Patterns generation problems arising in multiplicative integer systems

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1 Introduction

1.1 Some known results

Multiple ergodic average:

Let (X, T) be a topological dynamical system and $2 \leq l \in \mathbb{N}$ be a positive integer. The **multiple ergodic average**

$$\frac{1}{n} \sum_{k=1}^{n} f_1(T^k x) f_2(T^{2k} x) \cdots f_l(T^{lk} x),$$

where $f_1, \ldots, f_l$ are l given continuous functions.

H. Furstenberg, *J.d' Analyse Math. (1977)* : On the study of Szemerédi's theorem.

J. Bourgain, *J. Reine. Angew. Math.* (1990): For almost sure convergence.

B. Host and B. Kra, *Ann. Math. (2005)* : For L^2-norm convergence.

A. H. Fan, L. M. Liao and J. H. Ma, *Monatshefte für Mathematik* (2011) : If

$$f_1(x) = f_2(x) = \cdots = f_l(x) = x_1,$$

and $X \subseteq \mathbb{D}$, where

$$\mathbb{D} = \{+1, -1\}^{\mathbb{N}}.$$

Define

$$Y_\alpha = \left\{ x \in \mathbb{D} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{lk} = \alpha \right\},$$

then $\forall \alpha \in [-1, 1]$

$$\dim_H Y_\alpha = 1 - \frac{1}{l} + \frac{1}{l} H(\frac{1+\alpha}{2}),$$

where

$$H(t) = -t \log_2 t - (1-t) \log_2(1-t).$$

Let

$$X \subseteq \mathbb{E} = \{0,1\}^{\mathbb{N}},$$

and

$$Z_\alpha = \left\{ x \in \mathbb{E} : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} \cdots x_{lk} = \alpha \right\},$$

with *simplified form* $l = 2$ and $\alpha = 0$, that is

$$\widehat{Z}_0 = \{x \in \mathbb{E} : \ x_n x_{2n} = 0 \ \forall n\},$$

and show that

$$\dim_B(\widehat{Z}_0) = \frac{1}{2\log 2} \sum_{n=1}^{\infty} \frac{\log a_n}{2^n} \approx 0.8242936...,$$

R. Kenyon, Y. Peres and B. Solomyak, Ergodic Theory Dynam. Sys. (2011):

$$\dim_H(\widehat{Z}_0) = -\log_2 p = 0.81137...,$$

where

$$p^3 = (1-p)^2, 0 < p < 1.$$

Furthermore,

$$\dim_H \widehat{Z}_0 < \dim_B \widehat{Z}_0.$$

Y. Peres, J. Schmeling, S. Seuret and B. Solomyak, (2012): Consider

$$\mathbb{E}_m = \{0, \ldots, m-1\}^{\mathbb{N}}, \Omega \subseteq \mathbb{E}_m.$$

Let

$$S = \langle p_1, \ldots, p_J \rangle$$

be the semigroup generated by distinct primes $p_1, \ldots, p_J$

$$Z_{\Omega}^{(S)} = \{x \in \mathbb{E}_m : \ x|_{iS} \in \Omega \ \forall i, \ (i, S) = 1\},$$

they present the **Minkowski dimension formula** and **variational principle for Hausdorff dimension** of $Z_{\Omega}^{(S)}$.

Remark :

(i). **Different approach** on for some multi-dimensional systems.

(ii). Combinatorial method leads us to consider more general MS, e.g., coupled systems.

(iii). Based on the previous work of **patterns generation problems** for $\mathbb{Z}^d$ SFT.

1.2 Set up

(**A. H. Fan, R. Kenyon, L. M. Liao, J. H. Ma, Y. Peres and B. Solomyak, J. Schmeling and S. Seuret**) Consider

$$\mathbb{X}_2^0 = \left\{(x_1, x_2, \cdots) \in \{0,1\}^{\mathbb{N}} : x_k x_{2k} = 0, \forall k \geq 1\right\};$$

$$\mathbb{X}_{2,3}^0 = \left\{(x_1, x_2, \cdots) \in \{0,1\}^{\mathbb{N}} : x_k x_{2k} x_{3k} = 0, \forall k \geq 1 \right\}.$$

Goal : Compute $h(\mathbb{X}_2^0)$ or $h(\mathbb{X}_{2,3}^0)$.

Note :

$$\dim_M(\mathbb{X}) = \frac{1}{\log N} h(\mathbb{X}),$$

where N is the number of the symbols of the system $\mathbb{X}$.

1.3 Three types multiple shifts

Multi-dimensional system :

$$\mathbb{X}^0_{2,3} = \Big\{(x_1, x_2, \ldots) \in \{0,1\}^{\mathbb{N}} : x_k x_{2k} x_{3k} = 0,\ k \geq 1\Big\}.$$

Coupled systems :

$\mathbb{X}^A_2 = \{(x_1, x_2, \ldots) \in \Sigma_A : x_k x_{2k} = 0,\ k \geq 1\}$, i.e.,

$$\mathbb{X}^A_2 = \mathbb{X}^0_2 \cap \Sigma_A.$$

Multi-dimensional coupled systems :

$$\mathbb{X}_{2,3}^{A} = \{(x_1, x_2, \ldots) \in \mathbf{\Sigma}_A : x_k x_{2k} x_{3k} = 0,\ k \geq 1\},$$
ie.,

$$\mathbb{X}_{2,3}^{A} = \mathbb{X}_{2,3}^{0} \cap \mathbf{\Sigma}_A.$$

1.4 The approach of Fan, Liao and Ma

For $k \geq 1$,

Z_k : the blank lattice of k cells in $\mathbb{Z}^1$;

