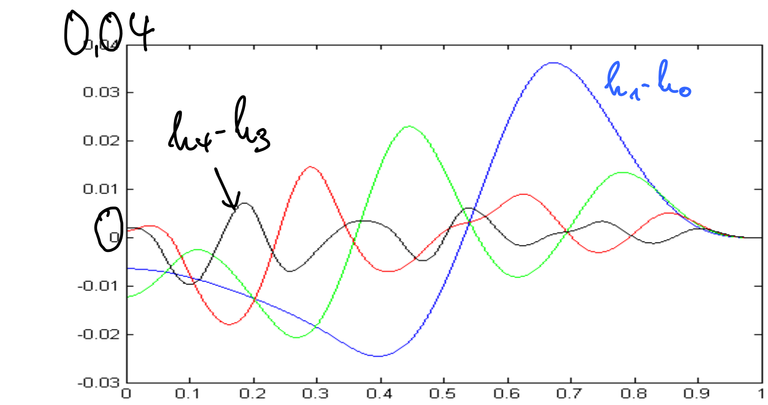
For any probability measure κ_0, the sequence $\kappa_n = H^n \kappa_0$ converges to the invariant measure ν of the IFS. This also holds for every h_0 and $h_n = \tilde{G}^n h_0$, but the limit need not be in $\mathcal{L}_1$.



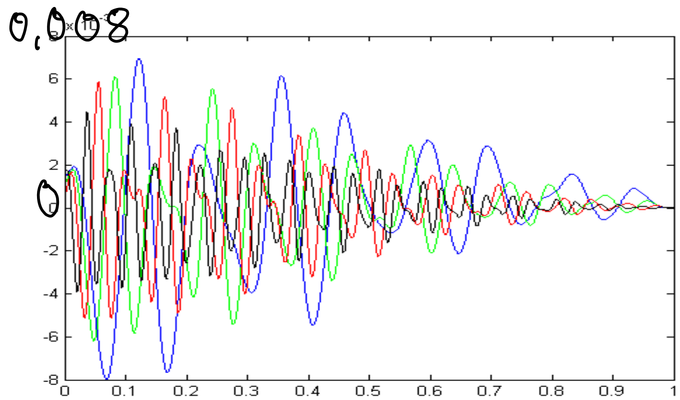
Approximation of $\rho = \nu * \nu$ for $\beta = 3/2$.



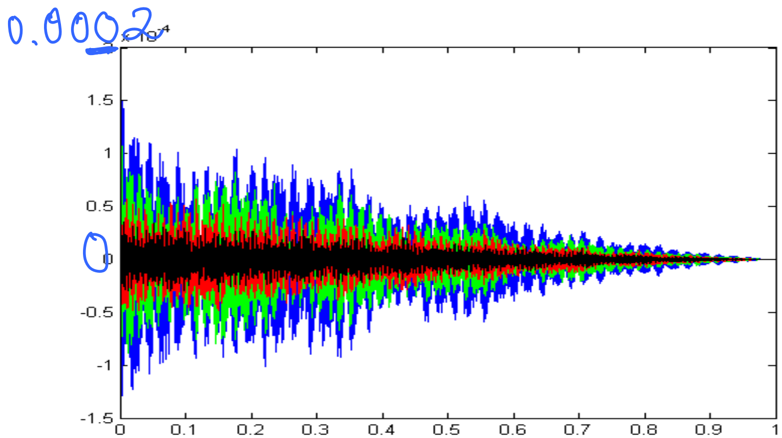
h_1 up to h_4 for a smooth starting function



Differences between first iterations



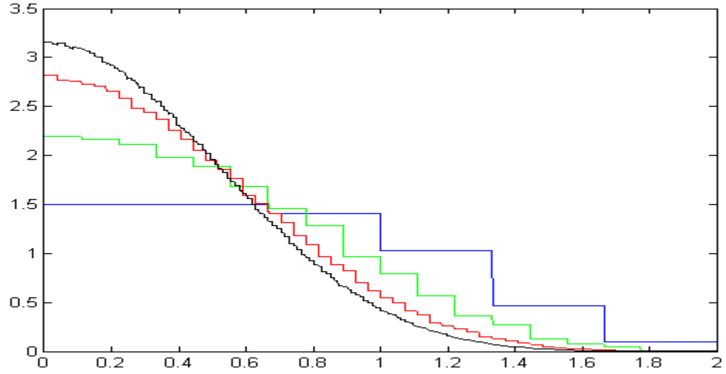
Differences between iterations h_5 to h_8



