The limit Rademacher functions and Bernoulli convolutions associated with Pisot numbers

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Abstract

We introduce an algebraic method to study the local properties of infinite Bernoulli convolution measures associated with the reciprocals of Pisot numbers. The distributions of these measures on so called "colored net intervals" are shown to be the products of matrixes, moreover for a class of Pisot numbers, these matrix products can be decomposed into the products of real numbers. The explicit values of some fractal dimensions for the limit Rademacher functions and Bernoulli convolutions associated with the positive root of $x^k + x^{k-1} + \cdots + x - 1$ $(k = 2, 3, \cdots)$ are obtained. Part of these results answer some open questions and conjectures.

Contents

1 Introduction

For $0 < \rho < 1$, the limit Rademacher function f_ρ is defined as:

$$f_\rho(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n R(2^n x), \quad x \in [0,1] \tag{1}$$

where R denotes the Rademacher function: $R(x)$ is defined on the line $\mathbb{R}$ with period 1, takes values 0 and 1 on the intervals $[0,1/2)$ and $[1/2,1)$ respectively.

The distribution of f_ρ induces a probability measure μ_ρ on $[0,1]$, which is called an *infinitely convolved Bernoulli measure* or simply a *Bernoulli convolution*. That is,

$$\mu_\rho(E) = \mathcal{L}\{x \in [0,1] : \ f_\rho(x) \in E\}, \ \forall E \subset [0,1] \text{ measurable}$$

where $\mathcal{L}$ denotes 1-dimensional Lebesgue measure. The Bernoulli convolution μ_ρ measures the density of points of the form $(1-\rho)\sum_{n=0}^{\infty} \rho^n r_n$ ($r_n = 0$, or 1). More precisely, for any interval $(a,b) \subset [0,1]$ and any positive integer m, let $\mu_{\rho,m}(a,b)$ denote the proportion of points of the form $(1-\rho)\sum_{n=0}^{m-1} \rho^n r_n$ ($r_n = 0$, or 1) that lie in (a,b), that is

$$\mu_{\rho,m}(a,b) = 2^{-m} \# \left\{ (r_0, r_1, \cdots, r_{m-1}) \in \{0,1\}^m : (1-\rho) \sum_{n=0}^{m-1} \rho^n r_n \in (a,b) \right\},$$

then

$$\mu_\rho(a,b) = \lim_{m \to \infty} \mu_{\rho,m}(a,b).$$

Project supported by China Postdoctoral Science Foundation .

Keywords: Rademacher function, Bernoulli convolution, Pisot number, Hausdorff dimension, information dimension, L^q-spectrum, matrix product.

1991 Mathematics Subject Classification. Primary 28A78, Secondary 28D20, 58F11, 54C30, 11R06

Equivalently μ_ρ can be expressed as an infinite convolution. Let $\beta(x)$ denote the measure with two atoms, each of weight $1/2$, at the point $0, x$. Then the measure μ_ρ can be expressed as the infinite convolution

$$\mu_\rho = \beta(1-\rho) * \beta((1-\rho)\rho) * \cdots * \beta((1-\rho)\rho^n) * \cdots .$$

μ_ρ is also a self-similar measure satisfying the following equation:

$$\mu_\rho = \frac{1}{2}\mu_\rho \circ \phi_{0,\rho}^{-1} + \frac{1}{2}\mu_\rho \circ \phi_{1,\rho}^{-1}, \tag{2}$$

where $\phi_{0,\rho}$ and $\phi_{1,\rho}$ are two similar contraction mappings defined by $\phi_{0,\rho}(x) = \rho x$ and $\phi_{1,\rho}(x) = \rho x + (1-\rho)$.

These Bernoulli convolutions have been studied for more than 60 years(for a good survey, see [PSS]). For $0 < \rho < 1/2$, the support of μ_ρ is a Cantor set of zero Lebesgue measure and μ_ρ is totally singular with respect to the Lebesgue measure. For $\rho = 1/2$, μ_ρ is just the 1-dimensional Lebesgue measure restriction on $[0,1]$. For $1/2 < \rho < 1$, μ_ρ is only partially understood still now. It follows from a theorem of Jesson and Wintner [JW] that μ_ρ is either absolute continuous or totally singular with respect to the Lebesgue measure(recently, Mauldin and Simon [MS] proved the converse result, that is, the Lebesgue measure is either absolute continuous or totally singular with respect to μ_ρ). Erdős proved that μ_ρ is absolutely continuous for almost all ρ closed enough to one [Er1]. He conjectured that the result should be true for almost all $1/2 < \rho < 1$. Solomyak [Sol] has recently proved this conjecture to be true. In spite of this, the only explicit values of ρ for which μ_ρ is known to be absolutely continuous are $\rho = 2^{-1/n}$($n = 1, 2, \cdots$) discovered by Wintner [Win], and a family of algebraic numbers discovered by Garsia [G1]. On the other hand, Erdös showed that when ρ is the reciprocal of a Pisot number, then not only μ_ρ is totally singular but actually its Fourier transformation $\hat{\mu}_\rho(x)$ does not even tend to zero at infinite [Er2]. We should recall that an algebraic integer is called a Pisot number if all its conjugates are less than 1 in modulus. Up to now, the reciprocals of Pisot numbers are the only $\rho > 1/2$ for which μ_ρ is known to be totally singular. The only quadratic ρ of this type is $\rho = (-1+\sqrt{5})/2$, the root of $x^2 + x - 1$. The only such ρ satisfying a cubic equation are the real roots of the following four polynomials: $x^3 + x^2 + x - 1$, $x^3 + x^2 - 1$, $x^3 + x - 1$, $x^3 - x^2 + 2x - 1$, with $\rho \approx 0.5436898, 0.7548777, 0.6823278, 0.5698403$, respectively. Other examples of such ρ are the roots of the polynomials $x^n + x^{n-1} + \cdots + x - 1$, where $n \geq 2$. The reader may refer to [Sa] and [BDGPS] for further information about Pisot numbers.