M_k : the numbered lattices of the first k elements in $\mathbb{M}_2$ on Z_k;

iM_k : the numbered lattices of the first k elements in $i\mathbb{M}_2$ on Z_k;

$$\mathcal{N}(2^n) = \bigcup_{i \in \mathcal{I}, 1 \leq i \leq 2^n} iM_{k_n(i)},$$

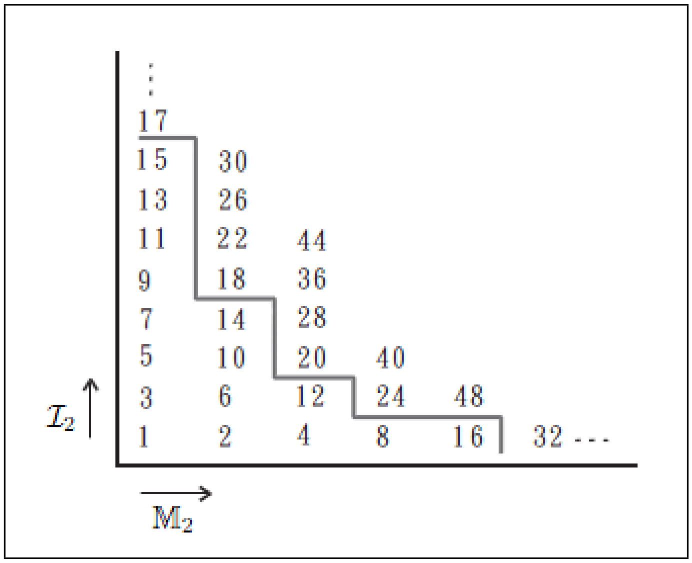

Figure 1: $\mathcal{I}_2$ and $\mathbb{M}_2$

where $\mathcal{N}(m) := \{k \in \mathbb{N} : 1 \leq k \leq m\}$ and $k_n(i) = \max\left\{k : i2^k \leq 2^n\right\}$

Proposition : For integer $Q \geq 2$ and $n \geq 1$,

$$Q^n = (n+1) + n(Q-2) + (Q-1)^2 \sum_{k=1}^{n-1} kQ^{n-1-k}.$$

In particular,

$$2^n = (n+1) + \sum_{k=1}^{n-1} k2^{n-1-k}.$$

- $X_m = \left\{ \begin{array}{c} (x_1, \ldots, x_m) \in \{0,1\}^{\mathbb{Z}_m} : x_k x_{2k} = 0, \\ \text{for all } k \geq 1,\ 2k \leq m \end{array} \right\}.$

- $h(\mathbb{X}_2^0) = \lim_{m\to\infty} \frac{1}{m} \log |X_m|$

Constraint : $x_k x_{2k} = 0 \Leftrightarrow$ The forbidden set on Z_2 is 11.

Theorem : For any $Q \geq 2$, denote the multiplicative integer system

$$\mathbb{X}_Q^0 = \left\{ (x_1, x_2 \ldots) \in \{0,1\}^{\mathbb{N}} : x_k x_{Qk} = 0\ \forall k \geq 1 \right\},$$

then

$$h(\mathbb{X}_Q^0) = (Q-1)^2 \sum_{k=1}^{\infty} \frac{1}{Q^{k+1}} \log a_k.$$

1.5 Main ingredient of the study on $\mathbb{X}_2^0$

(I). **Identify the numbered lattice** M_k in Z_k from the given system.

(II). **Compute the numbers of copies** of independent admissible lattices of the same length.

(III). **Determine the set of all admissible patterns** $\mathbf{\Sigma}_k$, which can be generated on Z_k, and compute the number of $|\mathbf{\Sigma}_k|$.

2 Multi-dimensional systems

Goal : Study the entropy of MDSs.

2.1 Step (I)

Goal : Identify the admissible numbered and blank lattices determined by the constraint $x_k x_{2k} x_{3k} = 0$ in $\mathbb{X}^0_{2,3}$.

- **Grouping lattices :** $\mathbb{M}_{2,3} := \left\{ 2^k 3^l : k, l \geq 0 \right\}$;

- **Decomposition of** $\mathbb{N}$ **:**

$$\mathbb{N} = \bigcup_{i \in \mathcal{I}_{2,3}} i\mathbb{M}_{2,3}$$

243	486	972	1944	3888	7776
81	162	324	648	1296	2592
27	54	108	216	432	864
9	18	36	72	144	288
3	6	12	24	48	96
1	2	4	8	16	32

q_{27}	q_{33}	q_{40}	q_{47}	q_{55}	q_{64}
q_{19}	q_{24}	q_{30}	q_{36}	q_{43}	q_{51}
q_{12}	q_{16}	q_{21}	q_{26}	q_{32}	q_{39}
q_{7}	q_{10}	q_{14}	q_{18}	q_{23}	q_{29}
q_{3}	q_{5}	q_{8}	q_{11}	q_{15}	q_{20}
q_{1}	q_{2}	q_{4}	q_{6}	q_{9}	q_{13}

$\mathbb{M}_{2,3}$

Figure 2: $\mathbb{M}_{2,3}$

- **Leading number :** $\mathcal{I}_{2,3} = \{n \in \mathbb{N} : \ 2 \nmid n \text{ and } 3 \nmid n\}$

  $= \{6k+1, 6k+5\}_{k=0}^{\infty} = \{1, 5, 7, 11, \ldots\}$.

