Periodic Orbits of Discretized Rotations

Shigeki Akiyama, Univ. of Tsukuba

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This is a joint work with Attila Pethő.

Discretized Rotation

Conjecture 1. *For any* $-2 < \lambda < 2$*, the integer sequence defined by* $0 \leq a_{n+1} + \lambda a_n + a_{n-1} < 1$ *is periodic.*

In other words, we are interested in the dynamics on $\mathbb{Z}^2$:

$$(x, y) \mapsto (y, -\lfloor x + \lambda y \rfloor)$$

Our transformation on $\mathbb{Z}^2 : (x, y) \mapsto (X, Y)$ is written as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \mu \end{pmatrix}$$

with $\mu \in [0, 1)$. Let $Q = \begin{pmatrix} -\sin\theta & \cos\theta \\ 0 & 1 \end{pmatrix}$. Since

$$Q \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} Q^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -\lambda \end{pmatrix},$$

we view this algorithm as

$$Q^{-1}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} Q^{-1}\begin{pmatrix} x \\ y \end{pmatrix} + Q^{-1}\begin{pmatrix} 0 \\ \mu \end{pmatrix}. \quad (1)$$

It is the dynamics acting on the lattice

$$\mathcal{L} = \begin{pmatrix} -\csc\theta \\ 0 \end{pmatrix} \mathbb{Z} + \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix} \mathbb{Z}$$

written as the composition of the Euclidean rotation of angle θ followed by a small translation

$$\mathbf{v} \mapsto \mathbf{v} + \mu \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix}$$

with $\mu \in [0, 1)$.

Why do we study this ?

- Problem on integers. Difficult. (Why $X^n + Y^n = Z^n$)
- Discretized version of dynamics. Is computer simulation reliable ?
- Reversible system. Rotation without information loss.
- Composition of two involutions. Common feature with interval exchange transformation, billiard dynamics, etc.

Mathematica code

```
r1 = 1/2;
a = {100, -15};
L = NestWhileList[Function[z,
 {Last[z],-Floor[First[z] + r1 Last[z]]}],
  a, ! (a == #) &, {2,1}];
Print[Length[L]-1];
Show[Graphics[Map[Point, L]],
AspectRatio -> Automatic, Axes -> True];
```

