Lecture 23

April 22, 2021

1 Green's function in Half-Space and Ball

From i) and iii) The Green's function $G(x, x_0)$ equals the fundamental solution $-\frac{1}{4\pi|x-x_0|}$ plus a harmonic function $H(x, x_0)$.

From ii) the boundary value vanishes. So we need to nd a function $H(x, x_0)$ which satises the Dirichelet problem

$$\begin{aligned} \triangle H(\cdot, x_0) &= 0 \quad in \quad \Omega, \\ H(\overrightarrow{x}, \overrightarrow{x_0}) &= -\frac{1}{4\pi|x-x_0|} \quad on \quad \partial\Omega. \end{aligned}$$

In some particular domains, we can find $H(x,x_0)$ by reflection method.

When $\Omega=\{z>0\}$, denote $\overrightarrow{x_0}=(x_0,y_0,z_0)$. The reflection point with respect to the plane $\{z=0\}$ is $\overrightarrow{x_0}^*=(x_0,y_0,-z_0)$ which is not in Ω. So let $H(\overrightarrow{x},\overrightarrow{x_0})=\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}^*|}$ which is harmonic in Ω. Moreover, on the boundary $\partial\Omega$, we can check that

$$H(\overrightarrow{x},\overrightarrow{x_0})=\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}} \quad = \quad \frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}|}.$$

So the Green's function on half-space is

$$G(\overrightarrow{x},\overrightarrow{x_0}) \quad = \quad -\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}|}+\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}^*|}.$$

When $\Omega=\{|x|<a\}$, the reflection point of $\overrightarrow{x_0}$ with respect to the sphere is $\overrightarrow{x_0}^*=\frac{a^2\overrightarrow{x_0}}{|\overrightarrow{x_0}|^2}$. We can check that $\triangle\frac{a}{4\pi|\overrightarrow{x_0}||\overrightarrow{x}-\overrightarrow{x_0}^*|}=0$ in Ω and $\frac{a}{4\pi|\overrightarrow{x_0}||\overrightarrow{x}-\overrightarrow{x_0}^*|}=\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}|}$ on $\partial\Omega$. So the Green's function on the sphere is

$$G(\overrightarrow{x},\overrightarrow{x_0})=-\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}|}+\frac{a}{4\pi|\overrightarrow{x_0}||\overrightarrow{x}-\overrightarrow{x_0}^*|} \quad = \quad -\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x_0}|}+\frac{1}{4\pi|\frac{|\overrightarrow{x_0}|}{a}\overrightarrow{x}-\frac{a\overrightarrow{x_0}}{|\overrightarrow{x_0}|}|}.$$

On $\partial B_a(0)$ we compute

$$\begin{aligned}\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}} &= \sum_{i=1}^{3}\frac{x^i}{r}\frac{\partial}{\partial x^i}G(x,x_0)\\ &= \sum_{i=1}^{3}\frac{x^i}{r}\Big[\frac{(x^i-x_0^i)}{4\pi|x-x_0|^3}-\frac{|x_0|}{a}\frac{\frac{|x_0|}{a}x^i-\frac{a}{|x_0|}x_0^i}{4\pi|\frac{|x_0|}{a}x-\frac{ax_0}{|x_0|}|^3}\Big].\end{aligned}$$

Because $|x-x_0|=|\frac{|x_0|}{a}x-\frac{ax_0}{|x_0|}|$ on $\partial B_a(0)$, we have

$$\begin{aligned}\frac{\partial G(x,x_0)}{\partial\overrightarrow{n}} &= \frac{|x|^2-\sum x^ix_0^i-\frac{|x_0|^2}{a^2}|x|^2+\sum x^ix_0^i}{4\pi r|x-x_0|^3}\\ &= \frac{a^2-|x_0|^2}{a^3}\frac{|x|^2}{4\pi|x-x_0|^3}.\end{aligned}$$

Thus by Theorem 2, we get the Poisson formula in dimension three

$$u(x_0) \quad = \quad \frac{a^2-|x_0|^2}{4\pi a}\iint_{\partial B_a(0)}\frac{u(x)}{|x-x_0|^3}dS.$$