Lecture 20

1 Maximum principle

Theorem 1. *(weak form) Let D be a bounded connected domain in $\mathbb{R}^n$. For a smooth solution u which satisfies $\triangle u = 0$. Then*

$$\max_{\overline{D}} u = \max_{\partial D} u$$

and

$$\min_{\overline{D}} u = \min_{\partial D} u.$$

Proof. Given a small $\epsilon > 0$, let $v(x) = u(x) + \epsilon |x|^2$. Then we have in D

$$\triangle v = \triangle u + 2n\epsilon = 2n\epsilon > 0. \tag{1}$$

If maximum of v attained at $x_0 \in D$, we have

$$D^2 v(x_0) \leq 0. \tag{2}$$

Thus (2) contradicts to (1). So the maximum point of v must be attained on the boundary which means $\max_{\overline{D}} v = \max_{\partial D} v$. So we get

$$\max_{\partial D} u \leq \max_{\overline{D}} u \leq \max_{\overline{D}} v = \max_{\partial D} v \leq \max_{\partial D} u + \epsilon l^2 \quad ,$$

where l is bounded by the diameter of the domain D.
Letting $\epsilon \to 0$, we have

$$\max_{\overline{D}} u = \max_{\partial D} u.$$

Similarly, for the proof of $\min_{\overline{D}} u = \min_{\partial D} u$, we consider $v(x) = u(x) - \epsilon |x|^2$. ☐

Exercise 2. Suppose u satisfies the equation $\triangle u + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0$ in $D \in \mathbb{R}^n$ where $c \leq 0$ and b_i are bounded constants. Prove that

$$\max_{D} u \leq \max_{\partial D} \{u, 0\}.$$

and

$$\min_D u \quad \geq \quad \min_{\partial D}\{u, 0\}.$$

Hint: let $v(x) = u(x) + \epsilon e^{\alpha x_1}$ for some large α.

Proposition 3. *(Mean value property) Let u be a harmonic function in a disk D, continuous in its closure $\overline{D}$. Then from Poisson's formula, we have the following mean value properties for any point $x_0 \in D$ and any ball $B_r(x_0) \subseteq D$*

$$u(x_0) \quad = \quad \frac{1}{2\pi r}\int_{|x'-x_0|=r} u(x')ds'.$$

and

$$u(x_0) \quad = \quad \frac{1}{\pi r^2}\int_{|x'-x_0|\leq r} u(x')dx'.$$

Remark. There are also mean value properties for n dimensional harmonic functions.

Proof. Without loss of generality, we assume that $x_0 = 0$.
Recall the Poisson's formula

$$u(x) \quad = \quad \frac{a^2 - |x|^2}{2\pi a}\int_{|x'|=a} \frac{u(x')}{|x - x'|^2}ds'.$$

Let $x = 0$, we have

$$\begin{aligned} u(0) &= \frac{a^2}{2\pi a}\int_{|x'|=a} \frac{u(x')}{|x'|^2}ds' \\ &= \frac{1}{2\pi a}\int_{|x'|=a} u(x')ds'. \end{aligned}$$

Mutiply both side by a and integrate from 0 to r

$$\begin{aligned} u(0)\frac{r^2}{2} = \int_0^r u(0)ada &= \frac{1}{2\pi}\int_0^r\int_{|x'|=a} u(x')ds' \\ &= \frac{1}{2\pi}\int_{|x'|\leq r} u(x')dx'. \end{aligned}$$

So we get

$$u(0) \quad = \quad \frac{1}{\pi r^2}\int_{|x'|\leq r} u(x')dx'.$$

□

Theorem 4. *(strong form) Let $u(x)$ be harmonic in D which is a bounded connected domain in $\mathbb{R}^n$. Then the maximum point $x_0 \notin D$ unless $u \equiv constant$. In other word, if maximum point $x_0 \in D$, then $u \equiv costant$.*