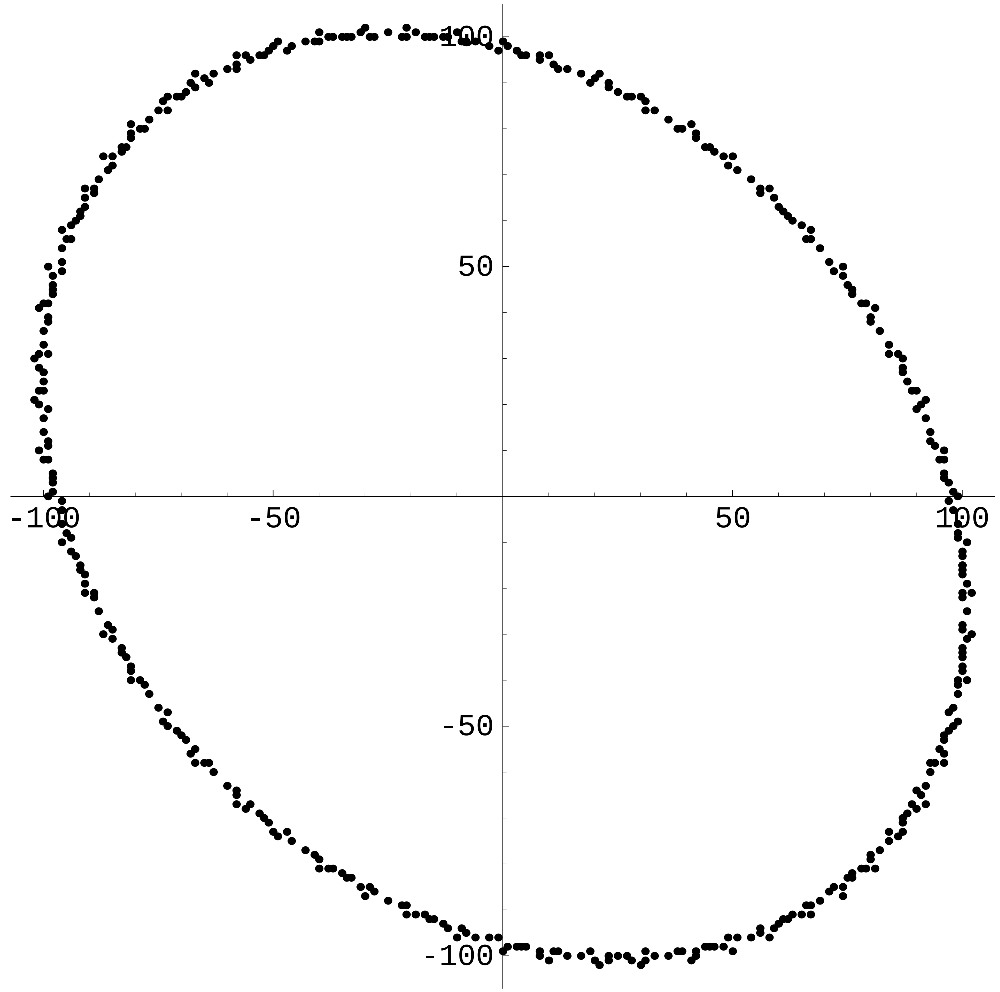


Figure 1: $\lambda = 1/2$: initial value $(100, -15)$

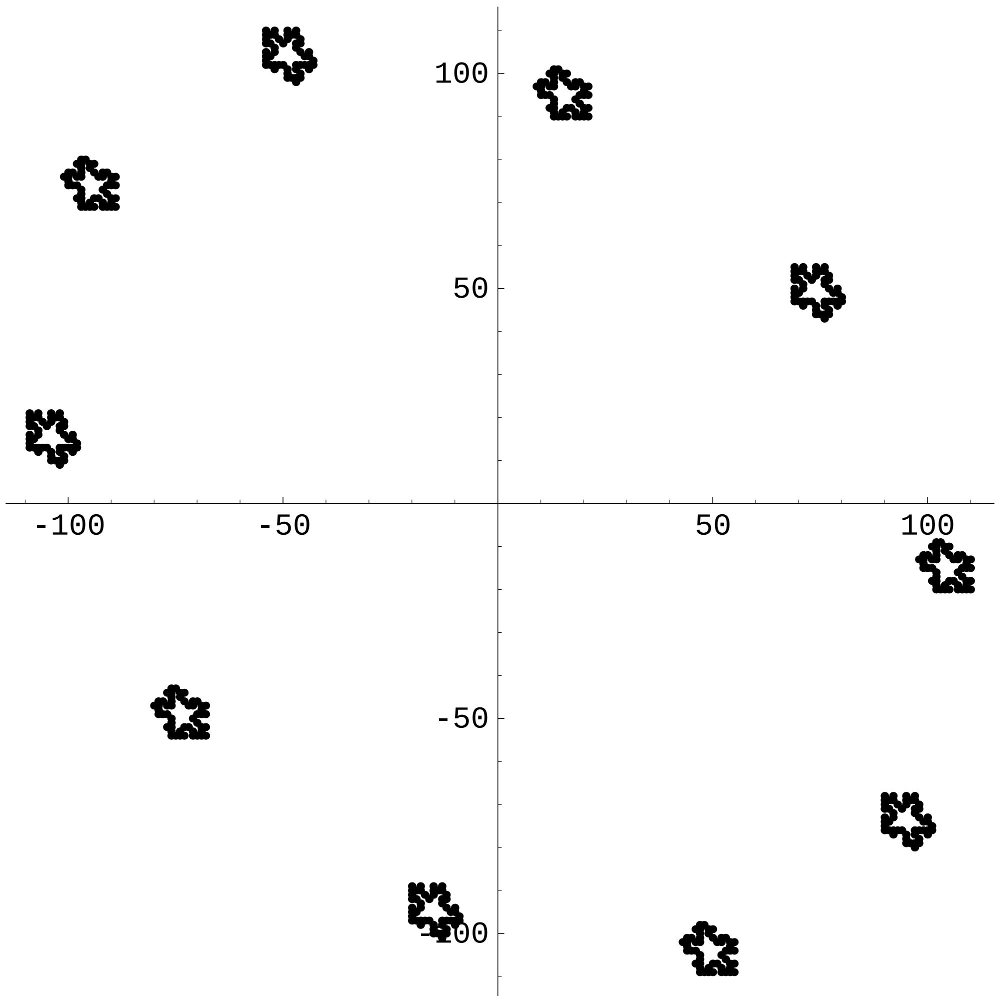


Figure 2: $\lambda = (\sqrt{5}-1)/2$: initial value $(100, -15)$

There are many researchers in dynamics interested in this system: Vivaldi, Kouptsov, Lowenstein, Goetz, Poggiaspalla, Vladimirov, Bosio, Shaidenko, ... However we know very little on this system. The Conjecture is true for 11 values:

$$\lambda = 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \frac{\pm 1 \pm \sqrt{5}}{2}.$$

First three cases are trivial. The others are exactly the cases when θ/π is rational and λ is quadratic. See [2, 7, 1].

In fact, if θ/π is rational then we can embed the system into domain exchange dynamics of the torus of dimension $2(d-1)$ where $d = \deg(\lambda)$. This makes the problem a little easier.

Let us take $\lambda = \omega = (1+\sqrt{5})/2$ and $\zeta = \exp(2\pi\sqrt{-1}/5)$. We have $\mathbb{Z}[\zeta] = \mathbb{Z}[\omega] + (-\zeta^{-1})\mathbb{Z}[\omega]$. Denote by $\langle x \rangle$ the fractional part of $x \in \mathbb{R}$. Putting $x_n = \langle \omega a_n \rangle$, we have

$$0 \le a_n + \omega a_{n+1} + a_{n+2} < 1$$

$$a_n + \omega a_{n+1} + a_{n+2} = \langle \omega a_{n+1} \rangle$$

$$\langle \omega a_n \rangle - \frac{1}{\omega}\langle \omega a_{n+1} \rangle + \langle \omega a_{n+2} \rangle \equiv 0 \pmod{\mathbb{Z}}$$

$$x_n - (\zeta + \zeta^{-1}) x_{n+1} + x_{n+2} \equiv 0 \pmod{\mathbb{Z}}$$

$$(x_{n+1} - \zeta^{-1} x_{n+2}) \equiv \zeta^{-1}(x_n - \zeta^{-1} x_{n+1}) \pmod{\zeta^{-1}\mathbb{Z}}$$

which gives a piecewise isometry acting on a lozenge X :

$$T(x) = \begin{cases} x/\zeta & \mathsf{Im}(x/\zeta) \geq 0 \\ (x-1)/\zeta & \mathsf{Im}(x/\zeta) < 0 \end{cases}.$$

and we have a commutative diagram by putting $x_n = \langle \omega a_n \rangle$:

$$\begin{array}{ccc} (a_{n+1}, a_n) \in \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^2 \ni (a_{n+2}, a_{n+1}) \\ \pi \Big\downarrow & & \pi \Big\downarrow \\ x_n - \zeta^{-1} x_{n+1} \in X & \xrightarrow{T} & X \ni x_{n+1} - \zeta^{-1} x_{n+2} \end{array}$$