Bernoulli convolutions have been studied since 1930's, originally because they are interesting examples of phenomena in harmonic analysis. The work of Alexander and Yorke[AY] relates to dynamics this old measure problem. They consider the transformation T_ρ : $\mathbb{R} \times [0,1] \circlearrowleft$:

$$T_\rho(x,y) = \begin{cases} (\rho x, 2y), & y \leq 1/2 \\ (\rho x + 1 - \rho, 2y - 1), & y > 1/2 \end{cases}$$

For $1/2 < \rho < 1$, T_ρ is the "fat" Baker's transformation, the map is not invertible and the attractor is the unit square $[0,1] \times [0,1]$, it possesses a Sinai-Bowen-Ruelle measure whose transverse component is just the Bernoulli convolution μ_ρ.

The nature of the measure μ_ρ will affect the Hausdorff dimension of the graph of f_ρ. As a class of Weierstrass-like functions, the limit Rademacher functions have been studied by many people. For $0 < \rho \leq 1/2$, both the Hausdorff dimension and box-counting dimension of the graph of f_ρ are equal to 1(see [PM] for a proof). For $1/2 < \rho < 1$, it is easy to show that the box-counting dimension of the graph of f_ρ is equal to $2 - \frac{\log \rho^{-1}}{\log 2}$ (see [PU]). Przytycki and Urbański [PU] showed that the Hausdorff dimension of the graph of f_ρ is equal to $2 - \frac{\log \rho^{-1}}{\log 2}$ when μ_ρ is absolutely continuous (this was also proved with an additional hypothesis in [HL1],[SS]), and it is strictly less than $2 - \frac{\log \rho^{-1}}{\log 2}$ when ρ^{-1} is a Pisot number. The Hausdorff dimension of t-level set of f_ρ is $1 - \frac{\log \rho^{-1}}{\log 2}$ for almost all t if μ_ρ is absolutely continuous [HL2]. The reader may refer to [Fal, Mat] for the definitions and properties of Hausdorff dimension and box-counting dimension.

This paper concerns the study of local properties of μ_ρ, and the explicit computations of some fractal dimensions associated with f_ρ and μ_ρ when ρ is the reciprocal of a Pisot number. As we have mentioned, in such case, μ_ρ is totally singular. To describe the degree of singularity of μ_ρ, we can study its local dimensions, information dimension, Hausdorff dimension, or L^q-spectrum ($q \in \mathbb{R}$). Suppose that ν is a Borel measure on the line. Recall that the *upper local dimension* of ν at x is defined by

$$\overline{d}(\nu, x) = \limsup_{r \to 0+} \frac{\log \nu([x-r, x+r])}{\log r},$$

and the *lower local dimension* $\underline{d}(\nu, x)$ is defined similarly by using the lower limits. When $\overline{d}(\nu, x) = \underline{d}(\nu, x)$, the common value is called the *local dimension* of ν at x and is denoted by $d(\nu, x)$. Now let us recall the definition of information dimension of ν. If κ is a finite partition of the line, let the diameter of κ , denoted diam κ, be the maximum of the diameters of the elements of κ. Set

$$\begin{cases} g(x) = x \log x^{-1} & 0 < x \leq 1, \\ g(0) = 0. \end{cases}$$

Define

$$h(\nu, \kappa) = \sum_{A \in \kappa} g(\nu(A)).$$

For $\epsilon > 0$, let

$$h(\nu, \epsilon) = \inf\{h(\nu, \kappa) : \text{ diam } \kappa \leq \epsilon\}.$$

Then we define upper and lower *information dimensions*:

$$\overline{\dim}_{\text{info}}(\nu) = \limsup_{\epsilon \to 0} \frac{h(\nu, \epsilon)}{\log \epsilon^{-1}}$$
$$\underline{\dim}_{\text{info}}(\nu) = \liminf_{\epsilon \to 0} \frac{h(\nu, \epsilon)}{\log \epsilon^{-1}}$$

and if they are equal $\dim_{\text{info}}(\nu)$ denotes the common value. Recall the *Hausdorff dimension* of ν is defined by

$$\dim_H(\nu) := \inf\{\dim_H(E) : \quad \nu(E) = 1\},$$

where $\dim_H(E)$ denotes the Hausdorff dimension of E. And the L^q*-spectrum* ($q \in \mathbb{R}$) of ν is defined by

$$\tau_\nu(q) = \liminf_{\delta \to 0+} \frac{\log \sup \sum_i \nu(B_\delta(x_i))^q}{\log \delta},$$

where $\{B_\delta(x_i)\}_i$ is a disjoint family of δ-balls with center $x_i \in \text{supp}(\mu)$. We refer the reader to [Pe] for more information about the above definitions.

Now, we recall some known results about the dimensions and L^q-spectrum for Bernoulli convolutions. In [Hu], T.-Y. Hu has obtained the explicit maximal and minimal values of the local dimensions of μ_ρ whenever $\rho = \frac{\sqrt{5}-1}{2}$(he also has given a generalized formula whenever ρ is the positive root of $1 - x - x^2 - \cdots - x^k (k \geq 3)$, however, as we will point out (see Theorem 51) that his generalized formula is incorrect). Recently, Lau and Ngai have given a closed formula of the L^q-spectrum of μ_ρ for $\rho = \frac{\sqrt{5}-1}{2}$ and $q > 0$ [LN2] (they asked a question that how to deal with the case $q < 0$), they also have given an algorithm to calculate the L^q-spectrum of μ_ρ when ρ is the reciprocal of a Pisot number and q is a positive integer [LN3]. For $\rho = \frac{\sqrt{5}-1}{2}$, the Hausdorff dimension and information dimension of μ_ρ have been studied by a number of authors ([AY], [AZ], [LP1], [Ng], [SV]), these two values are shown to be equal, and some different explicit theoretical formulas and numerical results for which have been obtained. Przytycki and Urbański [PU] proved that $\dim_H \ \mu_\rho < 1$ if ρ is the reciprocal of a Pisot number.