Proof. Denote $M = \max_{\overline{D}} u$. Set $\Sigma = \{x \in D; u(x) = M\}$. It is relatively closed in D. If $x_0 \in D$, We need to show $\Sigma = D$. From the mean value property, we have for $\overline{B}_r(x_0) \subseteq D$ for some $r > 0$

$$M = u(x_0) = \frac{1}{|B_r|} \int_{B_r(x_0)} u(x)dx \leq M.$$

Thus $B_r(x_0) \subseteq \Sigma$. This implies Σ is relatively open in D. In this way, using the assumption that D is connected, we deduce that $\Sigma = D$. □

Proposition 5. *Let u be a continuous harmonic function in any open set D of the plane. Then $u(x)$ is smooth in D. This also true for n-dimensional harmonic functions.*

Proof. For any point $x \in D$, there is a ball $B_a(x_0)$ such that $x \in B_a(x_0) \subseteq D$. The mean value property is

$$u(x) = \frac{a^2 - |x - x_0|^2}{2\pi a} \int_{|x' - x_0| = a} \frac{u(x')}{|x' - x|^2} ds'.$$

Because the denominator of the integrand $|x' - x|^2 \neq 0$ when $x \in B_a(x_0)$. It implies u is smooth in $B_a(x_0)$. □

Proposition 6. *The Dirichlet problem to the Laplace equation*

$$\begin{aligned} \triangle u = f \quad & in \quad D \\ u = h \quad & on \quad \partial D \end{aligned}$$

is unique.

Proof. Suppose u and v are solutions all satisfy the above Dirichlet problem. Let $w = u - v$ which satisfies

$$\begin{aligned} \triangle w = 0 \quad & in \quad D \\ w = 0 \quad & on \quad \partial D. \end{aligned}$$

By the maximum principle we have

$$0 = \min_{\partial D} u \leq \quad w_{\min} \leq w(x) \leq w_{\max} \quad \leq \max_{\partial D} u = 0.$$

So we get $w \equiv 0$ which proved the uniqueness. □

Proposition 7. *Suppose that $u \in C^2(\overline{B}_R(x_0))$ is harmonic. Then there holds*

$$|Du(x_0)| \leq \frac{n}{R} \max_{\overline{B}_R} |u|.$$

Proof. Because $\frac{\partial u}{\partial x_i}$ satisfies

$$\triangle \frac{\partial u}{\partial x_i} = 0.$$

Hence $\frac{\partial u}{\partial x_i}$ has the mean value inequality

$$\begin{aligned}\frac{\partial u}{\partial x_i}(x_0) &= \frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} \frac{\partial u}{\partial x_i}(y)dy \\ &= \frac{1}{|B_R(x_0)|}\int_{\partial B_R(x_0)} u(y)\frac{y_i}{R}dS_y.\end{aligned}$$

The last equation is due to divergence theorem. So we have

$$\begin{aligned}|Du(x_0)|^2 = \sum_i |\frac{\partial u}{\partial x_i}|^2(x_0) &\leq \frac{1}{|B_R(x_0)|^2}\sum_i (\int_{\partial B_R(x_0)} u(y)\frac{y_i}{R}dS_y)^2 \\ &\leq \frac{|\partial B_R(x_0)|}{|B_R(x_0)|^2}\int_{\partial B_R(x_0)} u^2(y)\sum_i (\frac{y_i}{R})^2 dS_y \\ &= (\frac{|\partial B_R(x_0)|}{|B_R(x_0)|}\max_{\overline{B}_R}|u|)^2 \\ &= (\frac{n}{R}\max_{\overline{B}_R}|u|)^2.\end{aligned}$$

□

Exercise 8. A bounded harmonic function in $\mathbb{R}^n$ is constant.