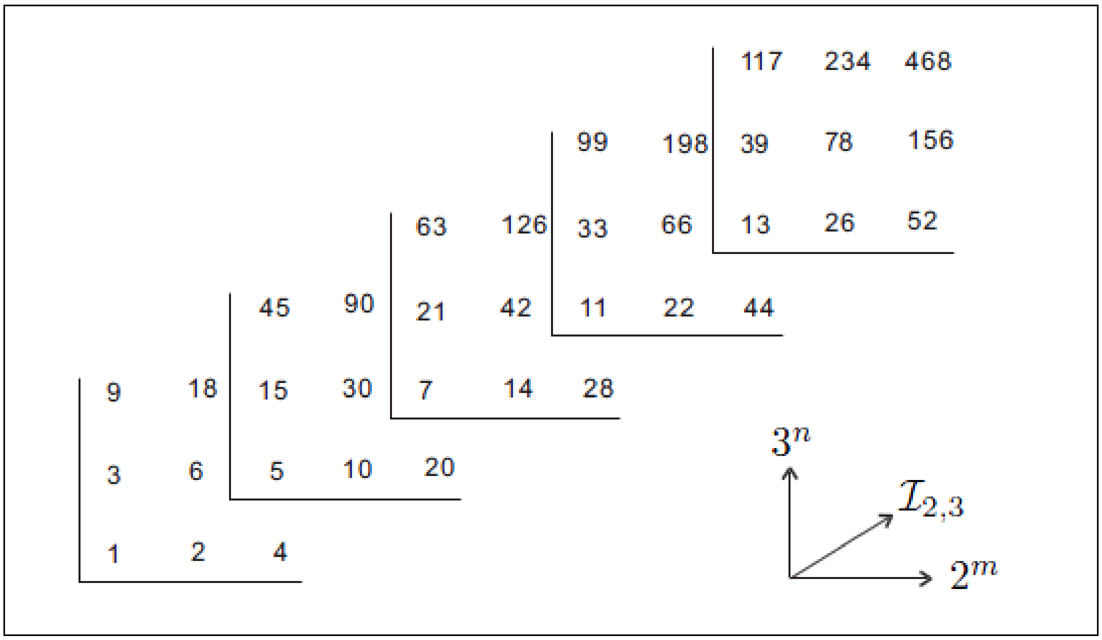

Figure 3: $\mathbb{N} = \bigcup_{i \in \mathcal{I}_{2,3}} i\mathbb{M}_{2,3}$

- **Decomposition of $\mathcal{N}(q_K)$:**

$$\mathcal{N}(q_K) = \bigcup_{i \in I_K(k)} iM_K,$$

  where $q_K = 2^m 3^n \in \mathbb{M}_{2,3}$.

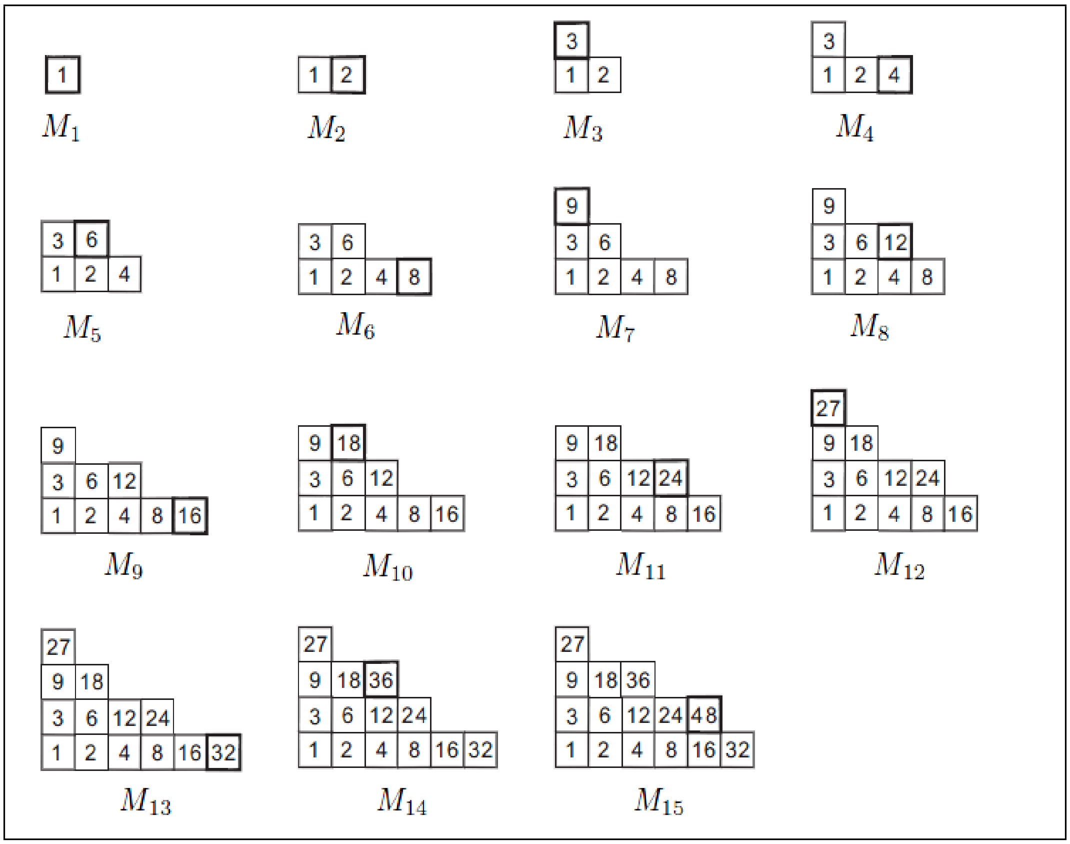

Figure 4: M_1 to M_{15}

- $I_K(k) = (\frac{q_K}{q_{k+1}}, \frac{q_K}{q_k}] \cap \mathcal{I}_{2,3}$.