In this paper, we give an efficient algebraic method to analyze the local properties of μ_ρ when ρ is the reciprocal of a Pisot number. As a result, we show that the measure μ_ρ on the neighborhoods ("net intervals") of any given point $x \in [0, 1]$ can be estimated explicitly by some products of matrixes (see Theorem 53), more importantly, these products of matrixes can be decomposed into the products of real numbers for a class of parameters ρ's (i.e. the reciprocals of Pisot numbers of *the first class*, see section 6), such is the case when $\rho = \lambda_k$ is the positive root of $1-x-x^2-\cdots-x^k (k \geq 2)$ (see Lemma 15, Lemma 40), or ρ is the positive root of $x^3 - x^2 + 2x - 1$. By using some combinatorial and ergodic techniques, we obtain the explicit formulas for the Hausdorff dimension of the graphs of f_{λ_k} $(k \geq 2)$, the Hausdorff dimension of almost all level sets (respect to Lebesgue measure) of f_{λ_k} $(k \geq 2)$, the Hausdorff dimension, information dimension and the L^q-spectrum ($q \in \mathbb{R}$) of μ_{λ_k} $(k \geq 2)$. We also give the formal formulas (in the form of matrix products) for the Hausdorff dimension of the graph of f_ρ and L^q-spectrum ($q \in \mathbb{R}$) of μ_ρ when ρ is the reciprocal of a general Pisot numbers(see Theorem 56 and Theorem 57). Our method is also valid to analyze biased Bernoulli convolutions associated with Pisot numbers and some other self-similar measure with some separate condition(see Section 7).

Set

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_\emptyset = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

For any $J = j_1 \cdots j_n \in \{0,1\}^n$, denote $M_J = M_{j_1} \circ \cdots \circ M_{j_n}$. For any 2×2 non-negative matrix B, denote by $||B|| = (1,1)B(1,1)'$.

We can formulate our main results as follows:

Theorem A. *For each integer $k \geq 2$, let $\alpha_k = -\dfrac{\log \lambda_k}{\log 2}$. Then the Hausdorff dimension of the graph of the limit Rademacher function f_{λ_k} satisfies that*

$$\dim_H \ \mathrm{Graph}(f_{\lambda_k}) = \frac{\log x_k}{\log \lambda_k},$$

where $0 < x_k \leq \lambda_{k-1}$ (defining $\lambda_1 = 1$) and x_k satisfies that

$$\frac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} \cdot \sum_{n=0}^{\infty} \Big(\sum_{|J|=n} ||M_J||^{\alpha_k} \Big) x^{kn+k+1} = 1.$$

Let $\alpha \in \mathbb{R}$, the α-level set of a function f is defined as $\{x : \ f(x) = \alpha\}$.

Theorem B. For $k \geq 2$, *the Hausdorff dimension and box-counting dimension of t-level set of the limit Rademacher function f_{λ_k} are equal to*

$$d_{\lambda_k} := \frac{(\lambda_k)^k (1 - 2(\lambda_k)^k)^2}{(2 - (k+1)(\lambda_k)^k) \log 2} \sum_{n=0}^{\infty} \Big((\lambda_k)^{kn} \sum_{|J|=n} \log ||M_J|| \Big)$$

for almost all $t \in [0,1]$ (respect to the Lebesgue measure).

Theorem C.(i) *For any $q \in \mathbb{R}$, the L^q-spectrum $\tau_{\lambda_2}(q)$ of μ_{λ_2} is equal to*

$$\frac{q \log 2}{\log \lambda_2^{-1}} + \frac{\log \mathbf{x}(2,q)}{\log \lambda_2^{-1}},$$

where

$$\mathbf{x}(2,q) = \sup\{x \geq 0 : \ \sum_{n=0}^{\infty} \Big(\sum_{|J|=n} ||M_J||^q \Big) x^{2n+3} \leq 1\}.$$

There exists a unique $q_0 < -2$ such that $\sum_{n=0}^{\infty} (\sum_{|J|=n} ||M_J||^{q_0}) = 1$. When $q > q_0$, $\mathbf{x}(2,q)$ is the positive root of $\sum_{n=0}^{\infty} (\sum_{|J|=n} ||M_J||^q) x^{2n+3} = 1$, and it is an infinitely differentiable function of q on $(q_0, +\infty)$. When $q \leq q_0$, $\mathbf{x}(2,q) = 1$. Moreover $\mathbf{x}(2,q)$ is not differentiable at $q = q_0$.

(ii) *For any integer $k \geq 3$ and any real number q, the L^q-spectrum $\tau_{\lambda_k}(q)$ of the Bernoulli convolution μ_{λ_k} is equal to*

$$\frac{q \log 2}{\log \lambda_k^{-1}} + \frac{\log \mathbf{x}(k,q)}{\log \lambda_k^{-1}},$$

where $0 < \mathbf{x}(k,q) < \lambda_{k-1}$, *and* $\mathbf{x}(k,q)$ *satisfies that*

$$\frac{1-2x^{k-1}+x^k}{1-2x+x^k}\cdot\sum_{n=0}^{\infty}(\sum_{|J|=n}||M_J||^q)x^{kn+k+1}=1.$$

Moreover $\mathbf{x}(k,q)$ *is an infinitely differentiable function on the whole line.*

Theorem D. *For any integer* $k\geq 2$, *the Hausdorff dimension and the information dimension of the Bernoulli convolution* μ_{λ_k} *satisfy that*

$$\dim_H \mu_{\lambda_k} = \dim_{\mathbf{info}} \mu_{\lambda_k} = -\frac{\log 2}{\log \lambda_k} + \left(\frac{2^k-3}{2^k-1}\right)^2 \cdot \frac{\sum_{n=0}^{\infty} 2^{-kn-k-1} \sum_{|J|=n} ||M_J|| \log ||M_J||}{\log \lambda_k}.$$

Theorem E. *For any integer* $k\geq 2$, *define*

$$\mathcal{R}(\mu_{\lambda_k}) := \{y:\ \exists x\in[0,1],\ d(\mu_{\lambda_k},x)=y\}.$$

where $d(\mu_{\lambda_k},x)=\lim_{r\to 0}\log(\mu_{\lambda_k}(x-r,x+r))/\log r$ *(if the limit exists). Then*

$$\mathcal{R}(\mu_\lambda)=[\frac{\log 2}{\log\lambda^{-1}}-\frac{1}{2},\frac{\log 2}{\log\lambda^{-1}}].$$

And

$$\mathcal{R}(\mu_{\lambda_k})=[\frac{k}{k+1}\cdot\frac{\log 2}{\log\lambda_k^{-1}},\quad \frac{\log 2}{\log\lambda_k^{-1}}].$$

for $k\geq 3$.