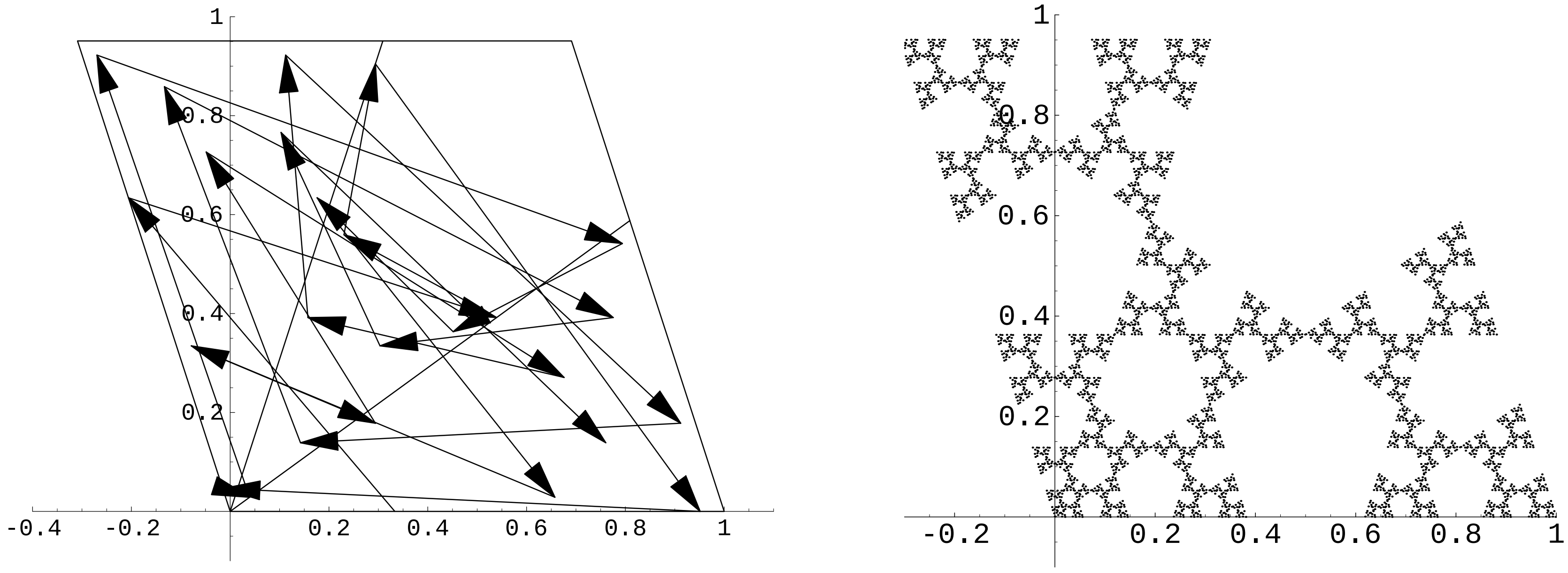


Figure 3: The orbit of $1/3$

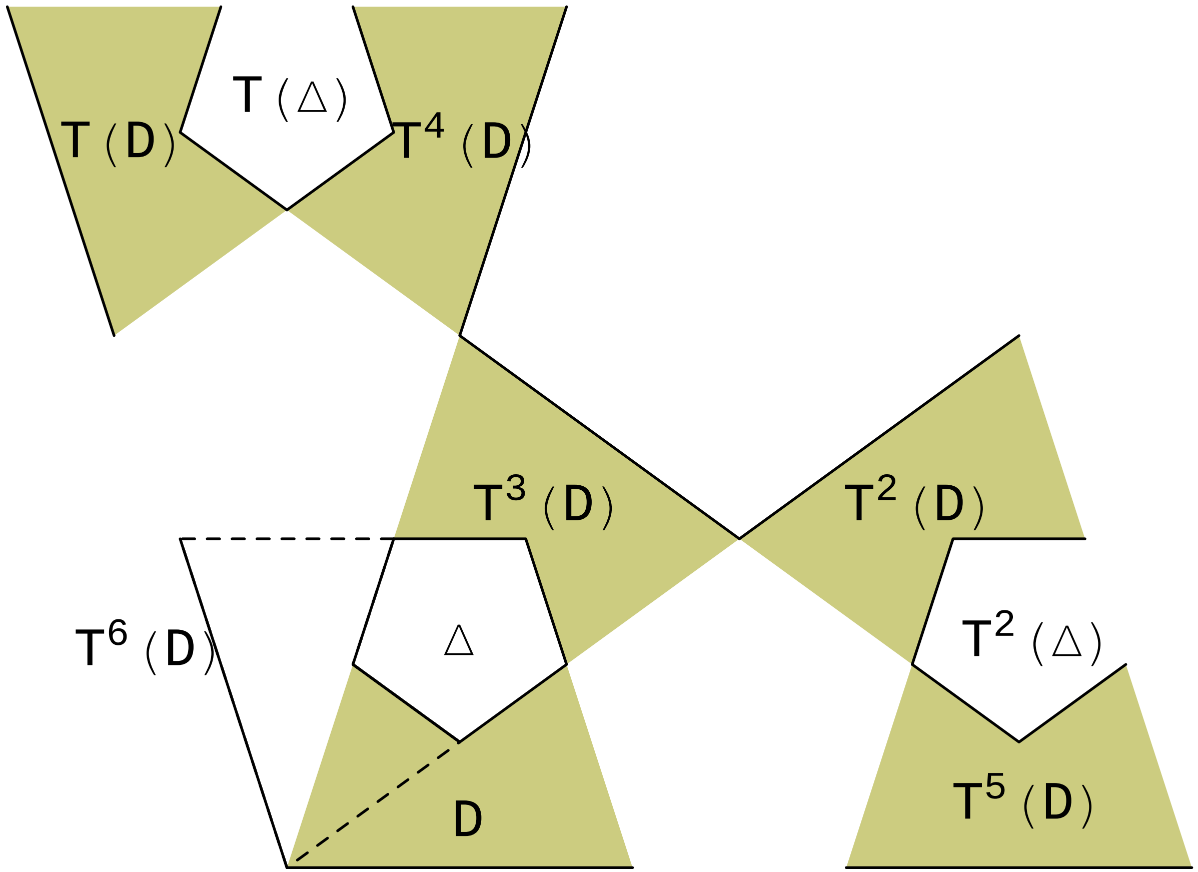
