Lecture 7

February 1, 2021

1 Diffusion equation on the whole line

Solve the diffusion equation on the whole line

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad -\infty < x < \infty, \quad 0 < t < \infty.$$

Key observation: If $u(x,t)$ is a solution to the diffusion equation, then $u(\lambda x, \lambda^2 t)$ is also a solution to the equation. Assume that $t > 0$ and let $\lambda = \frac{1}{\sqrt{t}}$ and $z = \frac{x}{\sqrt{t}}$, it suggests us to find a solution in the form of $u(x,t) = \frac{1}{\sqrt{t}}v(z)$. (One way to illustrate this: suppose $\lim_{x\to\pm\infty} u_x(x,t) = 0$. It is not hard to see that $\int_{-\infty}^{+\infty} u(x,t)dx$ is invariant on t. So we need to have

$$\int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} u(x,1)dx.$$

If $u(x,t) = \frac{1}{t^\alpha}v(z)$, we have

$$\int_{-\infty}^{+\infty} u(x,t)dx = \int_{-\infty}^{+\infty} \frac{1}{t^\alpha} v(\frac{x}{\sqrt{t}})dx = \int_{-\infty}^{+\infty} v(x)dx = \int_{-\infty}^{+\infty} u(x,1)dx.$$

Thus α would be $\frac{1}{2}$ in order to have $\int_{-\infty}^{+\infty} \frac{1}{t^\alpha} v(\frac{x}{\sqrt{t}})dx = \int_{-\infty}^{+\infty} v(x)dx$.)
We have

$$\begin{aligned}
u_x &= \frac{1}{\sqrt{t}}\frac{\partial z}{\partial x}v' = \frac{1}{t}v', \\
u_{xx} &= \frac{1}{t^{\frac{3}{2}}}v'', \\
u_t &= -\frac{1}{2t^{\frac{3}{2}}}v + \frac{1}{\sqrt{t}}\frac{\partial z}{\partial t}v' = -\frac{1}{2t^{\frac{3}{2}}}v - \frac{1}{2}\frac{x}{t^2}v'.
\end{aligned}$$

Thus reduced to the ODE

$$\frac{1}{2}v + \frac{1}{2}zv' + v'' = 0.$$

So a solution to the ODE is in the form

$$v(z) = ce^{-\frac{z^2}{4}}.$$

This motives the definition of the fundamental solution to the diffusion equation

$$\Phi(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} & t > 0 \\ 0. & t < 0 \end{cases}$$

Here c is chosen such that $\int_{-\infty}^{+\infty} \Phi(x,t)dx = 1$ for any fixed t. Note that $\Phi(x,t)$ is singular at $t = 0$.

The initial data problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & -\infty < x < \infty, 0 < t < \infty, \\ u(x,0) = \phi(x). \end{cases}$$

can be derived from the convolution of the fundamental solution and initial data as following: for $t > 0$

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \Phi(x-y,t)\phi(y)dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \phi(y)dy. \end{aligned} \tag{1}$$

There are several properties of u:

1. for $t > 0$, $\Phi(x-y,t)$ is infinitely differentiable with respect to x and t, so is $u(x,t)$.
2. Suppose for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $|x-y| \leq \delta$

   $$|\phi(x) - \phi(y)| \leq \epsilon.$$

   When $t \to 0$, we have

   $$\begin{aligned} |u(x,t) - \phi(x)| &= |\int_{-\infty}^{+\infty} \Phi(x-y,t)(\phi(y) - \phi(x))dy| \\ &\leq |\int_{x-\delta}^{x+\delta} \Phi(x-y,t)(\phi(y) - \phi(x))dy| \\ &\quad + |\int_{\mathbb{R}-[x-\delta,x+\delta]} \Phi(x-y,t)(\phi(y) - \phi(x))dy| \\ &\leq \epsilon |\int_{-\infty}^{+\infty} \Phi(y,t)dy| + \|\phi\| |\int_{\mathbb{R}-[x-\delta,x+\delta]} \Phi(x-y,t)| \\ &\leq \epsilon + C \int_{x+\delta}^{+\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy \\ &\leq \epsilon + C \int_{\frac{\delta}{2\sqrt{t}}}^{+\infty} \frac{1}{\sqrt{\pi}} e^{-z^2} dz \\ t \to 0 &\leq 2\epsilon \end{aligned}$$

3. It is the solution to the diffusion equation. Because for $t > 0$

$$\Phi_t(x-y,t) - \Phi_{xx}(x-y,t) \quad = \quad 0.$$

4. If $|u(x;t)| \leq Ae^{ax^2}$ for any $t \geq 0$ and $x \in \mathbb{R}$, then we have the uniqueness of the diusion equation on the whole line. If $u(x,t)$ does not satisfy this growth condition, then there admit other non-physical solutions. We do not prove this point in this course, see F.John, partial dierential equations [Chapter 7] for reference.