In the following table we give some numerical estimations of $\dim_H \mathrm{Graph}(f_{\lambda_k})$, d_{λ_k} and $\dim_H \mu_{\lambda_k}$ for $2\leq k\leq 10$.

Remark (i) Theorem B has also some explanation in the language of beta-expansion as follows: let ρ $(1/2<\rho<1)$ be the reciprocal of a Pisot number, for $x\in[0,1/(1-\rho)]$ we can define the "local ρ-expansion multiplicity" of x as

$$\Delta_\rho(x)=\lim_{m\to\infty}(\log N_{\rho,m}(x))/m$$

if the above limit exists, where $N_{\rho,m}(x)$ is equal to the cardinality of the set $\{(\epsilon_0,\cdots,\epsilon_{m-1})\in\{0,1\}^m:\ \exists(\epsilon_m,\epsilon_{m+1},\cdots)\in\{0,1\}^\infty, x=\sum_{i=0}^{+\infty}\epsilon_i\rho^i\}$. Then Theorem B deduces that for $\rho=\lambda_k$ $(k\geq 2)$,

$$\Delta_\rho(x)=\frac{(\lambda_k)^k(1-2(\lambda_k)^k)^2}{(2-(k+1)(\lambda_k)^k)}\sum_{n=0}^{\infty}((\lambda_k)^{kn}\sum_{|J|=n}\log||M_J||)$$

for almost all $x\in[0,1/(1-\rho)]$.

Table 1: numerical estimations

k	$\dim_H \mathbf{Graph}(\mathbf{f}_{\lambda_k})$	$\mathbf{d}_{\lambda_k}$	$\dim_H \boldsymbol{\mu}_{\lambda_k}$
2	1.304 ± 0.001	0.302 ± 0.001	0.9957 ± 10^{-4}
3	1.11875217 ± 10^{-8}	$0.1025001503 \pm 10^{-10}$	$0.98040931953 \pm 10^{-11}$
4	1.052565407 ± 10^{-9}	$0.041560454940769 \pm 10^{-14}$	$0.9869264743338 \pm 10^{-12}$
5	1.024596045 ± 10^{-9}	$0.01842625239655 \pm 10^{-14}$	$0.9925853002741 \pm 10^{-12}$
6	1.011844824 ± 10^{-9}	$0.00859023108854 \pm 10^{-14}$	$0.9960325915849 \pm 10^{-12}$
7	1.005796386 ± 10^{-9}	$0.00412363866083 \pm 10^{-14}$	$0.9979374455070 \pm 10^{-12}$
8	1.002862729 ± 10^{-9}	$0.00201383805752 \pm 10^{-14}$	$0.9989449154498 \pm 10^{-12}$
9	1.001421378 ± 10^{-9}	$0.00099344117302 \pm 10^{-14}$	$0.9994653680555 \pm 10^{-12}$
10	1.000707890 ± 10^{-9}	$0.00049294459129 \pm 10^{-14}$	$0.9997306068783 \pm 10^{-12}$

(ii) Theorem C gives an answer to Lau and Ngai's questions ([LN1], [LN2]) that how to determine the L^q-spectrum of μ_{λ_2} for $q < 0$, and how to determine L^q-spectrum of $\mu_{1/\beta}$ for Pisot parameters β other than $(\sqrt{5}+1)/2$;

(iii) Theorem D confirms an conjecture given by Alexander and Zagier [AZ], that "it seems likely" to give an explicit formula for $\dim_H(\mu_{\lambda_k})$, $k = 3, 4, \cdots$.

(iv) In Theorem E, the formula $\mathcal{R}(\mu_\lambda)$ was first given by Hu [Hu]; however, our result about $\mathcal{R}(\mu_{\lambda_k})$ ($k \geq 3$) correct Hu's corresponding result.

(v) In a subsequent paper [Fe], with some additive work we have given detailed mutifractal analysis of μ_{λ_k} ($k \geq 2$) by using Theorem C.

Now we would like to explain why μ_ρ ($1/2 < \rho < 1$) is difficult to understand, and how our method works when ρ is the reciprocal of a Pisot number. For $0 < \rho \leq 1/2$, μ_ρ is easy to understand because for any $x \in \mathrm{supp}(\mu_\rho)$ and $0 < r \leq 1$,

$$\mu_\rho([x-r, x+r]) \approx 2^{-m},$$

where m satisfies $\rho^{m+1} \leq r < \rho^m$. This result is easy to deduce from the fact that $\phi_{0,\rho}([0,1]) \cap \phi_{1,\rho}([0,1])$ has at most one point and thus has null μ_ρ measure. If $1/2 < \rho < 1$, $\phi_{0,\rho}([0,1]) \cap \phi_{1,\rho}([0,1])$ is an interval and has positive μ_ρ measure. In such case, to estimate $\mu_\rho(a,b)$ for an interval (a,b), one must estimate the number of points (including multiplicity) of the form $(1-\rho)\sum_{n=0}^{m-1} \rho^n r_n$ ($r_n = 0, 1$) which lie in (a,b) for $m \in \mathbb{N}$, however, this is difficult in general since the distribution of the points $(1-\rho)\sum_{n=0}^{m-1} \rho^n r_n$ ($r_n = 0,1$) is very complicated because of the overlap of $\phi_{0,\rho}$ and $\phi_{1,\rho}$. Nevertheless, when ρ is the reciprocal of a Pisot number, there is a rule followed from [G1] (which we call the *local finiteness of* ρ) on the distribution of these points: given any two points x, y of the form $(1-\rho)\sum_{n=0}^{m-1} \rho^n r_n$ ($r_n = 0, 1$), if the distance $|x-y| \leq \rho^m$, then the number of all possible different values for $\rho^{-m}|x-y|$ is finite (not depending on m), that is