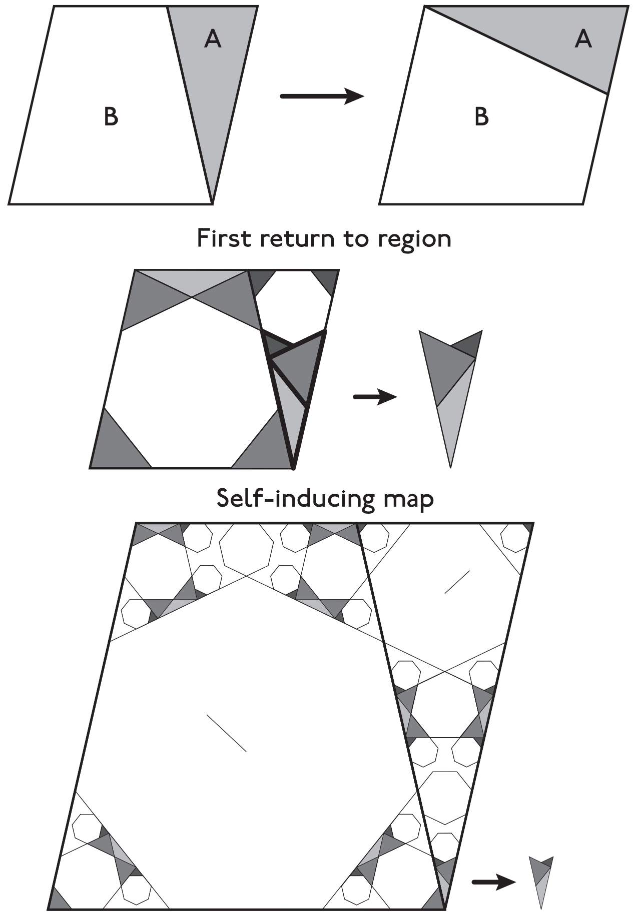
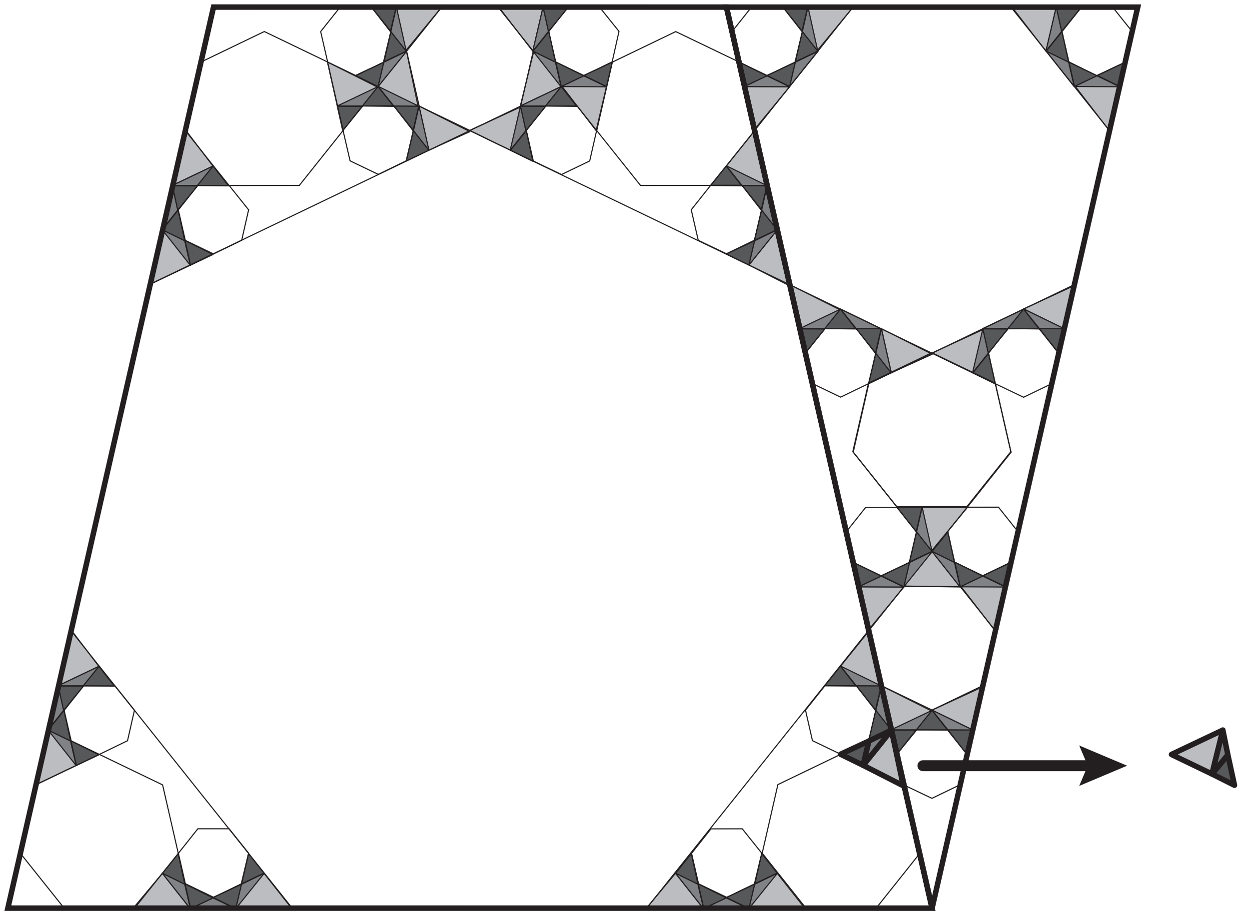