In conclusion, we have

Theorem 1. *Let $\phi \in C(\mathbb{R})$ be bounded and let $u(x,t)$ be given by the formula (1). Then*

- $u \in C^\infty(\mathbb{R} \times (0,\infty))$.
- u satisfies $u_t = u_{xx}$ for $-\infty < x < \infty$, $0 < t < \infty$.
- $\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0)$ for $x_0 \in \mathbb{R}$ and $t > 0$.
- $|u(x,t)| \leq Ae^{ax^2}$ for any $t \geq 0$
- and $x \in \mathbb{R}$,then we have the uniqueness.

2 Uniqueness by energy method in bounded interval

Now we consider the diusion equation in a bounded interval

$$\begin{cases} u_t - u_{xx} = f(x,t) & 0 \leq x \leq l, T \geq t > 0 \\ u(x,0) = 0, & 0 \leq x \leq l \\ u(0,t) = g(t), u(l,t) = h(t) & T \geq t > 0. \end{cases}$$

Proof. Suppose u_1 and u_2 are solutions to the initial value problem. $w = u_1 - u_2$ which satisfies

$$\begin{cases} w_t - w_{xx} = 0 & 0 \leq x \leq l, T \geq t > 0 \\ w(x,0) = 0, & 0 \leq x \leq l \\ w(0,t) = w(l,t) = 0, & T \geq t > 0. \end{cases}$$

Set $e(t) = \int_0^l w^2(x,t)dx$, then

$$\begin{aligned} \frac{de(t)}{dt} &= 2\int_0^l w_t w dx \\ &= 2\int_0^l w w_{xx} dx \\ &= 2ww_x(x,t)|_{x=0}^{x=l} - 2\int_0^l w_x^2 dx \\ &\leq 0. \end{aligned}$$

So

$$e(t) \leq e(0) = 0.$$

Thus

$$w(x,t) \equiv 0.$$

□

We can use this energy method to prove stability in the integral sense for $f = g = h = 0$.

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2 dx \leq \int_0^l [\phi_1(x) - \phi_2(x)]^2 dx.$$

3 Maximum Principle

If $u(x,t)$ smooth satisfies the diffusion equation

$$\begin{cases} u_t - u_{xx} = 0 & U_T = \{0 \leq x \leq l, T \geq t > 0\} \\ u(x,0) = \phi, & 0 \leq x \leq l \\ u(0,t) = g(t), & T \geq t > 0 \\ u(l,t) = h(t). & T \geq t \geq 0. \end{cases}$$

Then the maximum or minimum value of $u(x,t)$ is assumed either initially $t = 0$ or on the lateral sides $x = 0$ or $x = l$.

Proof. Consider $u^\epsilon(x,t) = u(x,t) - \epsilon t$ for $\epsilon > 0$. It satisfies

$$u_t^\epsilon - u_{xx}^\epsilon = -\epsilon < 0. \tag{2}$$

Because on the closed domain $\bar{U}_T = \{(x,t)|0 \leq x \leq l, T \geq t \geq 0\}$, there must be a maximum point of $u^\epsilon(x,t)$ say (x_0,t_0) in the space time $\bar{U}_T$. Denote $\Gamma_T = \partial U_T - \{t = T, 0 < x < l\}$. If we prove that $(x_0,t_0) \in \Gamma_T$, that is fine.

So we may suppose $(x_0,t_0) \in \{(x,t)|0 < x < l, T > t > 0\}$.Then at maximum point $u_t^\epsilon(x_0,t_0) = 0$ (max point is a critical point) and $-u_{xx}^\epsilon \geq 0$ (By Taylor expansion $u(x) = u(x_0) + u'(x_0)(x-x_0) + u''(x_0)(x-x_0)^2 + o((x-x_0)^2)$, because x_0 is max point and $u'(x_0) = 0$ thus $u''(x_0) \leq 0$.) which contradicts the equation (2) for any $\epsilon > 0$.

If $(x_0, t_0) \in \{t = T, 0 < x < l\}$, we also have $-u^\epsilon_{xx}(x_0, t_0) \geq 0$ and $u^\epsilon_t(x_0, t_0) \geq 0$ which is also a contradiction. So we have $\max_{\bar{U}_T} u^\epsilon = \max_{\Gamma_T} u^\epsilon$. Let $\epsilon \to 0$, we have

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

Similarly, we have

$$\min_{\bar{U}_T} u = \min_{\Gamma_T} u.$$

□

We can use maximum principle argument to prove the stability of the diffusion equation in the uniform norm sense. Suppose $f = g = h = 0$, we have

$$\max_{0 \leq x \leq l} |u_1(x,t) - u_2(x,t)| \leq \max_{0 \leq x \leq l} |\phi_1(x) - \phi_2(x)|.$$