- **The number of copies of M_k in $\mathcal{N}(q_K)$:** $\alpha_K(k) = |I_K(k)|$.

2.2 Step (II)

Goal : compute the numbers of copies of M_k for a given $\mathcal{N}(m)$

Proposition (Density of copies of M_k) : On $\mathbb{X}^0_{2,3}$ for an $k \geq 1$,

$$\lim_{K\to\infty} \frac{\alpha_K(k)}{q_K} = \beta_{2,3}\left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right),$$

where

$$\beta_{2,3} = \frac{\#\left\{\mathcal{I}_{2,3} \cap [1,[2,3]]\right\}}{[2,3]} = \frac{1}{3}.$$

2.3 Step (III)

Goal : computing the admissible patterns on L_k for all $k \geq 1$.

- **The basic set of admissible patterns on** L_3.

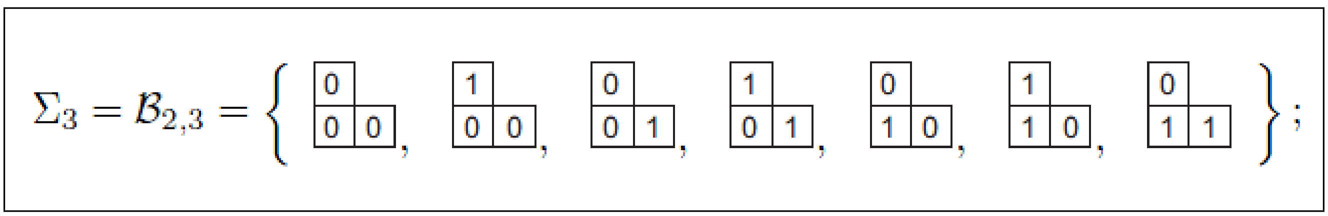

Figure 5: Basic patterns

- Let $\mathbf{\Sigma}_k = \mathbf{\Sigma}_k(\mathcal{B}_{2,3})$ and $|\mathbf{\Sigma}_k| = b_k$.

Remark :

(i). **Patterns generation problem** and 2-**dimensional transition matrices**.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
b_k	2	4	7	14	25	50	90	160	320	584	1039	1861	3722	6774

$b_{25} = 5,434,757$	$b_{42} = 172,749,984,030$
$b_{63} \approx 5.291646495998910 \times 10^{16}$	$b_{88} \approx 2.006283543836154 \times 10^{23}$
$b_{118} \approx 1.439075072036499 \times 10^{31}$	$b_{149} \approx 1.766912321512124 \times 10^{39}$

Figure 6: k and b_k for $\mathbb{X}^0_{2,3}$

J.-C. Ban and S.-S. Lin, **Discrete Contin. Dyn. Syst.** (2005);

J.-C. Ban, S.-S. Lin and Y.-H. Lin, **Asian J. Math.** (2007);

J.-C. Ban, S.-S. Lin and Y.-H. Lin, **International J. Bifurcation and Chaos.** (2008);

J.-C. Ban, C.-H. Chang, S.-S. Lin and Y.-H. Lin, **J. Differential Equations** (2009);

J.-C. Ban, C.-H. Chang and S.-S. Lin, **J. Differential Equations** (2012);

J.-C. Ban, W.-G. Hu, S.-S. Lin and Y.-H. Lin, **Memo. Amer. Math. Soc.** (2012);

W.-G. Hu and S.-S. Lin, **Proc. Amer. Math. Soc.** (2011) .

(ii). The L_k is **not regular lattice**, however, some idea are the same!

Theorem : The entropy $\mathbb{X}^0_{2,3}$ is given by

$$h(\mathbb{X}^0_{2,3}) = \sum_{k=1}^{\infty} \beta_{2,3} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\mathbf{\Sigma}_k| \, .$$

For $n \geq 1$, let

$$h^{(n)}(\mathbb{X}^0_{2,3}) = \sum_{k=1}^{n} \beta_{2,3} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k| .$$

Numerical result for $h^{(n)}(X^0_{2,3})$:

n	4	13	25	42
$h^{(n)}(\mathbb{X}^0_{2,3})$	0.319901	0.537229	0.620707	0.645733

63	88	118	149
0.652284	0.653865	0.654224	0.654303

Figure 7: $h^{(n)}(\mathbb{X}^0_{2,3})$

2.4 General multi-dimensional systems

$$\mathbb{X}^0_{\gamma_1\gamma_2} = \left\{(x_1, x_2, \ldots) \in \{0,1\}^{\mathbb{N}} : x_k x_{\gamma_1 k} x_{\gamma_2 k} = 0 \ \forall k \geq 1\right\}.$$

Theorem : For any two integers $\gamma_2 > \gamma_1 > 1$ with $\gamma_2 \neq \gamma_1^m$ for all $m > 1$. Then

$$h(\mathbb{X}^0_{\gamma_1,\gamma_2}) = \sum_{k=1}^{\infty} \beta_{\gamma_1,\gamma_2} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}}\right) \log \left|\mathbf{\Sigma}_k \left(\gamma_1, \gamma_2\right)\right|,$$

where

$$\beta_{\gamma_1,\gamma_2} = \frac{\#\left\{\mathcal{I}_{\gamma_1,\gamma_2} \cap [1, [\gamma_1, \gamma_2]]\right\}}{[\gamma_1, \gamma_2]}.$$

Theorem : For $Q, m \geq 2$, if $\gamma_1 = Q$ and $\gamma_2 = Q^m$, then

$$h(\mathbb{X}^0_{Q,Q^m}) = (Q-1)^2 \sum_{k=1}^{\infty} \frac{1}{Q^{k+1}} \log \left|a_k(Q, Q^m)\right|,$$

where $a_k = |A(Q, Q^m)|$ for $k \geq m$, $a_j = Q^j$, $1 \leq j \leq m$, where $A(Q, Q^m)$ is the associated transition matrix of $\mathcal{B}(Q, Q^m)$.