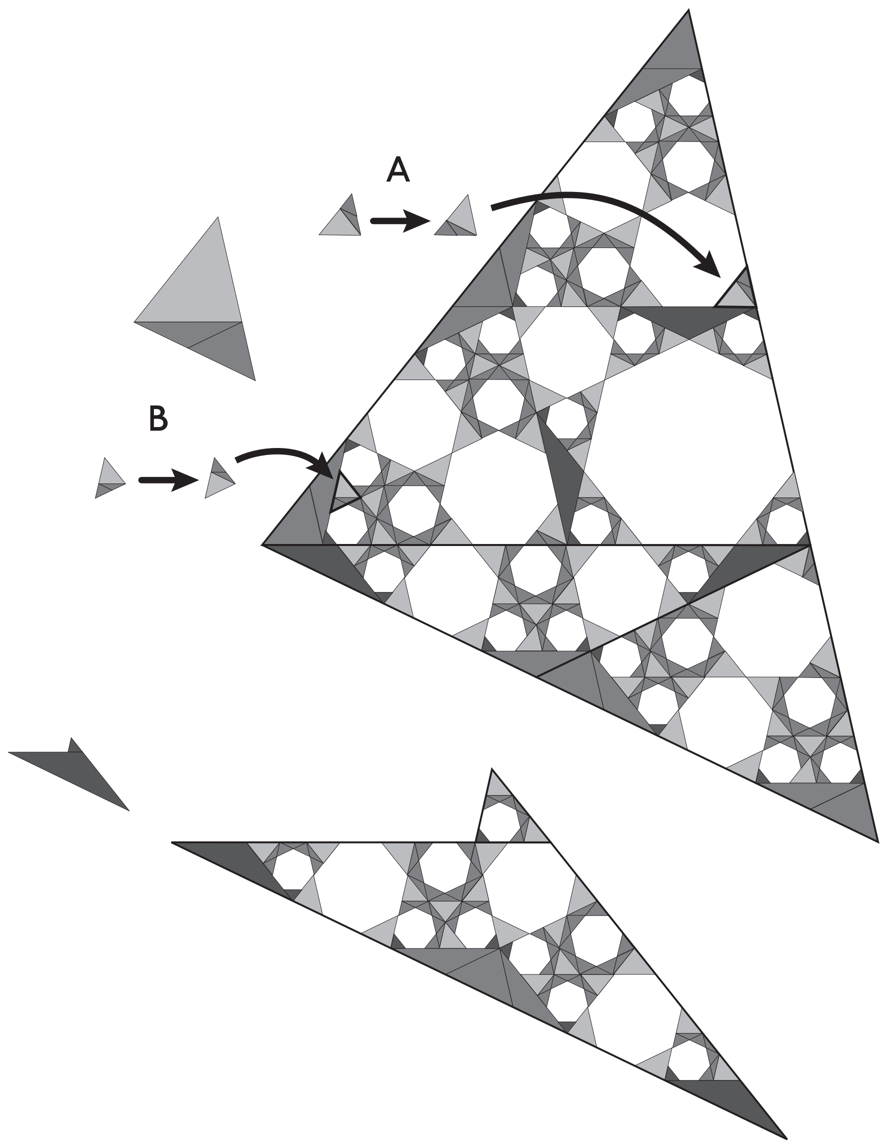
Figure 4: Self Inducing structure

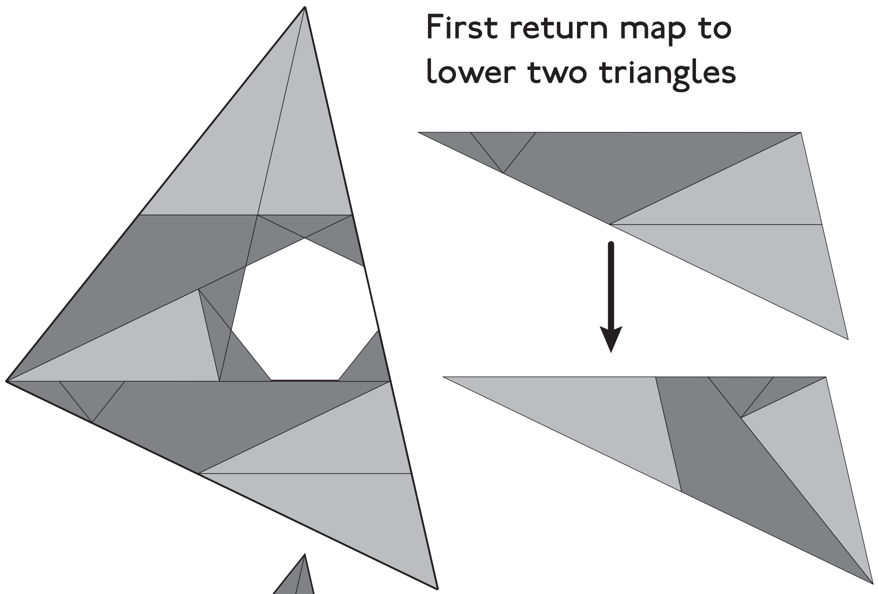
In higher dimension, it is hard to visualize self-inducing structure. If you project 4-dim discretized rotation to the plane, then we sometime see the self-inducing structure. However, by this projection, we lose connection to the original dynamics.

7-fold rotation.