$$\#\{\rho^{-m}|x-y| : x, y \text{ of form } (1-\rho)\sum_{n=0}^{m-1}\rho^n r_n (r_n = 0,1), |x-y| \le \rho^m\}$$

is uniformly bounded for $m \in \mathbb{N}$. It is just relying on this rule that we can analyze the local property of μ_ρ. For each $m \in \mathbb{N}$, we partition the interval $[0,1]$ into some subintervals, which will be called m-th net intervals, by the points $\rho^m r_m + (1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0, 1$ for $0 \le n \le m$). We associate each m-th interval with two special sort of vectors– its I-color and II-color. These two vectors contain respectively the information about the distribution and the multiplicity of the points of form $(1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0, 1$) which lie "nearly" to the net interval. In fact, the local properties for μ_ρ on a net intervals are completed determined by its colors.. When ρ^{-1} is a Pisot number, by the local finiteness of ρ we can show that the number of the possible different I-colors is finite, thus we can associate each m-th net interval with a Markov code of length m according to the I-colors, and due to this code we can determine the II-color of this net interval by the product of m matrixes (the number of different matrixes is finite). Thus we can determine the explicit value of the measure μ_ρ on each net interval. Furthermore, we can represent each point of $[0,1]$ by an infinite Markov code of finite states. Therefore we can use symbolic space to describe μ_ρ. For $x \in [0,1]$, and $m \in \mathbb{N}$, denote

$$N_{m,\rho}(x) = \#\{\mathbf{i} = i_1 i_2 \cdots i_m \in \{0,1\}^m : \quad x \in \phi_{\mathbf{i},\rho}([0,1])\},$$

where $\phi_{\mathbf{i},\rho} = \phi_{i_1,\rho} \circ \phi_{i_2,\rho} \circ \cdots \circ \phi_{i_m,\rho}$. Then the μ_ρ measure on the interval $(x-\rho^m, x+\rho^m)$ can be shown to satisfy the following relation

$$\mu_\rho(x-\rho^m, x+\rho^m) \approx 2^{-m} N_{m,\rho}(x)$$

when ρ^{-1} is a Pisot number. In such case, $N_{m,\rho}(x)$ is shown to be the norm of the product of m matrixes (the number of different matrixes is finite), and for a class of ρ's this product of matrixes can be decomposed into the product of some integers. We should point out that Ledrappier and Porzio have considered the local properties of μ_ρ for $\rho = (\sqrt{5}-1)/2$ [LP1], by using the technique of dynamics ("Markov partition") and the combinatorial properties of $(\sqrt{5}-1)/2$, they showed that $N_{m,\rho}(x)$ is the product of some matrixes (the number of different matrixes is countable infinite).

The paper is organized as follows: In Section 2, we introduce the definitions of net intervals and their I-colors and II-colors, and present some basic properties. In Section 3, we give some elementary properties of the graph and level sets of f_ρ. In section 4, we consider the case $\rho = \lambda_2 = \frac{\sqrt{5}-1}{2}$. We present in detail the generating relations of I-colors and II-colors, the process of labelling net interval by Markov codes, and the measure distribution on net intervals, and prove in detail our main theorems associated with this parameter. In Section 5, we deal with the case $\rho = \lambda_k (k \ge 3)$, and prove the relative results. In Section 6, we summarize some general results associated with other Pisot numbers. In Section 7, we point out that with some additive work our method can be used to analyze biased Bernoulli convolutions associated with Pisot numbers, or more generally, the self-similar measures generated by a family of similitudes satisfying "weak separate condition". In section 8, as an appendix, we present the generating relations of I-colors for ρ which is the positive root of $x^3 - x^2 + 2x - 1 = 0$.

2 Net intervals, I-colors and II-colors

In this section, we present the definitions and some basic properties of net intervals, I-colors and II-colors. We show that the number of all possible I-colors associated with the reciprocal of any given Pisot number is finite (see Lemma 2). Using this finiteness, we give the formal formulas (in the form of overlap times of net intervals) for the Hausdorff dimension of the graph f_ρ and the L^q-spectrum ($q \in \mathbb{R}$) of μ_ρ when ρ is the reciprocal of a Pisot number (see Corollary 5 and Lemma 9).

2.1 The definitions

Let $1/2 \le \rho \le 1$. The mappings $\phi_{0,\rho}, \phi_{1,\rho} : \mathbb{R} \to \mathbb{R}$ are defined as

$$\phi_{0,\rho}(x) = \rho x, \quad \phi_{1,\rho}(x) = \rho x + 1 - \rho.$$

For any $m \in \mathbb{N}$ and $\omega = (i_j)_{j=1}^m \in \{0,1\}^m$, we write

$$\phi_{\omega,\rho} = \phi_{i_1,\rho} \circ \cdots \circ \phi_{i_m,\rho}.$$

Denote by $P_{m,\rho}$ the set of endpoints of $\phi_{\omega,\rho}([0,1])$ ($\omega \in \{0,1\}^m$), that is

$$P_{m,\rho} = \{\phi_{\omega,\rho}(0) : \omega \in \{0,1\}^m\} \bigcup \{\phi_{\omega,\rho}(1) : \omega \in \{0,1\}^m\}.$$

One can see that $\phi_{\omega,\rho}(0) = (1-\rho)\sum_{n=0}^{m-1} \rho^n i_{n+1}$ and $\phi_{\omega,\rho}(1) = \rho^m + (1-\rho)\sum_{n=0}^{m-1} \rho^n i_{n+1}$ for $\omega = (i_j)_{j=1}^m$. Thus $P_{m,\rho}$ consists of all the points of the form

$$\rho^m r_m + (1-\rho)\sum_{n=0}^{m-1} \rho^n r_n \qquad (r_n = 0, \text{or } 1 \ \text{ for } 0 \le n \le m).$$