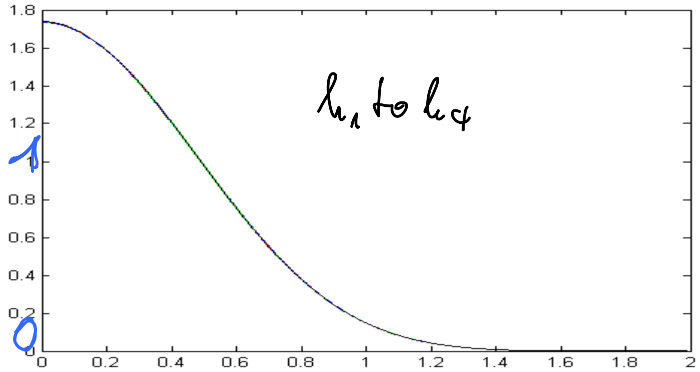
subsequent differences

Differences become small,
but perhaps they accumulate.

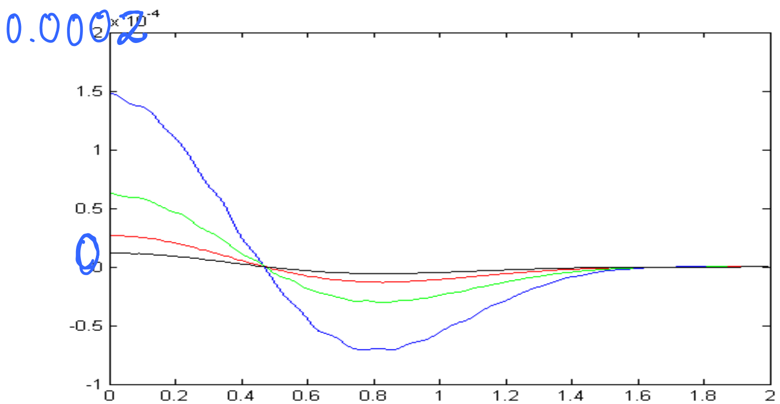
Let us consider the still more smooth measure $g * g$ for $\beta = 3/2$.



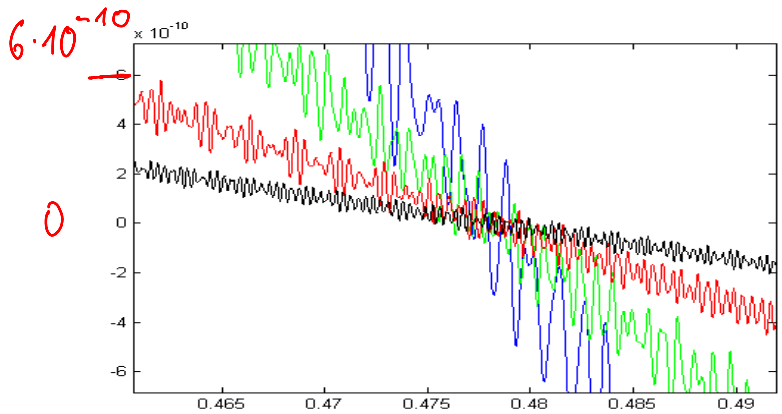
Successor functions S^1 to S^4 for $g*g$



$g*g$: $h_0 = (1+\cos \pi x)^\alpha$ is almost perfect



Differences of iterations



High-frequency corrections exist,
but they decrease very fast.