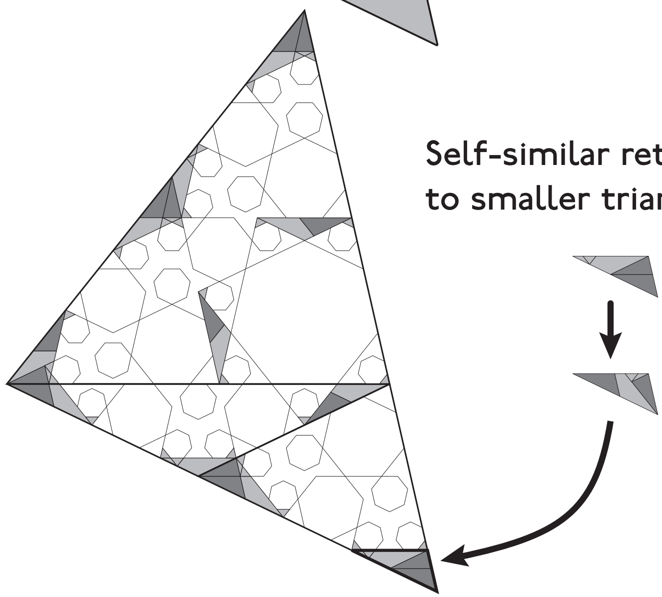





A
B

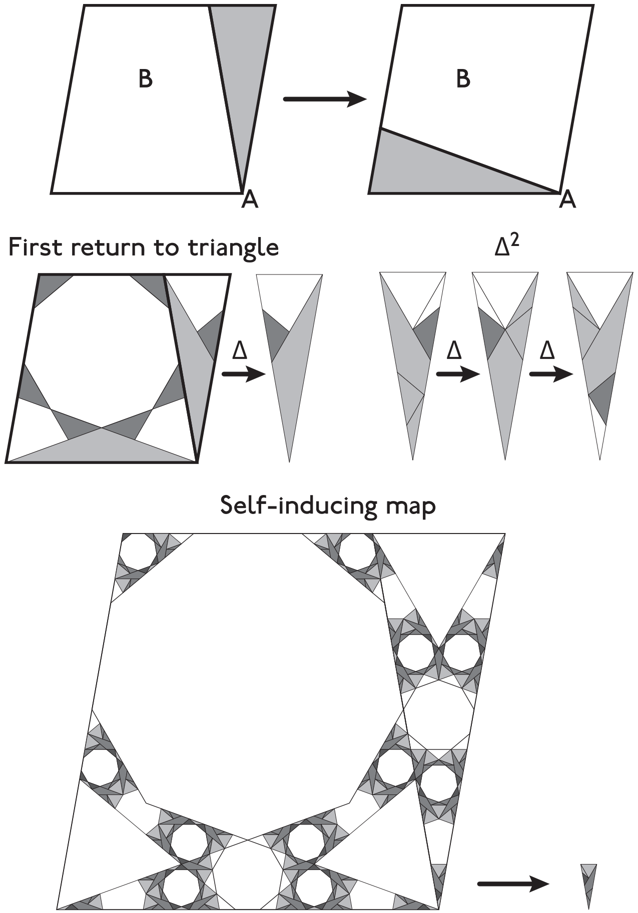


First return map to
lower two triangles



Self-similar return map
to smaller triangle

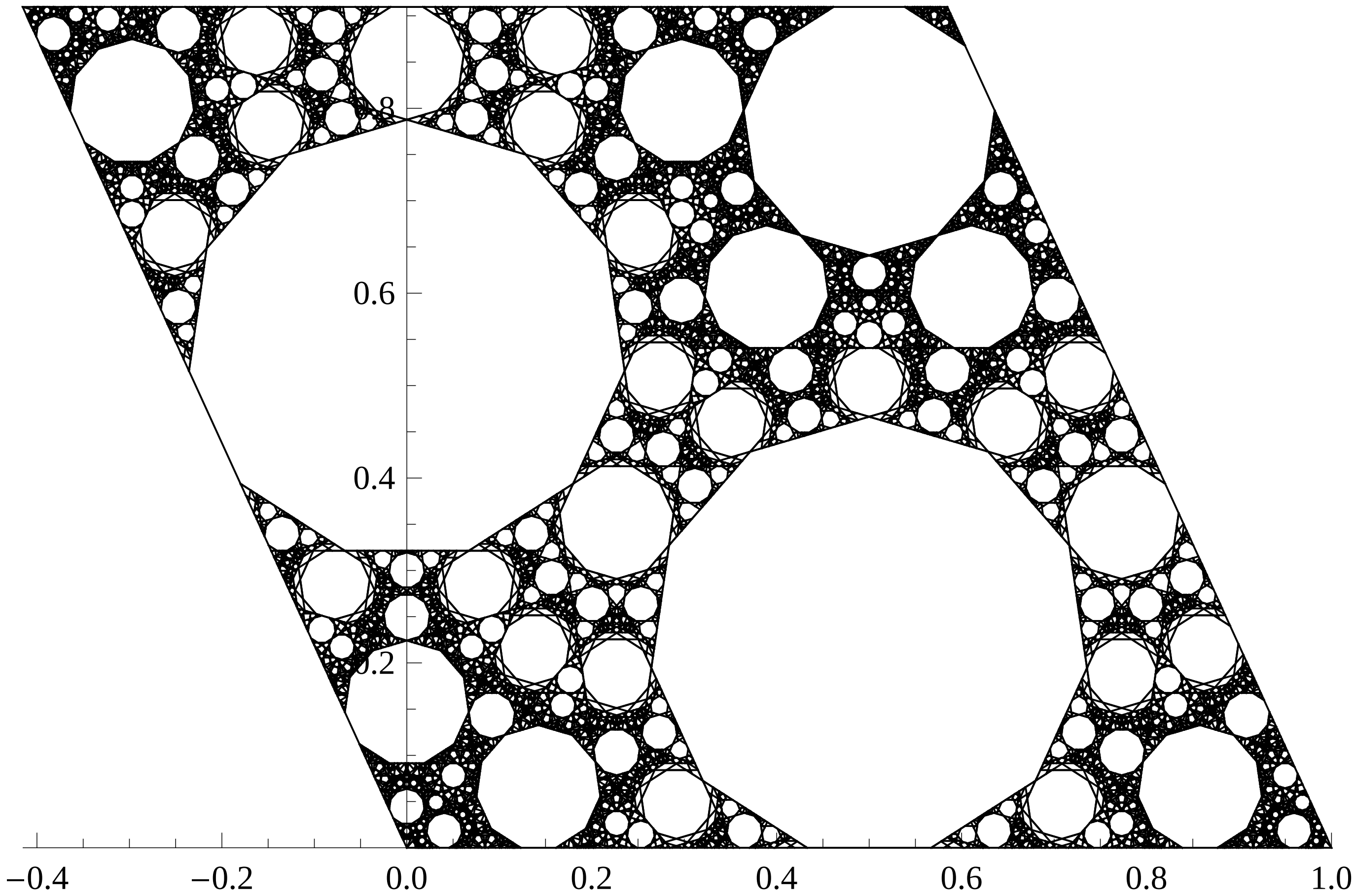
9-fold rotation.