$\mathbb{X}_{\Gamma}^{0}$

$$= \left\{ (x_1, x_2, \ldots) \in \{0,1\}^{\mathbb{N}} : x_k x_{\gamma_1 k} x_{\gamma_2 k} \cdots x_{\gamma_d k} = 0,\ k \geq 1 \right\}.$$

Theorem : Let $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, if $1 < \gamma_2 < \gamma_2 < \cdots < \gamma_d, d \geq 3$ and $\gamma_j \neq \gamma_i^m$ for all $m \geq 2$ and $1 \leq i \leq j \leq d$. Then the entropy of $\mathbb{X}_{\Gamma}^{0}$ is given by

$$h(\mathbb{X}_{\Gamma}^{0}) = \sum_{k=1}^{\infty} \beta_{\Gamma} \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log |\Sigma_k| \,,$$

where

$$\beta_{\Gamma} = \frac{\mathcal{I}_{\Gamma} \cap [1, [\gamma_1, \gamma_2, \ldots, \gamma_d]]}{[\gamma_1, \gamma_2, \ldots, \gamma_d]}.$$

Note : the numbered lattice is d-dimensional.

$\mathbb{X}_{\Gamma}(N, \mathcal{C})$

$$= \left\{ (x_1, x_2, \ldots) \in \{0,1,\ldots,N\}^{\mathbb{N}} : x_k x_{\gamma_1 k} \cdots x_{\gamma_d k} \in \mathcal{C} \right\}.$$

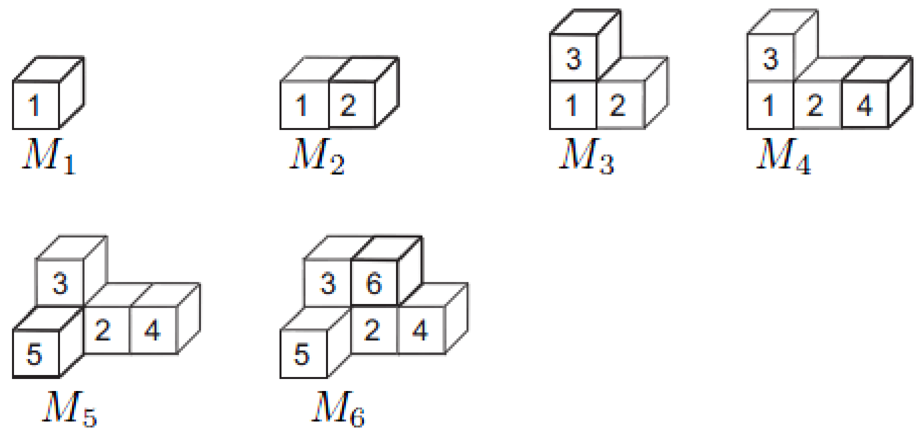

Figure 8: The numbered lattice for $\mathbb{M}_{2,3,5}$

Theorem : Let $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ satisfy conditions as above and $\mathcal{C} \subseteq \left\{0, 1, \ldots, (N-1)^d\right\}$. Then the entropy of $\mathbb{X}_\Gamma(N, \mathcal{C})$ is given by

$$h(\mathbb{X}_\Gamma(N, \mathcal{C})) = \sum_{k=1}^{\infty} \beta_\Gamma \left(\frac{1}{q_k} - \frac{1}{q_{k+1}} \right) \log \left| \Sigma_k(\mathcal{B}_\Gamma(N, \mathcal{C})) \right|,$$

where $\Sigma_k(\mathcal{B}_\Gamma(N, \mathcal{C}))$ is the set of d-dimensional admissible local patterns that can be generated by $\mathcal{B}_\Gamma(N, \mathcal{C})$ on L_k.

3 Coupled systems

Goal: compute the entropy of $\mathbb{X}_Q^A = \mathbb{X}_Q^0 \cap \boldsymbol{\Sigma}_A$.

Coupled systems :

$\mathbb{X}_2^A = \mathbb{X}_2^0 \cap \boldsymbol{\Sigma}_A = \{(x_1, x_2, \ldots) \in \boldsymbol{\Sigma}_A : x_k x_{2k} = 0 \ \forall k \geq 1\}.$