Exercise 9. Suppose $u \in C^2(\overline{D})$ satisfies

$$\begin{aligned}\triangle u + cu = f(x) \quad & in \quad D \\ u = \varphi(x) \quad & on \quad \partial D\end{aligned}$$

for some $f \in C(\overline{D})$ and $\varphi \in C(\partial D)$. If $c \leq 0$, then show that

$$|u(x)| \leq \max_{\partial D}|\varphi| + C\max_{\overline{D}} f$$

for any $x \in D$. Where C is a positive constant which depends on diameter of D.

Example 10. Solve laplace equation in a Wedge

$$\begin{aligned}\triangle_2 u = 0 \quad in \quad W = \{(r,\theta); 0 < r < a, 0 < \theta < \beta\} \\ u(r,0) = 0 = u(r,\beta) \\ \frac{\partial u}{\partial r}(a,\theta) = h(\theta).\end{aligned}$$

Proof. By separation of variable in polar coordinate, we get the solution

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{n\pi/\beta} \sin\frac{n\pi\theta}{\beta}$$

with coefficients given by

$$A_n = a^{1-n\pi/\beta}\frac{2}{n\pi}\int_0^{\beta} h(\theta)\sin\frac{n\pi\theta}{\beta}d\theta.$$

□

Example 11. Solve laplace equation in annulus

$$\begin{aligned} \triangle_2 u = 0 \quad & in \quad A = \{0 < a^2 < x^2+y^2 < b^2\} \\ u = g(\theta) \quad & on \quad x^2+y^2 = a^2 \\ u = h(\theta). \quad & on \quad x^2+y^2 = b^2. \end{aligned}$$

Proof. By separation of variable in polar coordinate, we get the solution

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0\log r) + \sum_{n=1}^{\infty}(C_n r^n + D_n r^{-n})\cos n\theta + (A_n r^n + B_n r^{-n})\sin n\theta.$$

Note that in this case we don't throw out the function r^{-n} and $\log r$.
From the boundary condition, the coefficients need to satisfy

$$\begin{aligned} C_0 + D_0\log a &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)d\theta \\ C_0 + D_0\log b &= \frac{1}{\pi}\int_0^{2\pi} h(\theta)d\theta, \end{aligned}$$

and

$$\begin{aligned} C_n a^n + D_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta d\theta \\ C_n b^n + D_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta d\theta, \end{aligned}$$

and

$$\begin{aligned} A_n a^n + B_n a^{-n} &= \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta d\theta \\ A_n b^n + B_n b^{-n} &= \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin n\theta d\theta. \end{aligned}$$

Thus

$$\begin{aligned} D_0 &= \frac{1}{\pi\log\frac{b}{a}}\int_0^{2\pi}(h(\theta)-g(\theta))d\theta \\ C_0 &= \frac{1}{\pi}\int_0^{2\pi}[\frac{\log b}{\log b-\log a}g(\theta) - \frac{\log a}{\log b-\log a}h(\theta)]d\theta, \end{aligned}$$

and

$$\begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos n\theta d\theta \\ \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos n\theta d\theta \end{bmatrix},$$

and

$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin n\theta d\theta \\ \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin n\theta d\theta \end{bmatrix}.$$

□

Example 12. Solve laplace equation in the exterior of a disk

$$\begin{aligned} \triangle_2 u = 0 \qquad & \textit{in} \qquad x^2+y^2 > a^2 \\ u = h(\theta) \qquad & \textit{on} \qquad x^2+y^2 = a^2 \\ u \quad & \textit{bounded as} \quad x^2+y^2 \to \infty. \end{aligned}$$

Proof. Because u is bounded as $r \to \infty$, in this case we throw out the function r^n and $\log r$. By separation of variable in polar coordinate, we get the solution

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n \cos n\theta + B_n \sin n\theta)$$

with the coefficients given by

$$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\cos n\theta d\theta$$

and

$$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} h(\theta)\sin n\theta d\theta.$$

So we get Poisson's formula in this case

$$u(r,\theta) = (r^2 - a^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}\frac{d\phi}{2\pi}$$

for $r > a$. □