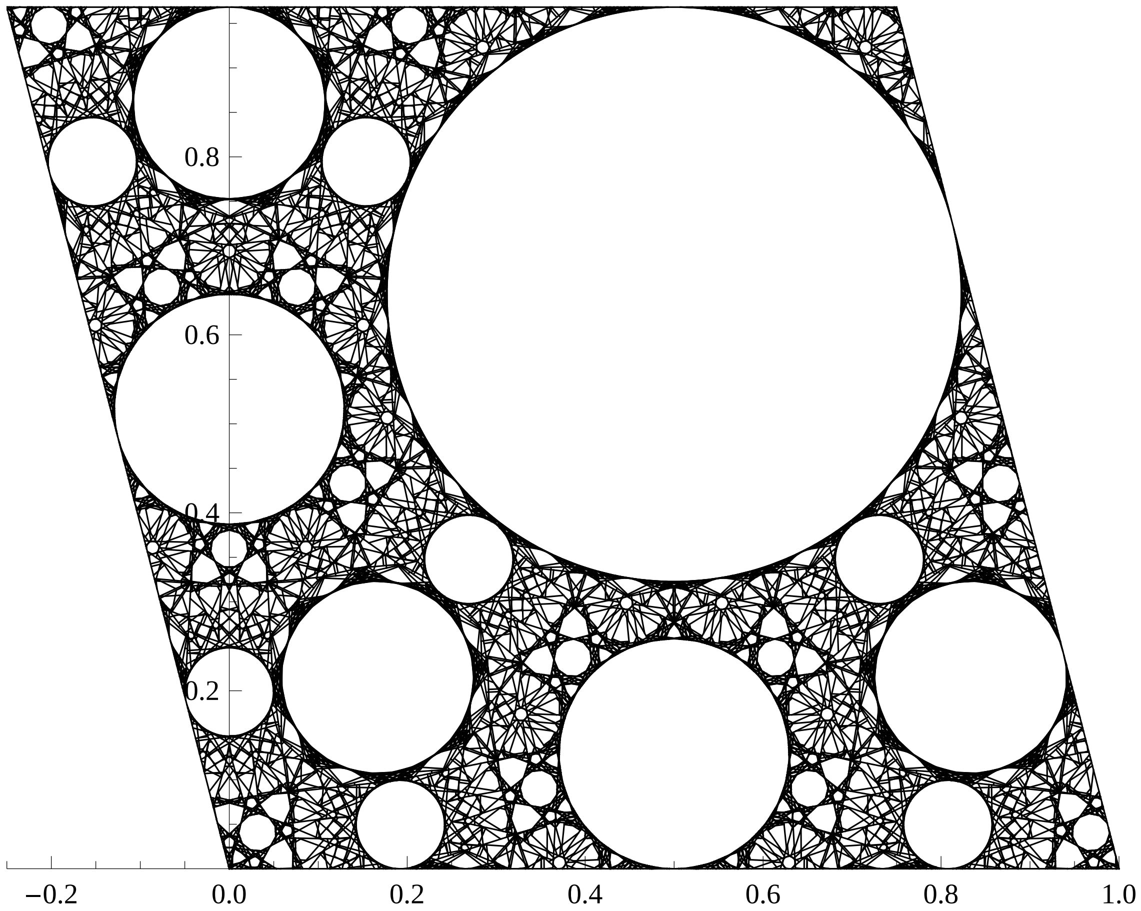
When λ is a rational number with a prime power denominator, there is an interesting attempt to embed discretized rotation to a p-adic space by Bosio-Vivaldi [5]. They represent the dynamics as composition of multiplication of p-adic unit and symbolic shift.

Domain exchange is an invertible dynamics with zero entropy ([6]). So we do not know the behavior of periodic orbits. Changing the angle of the domain exchange, we obtain interesting experiments.

11-fold rotation.



Irrational rotation case.



However this is nothing to do with original problem.

Summary.

We know almost nothing on the original discretized rotation problem.

At this stage, we are interested in giving a weak result on the discretized rotation dynamics.

Theorem 1 ([3]). *For all fixed* $\lambda \in (-2, 2)$ *there are infinitely many periodic orbits of the dynamics*

$$(x, y) \to (y, -\lfloor x + \lambda y \rfloor)$$

on $\mathbb{Z}^2$.

In fact, we could show the same statement for:

$$(x, y) \to (y, -\lfloor x + \lambda y + \mu \rfloor)$$

First Tool.

Lemma 2 (Vinogradoff). *Let* $f \in C^2[a, b], k \geq 1$ *and* $A > 29$. *Assume that*

$$\frac{1}{kA} < f''(x) < \frac{1}{A}$$

for $a \leq x \leq b$. *Then we have*

$$\sum_{a<x<b} \langle f(x) \rangle = \frac{b-a}{2} + G$$

with

$$|G| < 2k \left(\frac{b-a}{A} + 1 \right) (A \log A)^{2/3}.$$

This is used to count lattice points in the region defined by curves of positive curvature, like the circle problem of Gauss:

$$\mathrm{Card}\{(x,y) \in \mathbb{Z}^2 \mid x^2+y^2 \le R^2\} = \pi R^2 + O(R^{2/3+\varepsilon})$$

is derived by the lemma. Many number theorists made efforts to improve the error term. The expected exponent is $1/2+\varepsilon$.

Second Tool.

Identify the orbits of our dynamics with bi-infinite sequence (a_n). We say that (a_n) is *periodic* if there is an integer $p > 0$ that $a_{n+p} = a_n$, and is *symmetric* at $b/2$ if $a_{b-n} = a_n$. If the orbits is symmetric at $b_1/2$ and $b_2/2$, then we call it *doubly symmetric*. Here we state a pretty trivial

Lemma 3. *Doubly symmetric orbits are periodic. Symmetric periodic orbits are doubly symmetric.*

The idea to observe symmetric orbits date back to G. Birkhoff [**?**, 4] who proved that there are infinitely many symmetric periodic orbits in restricted problem of three bodies.