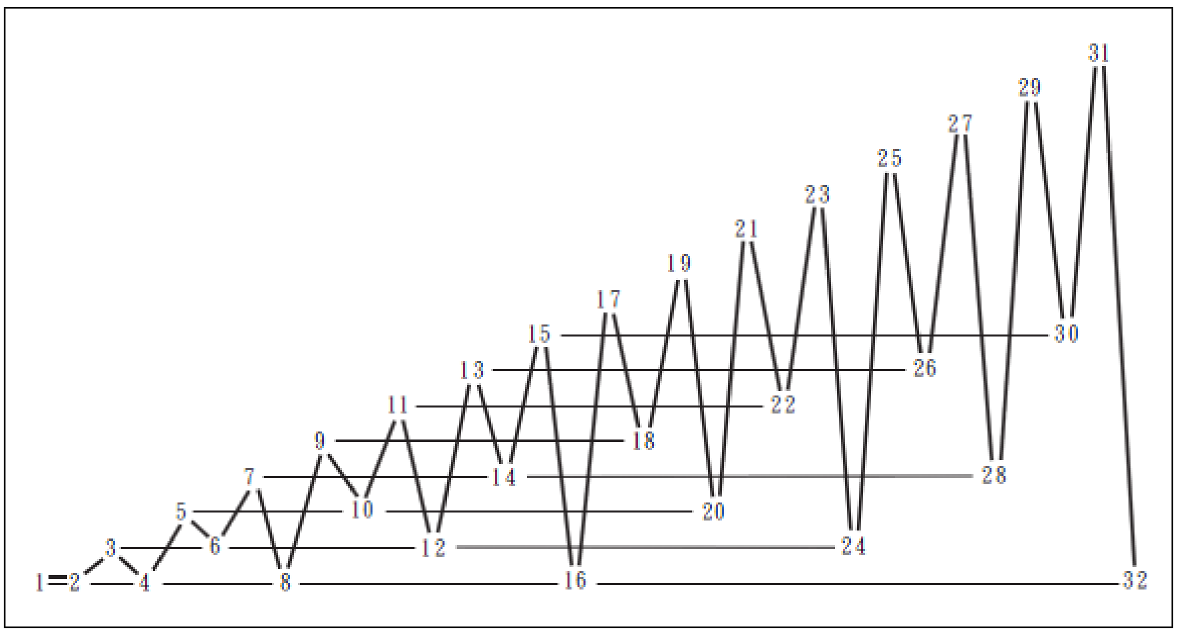

Figure 9: The effect of $\boldsymbol{\Sigma}_A$

Zigzag line : connects all natural integers comes from $\boldsymbol{\Sigma}_A$;

Horizontal line : connect the integers in in $i\mathbb{M}_2$ for each $i \in \mathcal{I}_2$.note: $i\mathbb{M}_2$ and $j\mathbb{M}_2$ are no longer mutually independent!! Therefore, it is regarded as a *coupled system*.

Idea : Decouple !!

3.1 Strategy: decouple system $\mathbb{X}_2^A$

Strategy:

(I). To **decouple** the whole system into **disjoint pieces** by eliminating $\mathbb{M}_2$ such that only

$$\widetilde{\mathbb{X}}_2^A = \left(\bigcup_{1<i\in\mathcal{I}_2} i\mathbb{M}_2 \right) \bigcap \Sigma_A$$

is considered.

(II). From the reduced system $\widetilde{\mathbb{X}}_2^A$, a sequence $\left\{ \mathbb{X}_2^A(m) \right\}_{m=1}^{\infty}$ of **independent branches** are chosen.

(III) The **entropy** of the decoupled independent system $\mathbb{X}_2^A(m)$ can be computed easily.

(IV). An appropriate choice of $\mathbb{X}_2^A(m)$ is demonstrated to enable the **recovery of the entropy** of $\mathbb{X}_2^A$, i.e.,

$$\lim_{m\to\infty} h(\mathbb{X}_2^A(m)) = h(\mathbb{X}_2^A).$$

3.2 Lower and upper bounds for $h(\mathbb{X}_2^A)$

Theorem : The entropy $h(\mathbb{X}_2^A)$ is given by

$$h(\mathbb{X}_2^A) = \lim_{k\to\infty} \frac{1}{2(2^k-1)} \log |\mathbf{\Sigma}_k| ,$$

where $\mathbf{\Sigma}_k$ the admissible patterns on L_k. Furthermore,

$$\begin{aligned} & \frac{1}{2(2^k-1)} \log |\mathbf{\Sigma}_k| \leq h(\mathbb{X}_2^A) \\ \leq \; & \frac{1}{2(2^k-1)} \log |\mathbf{\Sigma}_k| + \frac{k}{2(2^k-1)} \log 2. \end{aligned}$$

Numerical result for $h^{(n)}\left(\mathbb{X}_2^A\right)$:

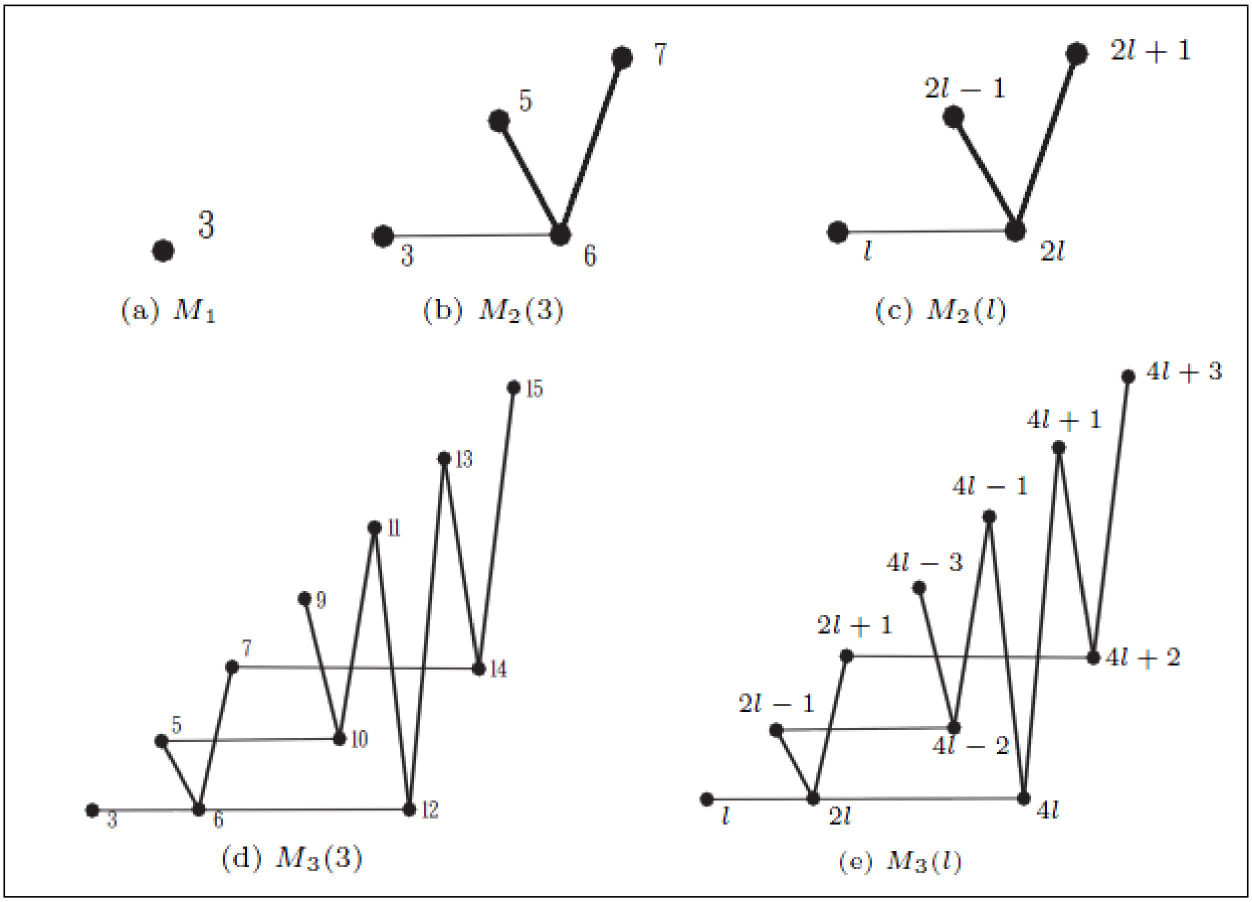

Figure 10: The admissible numbered lattice M_k in $\widetilde{\mathbb{X}}_2^A$

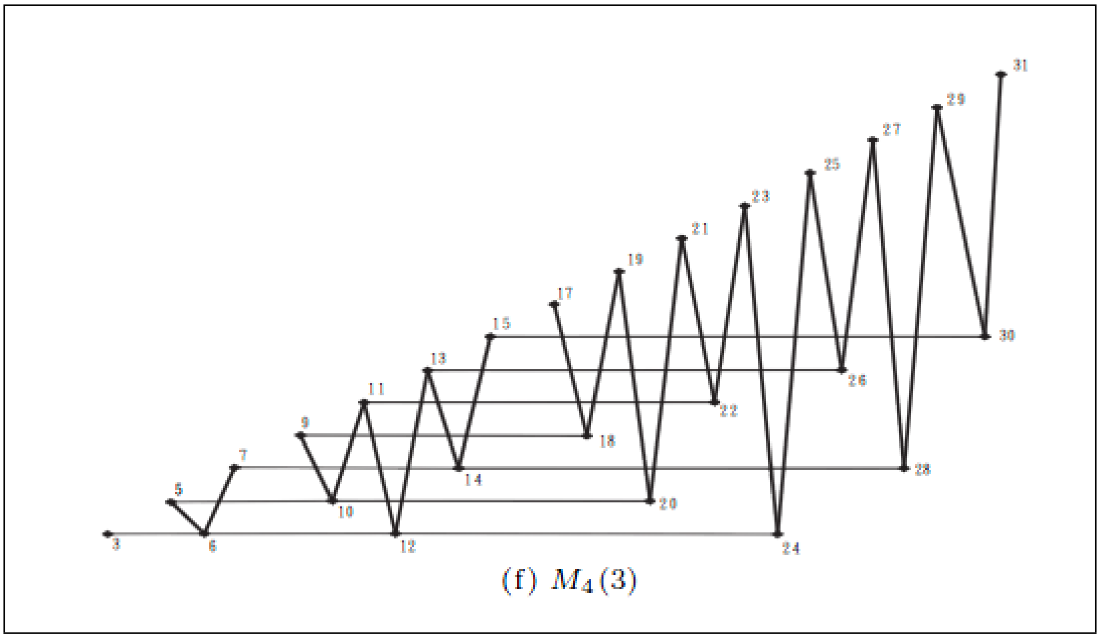

Figure 11: $M_4(3)$

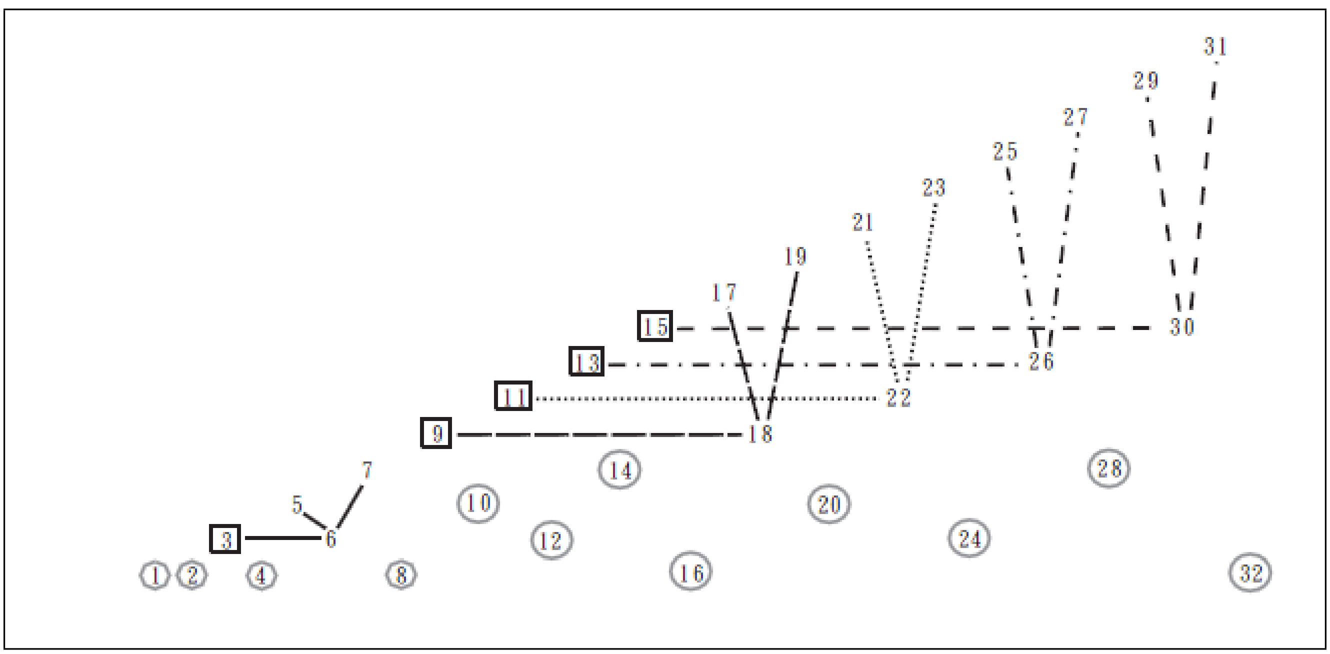

Figure 12: The decoupled system by M_2

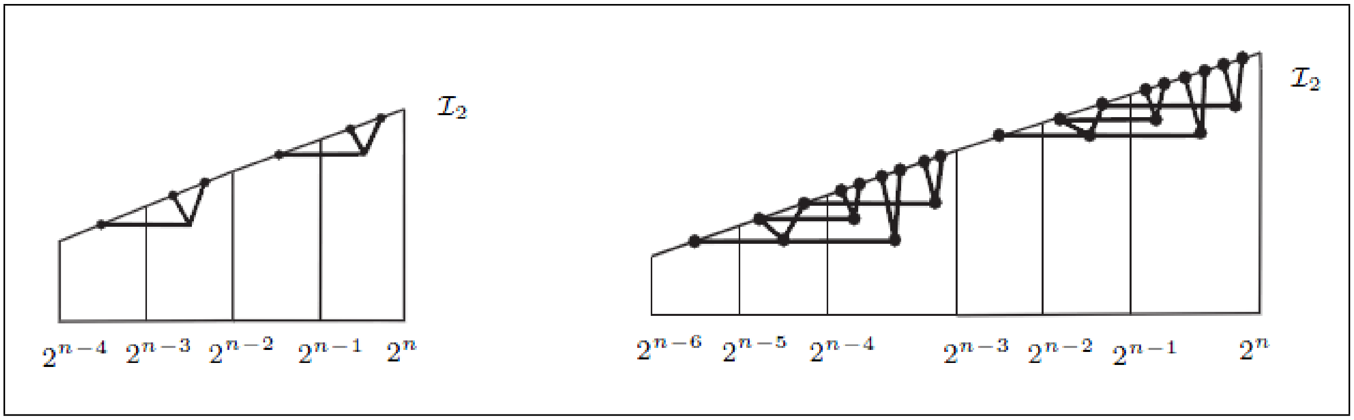

Figure 13: Distribution of M_k

n	2	3	4
$\lvert\Sigma_n\rvert$	9	237	213624
$h^{(n)}(\mathbb{X}_2^A)$	0.366204	0.390576	0.409066
$\bar{h}^{(n)}(\mathbb{X}_2^A)$	0.597253	0.539107	0.501485

Figure 14: $\lvert\Sigma_n\rvert$ and $h^{(n)}\left(\mathbb{X}_2^A\right)$

3.3 General coupled systems

Theorem : For any $Q \geq 3$ and $k \geq 2$,

$$\frac{Q-1}{Q(Q^k-1)} \log \left| \mathbf{\Sigma}_{Q;k} \right| \leq h(\mathbb{X}_Q^A)$$
$$\leq \frac{Q-1}{Q(Q^k-1)} \left(\log \left| \mathbf{\Sigma}_{Q;k} \right| + k \log 2 \right),$$

and

$$h(\mathbb{X}_Q^A) = \lim_{k\to\infty} \frac{Q-1}{Q(Q^k-1)} \log \left| \mathbf{\Sigma}_{Q;k} \right|,$$