④ Smooth cases

Prop. (cf. Wintner 1935)
If ν_β admits a density, then $\nu_{\sqrt{\beta}}$ also has a density since $\nu_{\sqrt{\beta}} = \nu_\beta * \nu_\beta \circ \beta$

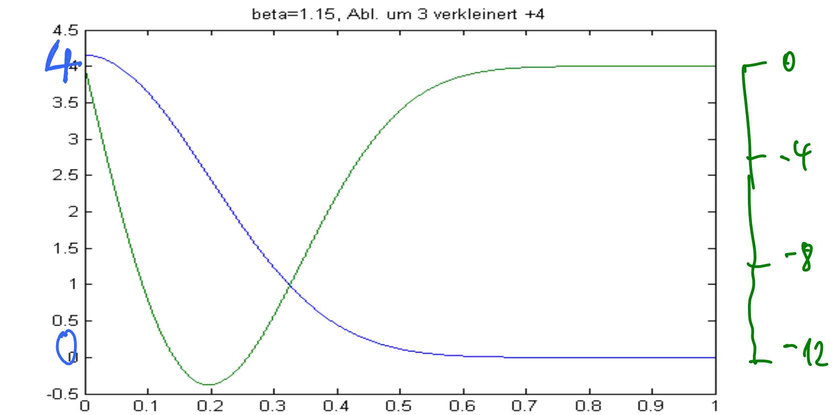
where $\nu_\beta \circ \beta(A) = \nu_\beta(\beta A)$

Thus ν_β becomes more smooth (more continuous, more derivatives) when β comes nearer to 1.
However, there is a small result into the other direction:

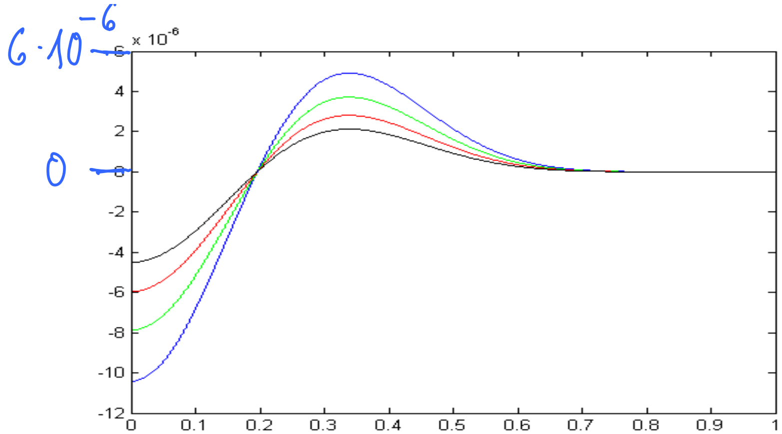
Smoothing Lemma. Let ν_β have no density, and let β not belong to a certain countable set B.
Then $\nu * \nu$ does not have a bounded density.

This motivates the study of smooth cases. In the following, we take $\beta = 1{,}15$ and the correlation measure

$$\varrho = \nu * \nu .$$



density h of ν and its derivative h'



Differences $h_{n+1} - h_n$ of approximations

Observation: up to a constant, the differences $h_{n+1} - h_n$ approach the second derivative h'' of h.
$\tilde{G}$ becomes a $\|\cdot\|_\infty$-contraction on certain subspaces of $C[0,1]$ and even $C^k[0,1]$.

Problem. Prove the geometrical convergence of the corrections $h_{n+1} - h_n$ to zero, by means of a recursive estimate, and computer support to find the proper starting function.

Why is $h_{n+1} - h_n$ near to h'' ?

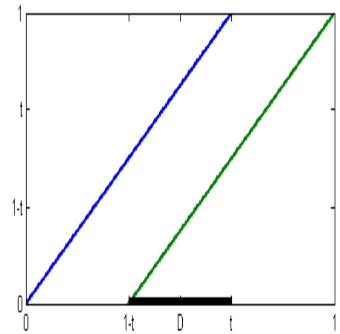
Prop. IFS $f_i(x) = tx + v_i$, arbitrary p_i, $i = 1 \dots m$

$\beta = 1/t$, $G = \{g_1 \dots g_m\}$ inverse maps.