It is easy to see $P_{m,\rho} \subset P_{m+1,\rho}$ from the fact that $\phi_{\omega,\rho}(0) = \phi_{\omega\circ 0,\rho}(0), \phi_{\omega,\rho}(1) = \phi_{\omega\circ 1,\rho}(1)$. The points in $P_{m,\rho}$ partition $[0,1]$ into some non-overlap closed intervals, each of which is called *a m-th net interval associated with ρ*. For example, the 1-th net intervals associated with any ρ are

$$[0, 1-\rho], \ [1-\rho, \rho], \ [\rho, 1]$$

respectively; The 2-th net intervals associated with $\rho = (\sqrt{5}-1)/2$ are

$$[0, \rho^3], \ [\rho^3, \rho^2], \ [\rho^2, \rho], \ [\rho, 2\rho^2], \ [2\rho^2, 1].$$

The 3-th net intervals associated with $\rho = (\sqrt{5}-1)/2$ are

$$[0,\rho^4], [\rho^4,\rho^3], [\rho^3,\rho^2], [\rho^2, 2\rho^3], [2\rho^3, \rho^2+\rho^4], [\rho^2+\rho^4, \rho], [\rho, 2\rho^2],$$

and $[2\rho^2, \rho^2+2\rho^3]$, $[\rho^2+2\rho^3, 1]$.

Since $P_{m,\rho} \subset P_{m+1,\rho}$, it follows that each m-th net interval is the union of some $(m+1)$-th net intervals, and each $(m+1)$-th net interval is contained in a unique m-th net interval. Denote by $\mathcal{I}_{m,\rho}$ the collection of all m-th net intervals. Now, we define a mapping $\Gamma_{m,\rho} : \mathcal{I}_{m,\rho} \to 2^{\mathbb{R}} \times \mathbb{R}$ by

$$[a,b] \mapsto \left(\left\{ \frac{\phi_{\omega,\rho}(0)-a}{\rho^m} : \omega \in \{0,1\}^m \text{ such that } -\rho^m < \phi_{\omega,\rho}(0) - a \le 0 \right\}, \frac{b-a}{\rho^m} \right).$$

We call $\Gamma_{m,\rho}$ to be the m-th *I-color mapping*, and call $\Gamma_{m,\rho}([a,b])$ to be the m-th *I-color* of $[a,b]$. We can see from the definition that $\Gamma_{m,\rho}([a,b])$ contains the following information about the net interval $[a,b]$: (i) the various relative distances (with a ratio ρ^{-m}) between the point a and the points of the form $(1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0$,or 1) which lie on the left side of a and have distance less than ρ^m from a; (ii) the relative length of $[a,b]$(with a ratio ρ^{-m}).

For $\omega \in \{0,1\}^m$, write

$$<\omega>_\rho := \{v \in \{0,1\}^m : \phi_{v,\rho}(0) = \phi_{\omega,\rho}(0)\}.$$

we use $\# <\omega>_\rho$ to denote the cardinal of $<\omega>_\rho$.
Define a mapping $\Upsilon_{m,\rho} : \mathcal{I}_{m,\rho} \to 2^{\mathbb{R}\times\mathbb{N}} \times \mathbb{R}$ by

$$\begin{aligned}[a,b] \mapsto \quad & (\{(\frac{\phi_{\omega,\rho}(0)-a}{\rho^m}, \# <\omega>_\rho) : \omega \in \{0,1\}^m \text{ such that} \\ & -\rho^m < \phi_{\omega,\rho}(0) - a \le 0\}, \frac{b-a}{\rho^m})\end{aligned}$$

We call $\Upsilon_{m,\rho}$ to be the m-th *II-color mapping*, and call $\Upsilon_{m,\rho}([a,b])$ to be the m-th *II-color* of $[a,b]$. Compared with the I-color of $[a,b]$, the II-color $\Upsilon_{m,\rho}([a,b])$ contains the following extra information: (iii) the multiplicity of the points of the form $(1-\rho)\sum_{n=0}^{m-1}\rho^n r_n$ ($r_n = 0$ or 1) which lie on the left side of a and have distance less than ρ^m from a.

Let us take an example. Suppose $\rho = (\sqrt{5}-1)/2$, let us consider the I-color and II-color for the 3-th net interval $[a,b] = [\rho^2, 2\rho^3]$. Since the points (with the multiplicity) of the form $(1-\rho)\sum_{n=0}^{2}\rho^n r_n$ ($r_n = 0$,or 1) can be written as:

$$0,\ \rho^4,\ \rho^3,\ \underbrace{\rho^2,\ \rho^2}_{2\text{'s}},\ \rho^2+\rho^4,\ \rho^2+\rho^3,\ 2\rho^2.$$

Among the above points, only ρ^3(with multiplicity 1) and ρ^2 (with multiplicity 2) lie on the left side of $a = \rho^2$ and have distance less than ρ^3 from a. Thus the 3-th I-color of $[\rho^2, 2\rho^3]$ is

$$(\{\frac{\rho^3-\rho^2}{\rho^3}, \frac{\rho^2-\rho^2}{\rho^3}\}, \frac{2\rho^3-\rho^2}{\rho^3}) = (\{-\rho, 0\}, 1-\rho),$$

and the 3-th II-color of $[\rho^2, 2\rho^3]$ is

$$(\{(-\rho, 1), (0, 2)\}, 1 - \rho).$$

For $x \in [0,1]$ and $m \in \mathbb{N}$, define

$$N_{m,\rho}(x) = \#\{\omega \in \{0,1\}^m : \quad x \in \phi_{\omega,\rho}([0,1])\}.$$

we call $N_{m,\rho}(x)$ the *m-th overlap times* at x. Given a m-th net interval $[a,b]$ associated with ρ, assume its II-color to be

$$(\{t_1, n_1\}, \cdots, \{t_r, n_r\}, \gamma).$$

For convenience, we say that the integral vector $(n_1, \cdots, n_r)$ is the *II-characteristic vector* of $[a,b]$. It is an elementary fact that

$$N_{m,\rho}(x) = \sum_{i=1}^{r} n_r, \tag{4}$$

when $x \in (a,b)$. For this reason, we call $\sum_{i=1}^{r} n_r$ the *m-th overlap times* of $[a,b]$ and denote it by $N_{m,\rho}([a,b])$. The definitions of net interval and II-color imply the following property:

$$\begin{aligned} N_{m,\rho}([a,b]) &= \#\{\omega \in \{0,1\}^n : \ \phi_{\omega,\rho}([0,1]) \cap (a,b) \neq \emptyset\} \\ &= \#\{\omega \in \{0,1\}^n : \ \phi_{\omega,\rho}([0,1]) \supset [a,b]\} \end{aligned}$$