where $\mathbf{\Sigma}_{Q;k}$ is the set of all admissible patterns on $L_{Q;k}$, and $L_{Q;k}$ is the degree k blank lattice.

Theorem : For any $Q \geq 3$, $\mathcal{C} \subseteq \left\{0, 1, \ldots, (N-1)^d\right\}$ and $k \geq 2$,

$$\frac{Q-1}{Q(Q^k-1)} \log \left| \mathbf{\Sigma}_k(Q; A; N, \mathcal{C}) \right| \leq h\left(\mathbb{X}_Q^A(N, \mathcal{C}) \right)$$
$$\leq \frac{Q-1}{Q(Q^k-1)} \left(\log \left| \mathbf{\Sigma}_k(Q; A; N, \mathcal{C}) \right| + k \log N \right),$$

and

$$h\left(\mathbb{X}_Q^A(N,\mathcal{C})\right) = \lim_{k\to\infty} \frac{Q-1}{Q(Q^k-1)} \log \left|\Sigma_k\left(Q;A;N,\mathcal{C}\right)\right|,$$

where $\Sigma_k\left(Q;A;N,\mathcal{C}\right)$ is the set of all admissible patterns on $L_{Q;k}$ the constraint of the vertices on the bold lines in $L_{Q;k}$ is given by A and the constraint of the vertices on the lines in $L_{Q,k}$ is given by N and $\mathcal{C}$.