If the self-sim. measure ν has a density h with k derivatives, then $h^{(j)}$ is an eigenvector of $\tilde{G}$ with eigenvalue β^{-j} $j = 0, \dots, k$.

Question: What is the spectrum of $\tilde{G}$?

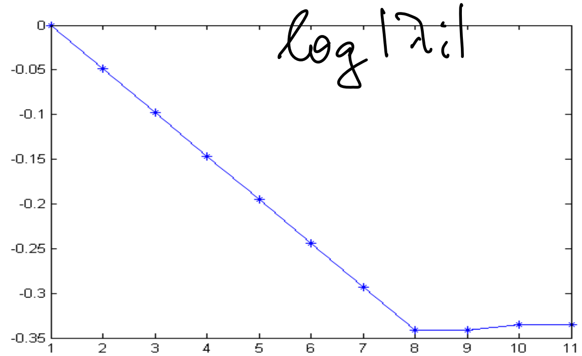
Without factor $1/8$, $\tilde{G}$ is just a centrally symmetric 0-1-matrix.



Experiment: Model $\tilde{G}$ by $n \times n$-matrix of zeros and ones. (Matlab, $n = 1500$)

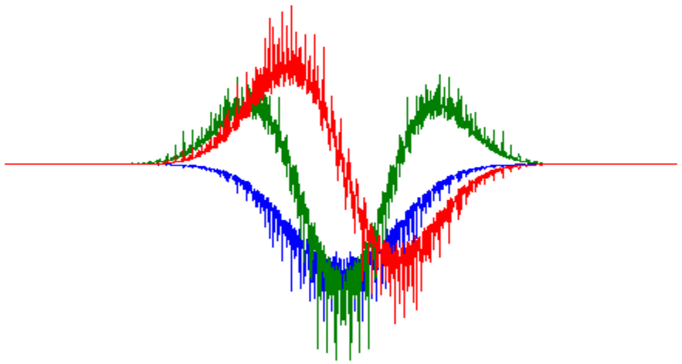
Determine the leading eigenvalues and corresponding eigenvectors.

Take ν for $\beta = 1{,}05$.



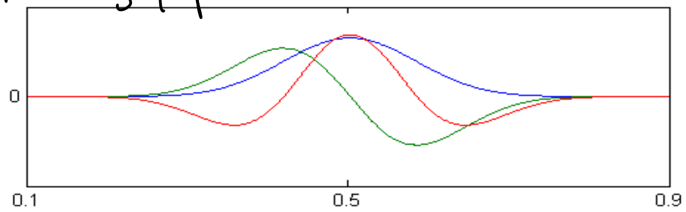
For $i = 1, \dots, 7$, λ_i is real and $\lambda_i = \beta^{1-i}$

(error $\leq 10^{-4}$)

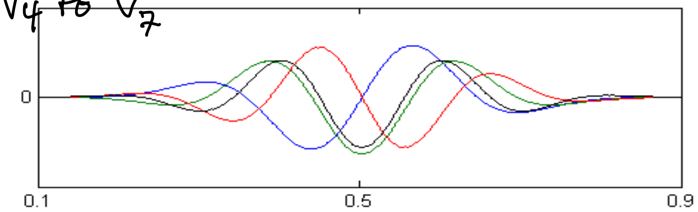


The first 3 eigenvectors in raw form

V_1 to V_3, polished



V_4 to V_7



Conjecture. Let $\beta_1 = 1.32$. There are numbers $\beta_1 > \beta_2 > \beta_3 > \dots > 1$ so that $\tilde{G}$ for $\beta \in (\beta_{n+1}, \beta_n)$ has the leading eigenvalues $1, \beta^{-1}, \dots \beta^{1-n}$, with eigenfunctions similar to those of Sturm-Liouville operators.

Difficulties of the Erdős problem

- Definition of ν recursive, not explicit
- Action of $\tilde{G}$ on $\mathcal{L}_1$, not on $\mathcal{L}_2$
- densities h are $\mathcal{C}^k$, not $\mathcal{C}^\infty$

Still, seems solvable.