2.2 The properties of I-colors and II-colors

Lemma 1 *Let $J = [a,b]$ be a m-th net interval associated with ρ. Denote by $J_1, \cdots, J_l$ all the $(m+1)$-th net intervals which are contained in $[a,b]$. Then the $(m+1)$-th I-colors of $J_1, \cdots, J_l$ are completely determined by the m-th I-color of $[a,b]$.*

Proof. Assume the I-color of $[a,b]$ to be $(\{t_1, \cdots, t_r\}, \gamma)$. By the definition of I-color, we have

$$(a - \rho^m, b) \bigcap \{\phi_{\omega,\rho}(0) : \omega \in \{0,1\}^m\} = \{a + t_1\rho^m, \cdots, a + t_r\rho^m\},$$

$$\begin{gathered} (a - \rho^m, b) \bigcap \{\phi_{\omega,\rho}(0) : \omega \in \{0,1\}^{m+1}\} = (a - \rho^m, b) \bigcap \\ \{a + t_i\rho^m + \rho^m\theta : \quad 1 \leq i \leq r, \theta = 0 \text{ or } 1 - \rho\}, \end{gathered}$$

and

$$\begin{aligned} &[a,b] \bigcap P_{m+1,\rho} = [a,b] \bigcap \{a + t_i\rho^m + \rho^m\theta : \quad 1 \leq i \leq r, \theta = 0, 1-\rho, \rho \text{ or } 1\} \\ &= \{a + t_i\rho^m + \rho^m\theta : \quad 1 \leq i \leq r, \ \theta = 0, 1-\rho, \rho \text{ or } 1, \ 0 \leq t_i + \theta \leq \gamma\}. \end{aligned} \tag{5}$$

Since there are l different $(m+1)$-th net intervals $J_1, \cdots, J_l$ contained in $[a,b]$, it follows that there are just $l+1$ different elements (including a and b) in the set $[a,b] \bigcap P_{m+1,\rho}$, therefore by (5) the set

$$\{t_i + \theta : \quad 1 \leq i \leq r, \theta = 0, 1-\rho, \rho \text{ or } 1, 0 \leq t_i + \theta \leq \gamma\} \tag{6}$$

consists of $l+1$ different points: $0 = h_1 < h_2 < \cdots < h_{l+1} = \gamma$. Hence $J_i = [a + h_i\rho^m, a + h_{i+1}\rho^m]$ $(1 \le i \le l)$. By the definition, the I-color of J_i is

$$\begin{array}{c}(\{\rho^{-1}(t_j + \theta - h_i) : \quad -\rho < t_j + \theta - h_i \le 0, \quad 1 \le j \le r, \\ \theta = 0 \text{ or } 1-\rho\}, \rho^{-1}(h_{i+1} - h_i)).\end{array} \tag{7}$$

It follows from the formulas (6) and (7) that the I-colors of J_i $(1 \le i \le l)$ are completely determined by the I-color of $[a, b]$. ■

Denote

$$\mathcal{C}_\rho := \bigcup_{m \ge 1} \{\Gamma_{m,\rho}([a, b]) : \quad [a, b] \in \mathcal{I}_{m,\rho}\}.$$

That is, $\mathcal{C}_\rho$ consists of all the possible I-colors of net intervals associated with ρ. The following lemma is our start point:

Lemma 2 *If ρ is the reciprocal of a Pisot number, then $\mathcal{C}_\rho$ is a finite set.*

Proof. When ρ^{-1} is a Pisot number, Garsia's result (Lemma 1.51 of [G1]) implies that for each positive integer d there exists a positive constants c_d, such that if each r_i $(i = 1, \cdots, n)$ takes only the value $\pm d, \pm(d-1), \cdots, \pm 1$ or 0, then either

$$\sum_{i=1}^{n} \rho^{-n} r_n = 0$$

or

$$|\sum_{i=1}^{n} \rho^{-n} r_n| \ge c_d.$$

The above result implies that the number of different points of the form $\sum_{i=1}^{m} \rho^{-n} r_n$ $(r_n = \pm 1, 0)$ which lie in a given interval (a, b) is not greater than $\dfrac{b-a}{c_2}$ (to see this, note that the distance between any different two of these points is of the form $\sum_{i=1}^{m} \rho^{-n} r_n$ $(r_n = \pm 2, \pm 1, 0)$ and thus not less than c_2). Therefore, the sets

$$\bigcup_{m \ge 0} \left\{ \frac{\phi_{\omega,\rho}(0) - \phi_{v,\rho}(0)}{\rho^m} : \quad |\phi_{\omega,\rho}(0) - \phi_{v,\rho}(0)| \le \rho^m, \ \omega, v \in \{0,1\}^m \right\}$$

and

$$\bigcup_{m \ge 0} \left\{ \frac{\phi_{\omega,\rho}(0) - \phi_{v,\rho}(1)}{\rho^m} : \quad |\phi_{\omega,\rho}(0) - \phi_{v,\rho}(1)| \le \rho^m, \ \omega, v \in \{0,1\}^m \right\}$$

contain only finite many elements. This fact and the definition of $\mathcal{C}_\rho$ yield the desired result. ■

Corollary 3 *For $1/2 < \rho < 1$, if ρ is the reciprocal of a Pisot number, then there exists positive constant c (depending on ρ) such that*

$$c\rho^m \le |J| \le \rho^m \tag{8}$$

for any m-th net interval J, where $|J|$ denotes the length of J.

Proof. The right hand side of (8) follows from the fact that each m-th net interval is contained in $\phi_{\omega,\rho}([0,1])$ for some $\omega \in \{0,1\}^m$. On the other hand, the finiteness of $\mathcal{C}_\rho$ implies the left hand side. ■

Lemma 4 *For $1/2 < \rho < 1$, suppose J is a m-th net interval with II-color $\{(t_1, n_1), \cdots, (t_r, n_r), \gamma\}$, then*

(i) $\mu_\rho(J) = 2^{-m} \sum_{i=1}^{r} n_i \mu_\rho([-t_i, -t_i + \gamma])$.