Sketch of the proof.

Define a *trap region* $T(R)$ by

$$T(R) = \left\{ x + y \begin{pmatrix} \cot\theta \\ 1 \end{pmatrix} \;\middle|\; x \in B(R), y \in [0,1) \right\} \setminus B(R).$$

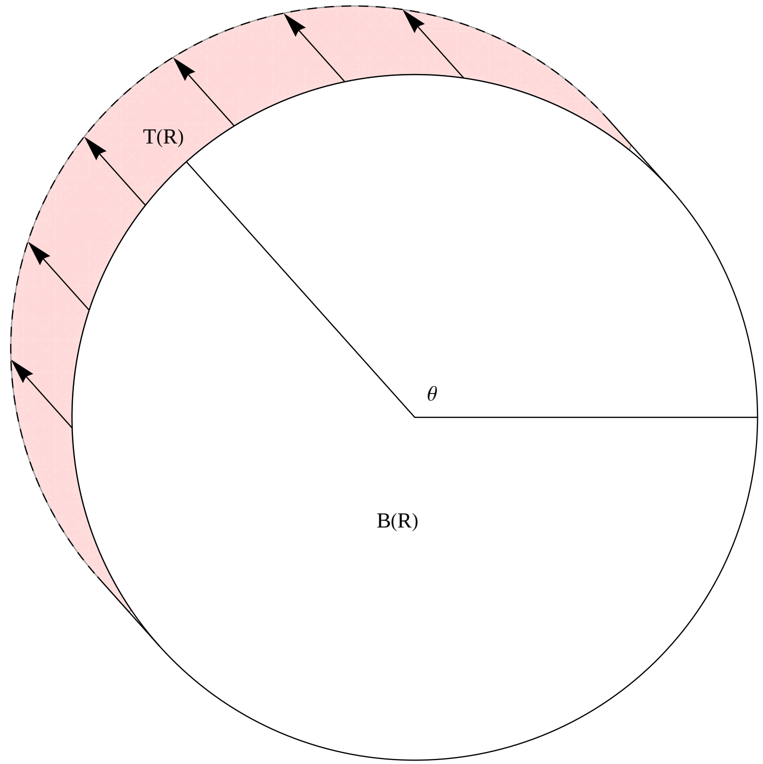


Figure 5: Trap Region

Then symmetric unbounded orbits must visit at least once the trap region. We compare an upper bound of lattice points in the trap region and the lower bound of symmetric points in $B(R)$.

There are two kinds of symmetric orbits:

$$(\dots, a_{-1}, a_{-1}, X, X.a_1, a_2, \dots)$$

and

$$(\dots, a_{-1}, a_{-1}, X, Y, X.a_1, a_2, \dots).$$

There are $2R\cos(\theta/2) - C_1$ unbounded orbits of (X, X) type.

There are $R - C_2$ unbounded orbits of (X, Y, X) type.

There are $2R + O(R^{2/3+\varepsilon})$ points in the trap region.

If $\theta < 2\pi/3$, then $2R\cos(\theta/2) - C_1 + R - C_3 > 2R + O(R^{2/3+\varepsilon})$ holds for $R \ll 1$.

To study the case $\theta \geq 2\pi/3$, we have to look into the symmetry of the periodic orbits. If $(a_n, a_{n+1}) = (C, D)$ occurs, then there is m with $(a_m, a_{m+1}) = (D, C)$. We are double counting some part of lattice points in the trap region.

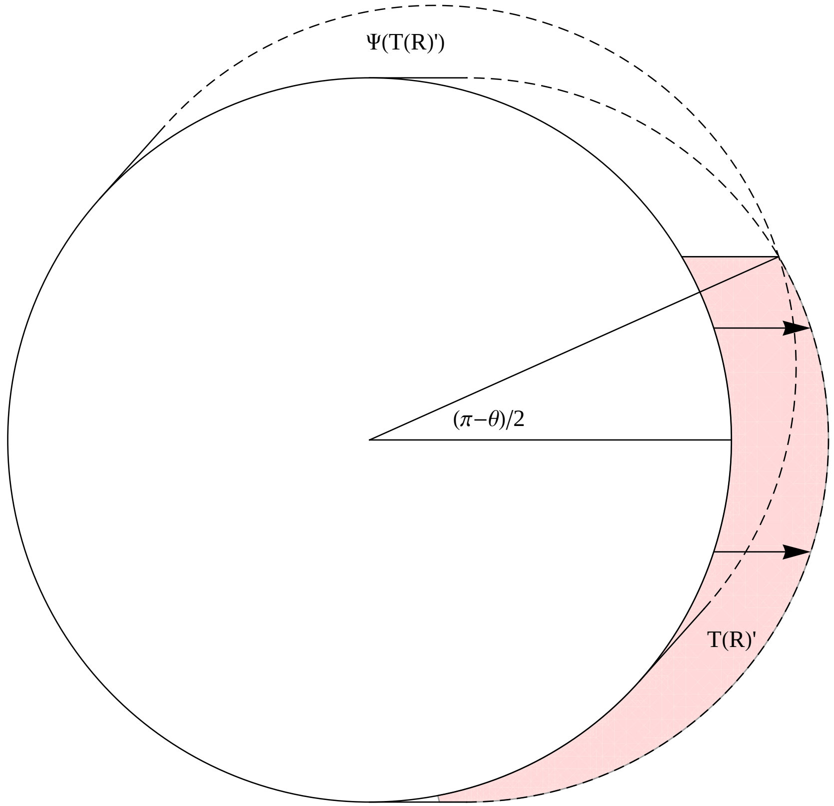


Figure 6: Symmetry of the trap region

Eliminating double counts there are $R + R\cos(\theta/2) + O(R^{2/3+\varepsilon})$ lattice points in the trap region!

$$2R\cos(\theta/2) - C_1 + R - C_3 > R + R\cos(\theta/2) + O(R^{2/3+\varepsilon})$$

holds for sufficiently large R.

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