(ii) If ρ^{-1} is a Pisot number, then there exists a constant $D > 0$ such that

$$D 2^{-m} N_{m,\rho}(J) \leq \mu_\rho(J) \leq 2^{-m} N_{m,\rho}(J)$$

Proof. For $1/2 < \rho < 1$, it is clear that μ_ρ is a non-atomic measure and has positive measure on any subinterval of $[0,1]$. Now suppose that $J = [a,b]$ is a m-th net interval with II-color $\{(t_1, n_1), \cdots, (t_r, n_r), \gamma\}$, then by the similarity of μ_ρ (see Formula (2)) we have

$$\mu_\rho(J) = 2^{-m} \sum_{\omega \in \{0,1\}^m} \mu_\rho(\phi_{\omega,\rho}^{-1}(J)).$$

By the definition of net intervals, if $\phi_{\omega,\rho}(0) - a \leq -\rho^m$ or $\phi_{\omega,\rho}(0) > a$, then $\phi_{\omega,\rho}([0,1]) \bigcap \operatorname{int}(J) = \emptyset$ and thus $\mu_\rho(\phi_{\omega,\rho}^{-1}(J)) = 0$ since $\phi_{\omega,\rho}^{-1}(J) \bigcap (0,1) = \emptyset$. On the contrary, if $-\rho^m < \phi_{\omega,\rho}(0) - a \leq 0$, then $J \subset \phi_{\omega,\rho}([0,1])$, therefore

$$\phi_{\omega,\rho}^{-1}(J) = [\phi_{\omega,\rho}^{-1}(a), \phi_{\omega,\rho}^{-1}(b)] = [-\frac{\phi_{\omega,\rho}(0) - a}{\rho^m}, -\frac{\phi_{\omega,\rho}(0) - a}{\rho^m} + \frac{b-a}{\rho^m}] \subset [0,1].$$

By the above analysis we have

$$\mu_\rho(J) = 2^{-m} \sum_{\omega \in \{0,1\}^m, -\rho^m < \phi_{\omega,\rho}(0) - a \leq 0} \mu_\rho([-\frac{\phi_{\omega,\rho}(0) - a}{\rho^m}, -\frac{\phi_{\omega,\rho}(0) - a}{\rho^m} + \frac{b-a}{\rho^m}]).$$

and thus

$$\mu_\rho(J) = 2^{-m} \sum_{i=1}^{r} n_i \mu_\rho([-t_i, -t_i + \gamma])$$

by the definition of the II-color. Therefore we complete the proof of the statement (i).

Now suppose ρ^{-1} is a Pisot number, since the collection $\mathcal{C}_\rho$ of all possible I-colors is a finite set, it follows that the number of all the possible different intervals $[-t_i, -t_i + \gamma]$ is finite. Denote by D the minimal value of μ_ρ measure on $[-t_i, -t_i + \gamma]$. Then $0 < D \leq 1$, therefore by the statement (i), we have

$$D 2^{-m} N_{m,\rho}(J) \leq \mu_\rho(J) \leq 2^{-m} N_{m,\rho}(J).$$

which proves the statement (ii). ■

Corollary 5 *For $1/2 < \rho < 1$, if ρ is the reciprocal of a Pisot number, then the L^q-spectrum $\tau_{\mu_\rho}(q)$ of μ_ρ is equal to*

$$\liminf_{m \to \infty} \frac{\log(\sum_{J \in \mathcal{I}_{m,\rho}} (\mu_\rho(J)^q)}{m \log \rho} = -\frac{\log 2}{\log \rho} + \liminf_{m \to \infty} \frac{\log \sum_{J \in \mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^q}{m \log \rho} \tag{9}$$

Proof. By the definition of L^q-spectrum (see Section 1) and Lemma 4(ii), it suffices to prove that the left hand side of (9) is equal to

$$\liminf_{\delta\downarrow 0} \frac{\log \sup \sum_i (\mu_\rho[x_i-\delta, x_i+\delta])^q}{m\log\rho}, \tag{10}$$

where the superum in (10) takes over all disjoint family of intervals $[x_i-\delta, x_i+\delta]_i$ with $x_i \in [0,1]$. Let c be the constant in Corollary 3.

We first show (9)$\geq$(10). Fix $m \in \mathbb{N}$, take $\delta = \frac{1}{2}c\rho^m$. For each $J \in \mathcal{I}_{m,\rho}$, select a interval $s(J) \subset J$ such that $|s(J)| = 2\delta$ and $\mu_\rho(s(J)) \geq \frac{1}{4}c\mu_\rho(J)$. Then

$$\sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q \leq \begin{cases} (\frac{1}{4}c)^{-q}\sum_{J\in\mathcal{I}_{m,\rho}}(\mu_\rho(s(J)))^q, & \text{if } q\geq 0 \\ \sum_{J\in\mathcal{I}_{m,\rho}}(\mu_\rho(s(J)))^q, & q<0 \end{cases}$$

which implies (9)$\geq$(10).

Now we show the converse relation. For any small $\delta > 0$, let m be the integer so that $\rho^m < \delta \leq \rho^{m-1}$. Suppose that $[x_i-\delta, x_i+\delta]_i$ is a disjoint family of intervals with $x_i \in [0,1]$. Since for each i, $[x_i-\delta, x_i+\delta]$ intersects at most $\frac{2}{c\rho}+1$ many m-th net intervals, it follows that when $q \geq 0$,

$$\begin{aligned} \mu_\rho([x_i-\delta, x_i+\delta])^q &\leq (\mu_\rho(\bigcup_{J\in\mathcal{I}_{m,\rho}, J\cap[x_i-\delta,x_i+\delta]\neq\emptyset} J))^q \\ &\leq (\frac{2}{c\rho}+1)^q \cdot \sum_{J\in\mathcal{I}_{m,\rho}, J\cap[x_i-\delta,x_i+\delta]\neq\emptyset} (\mu_\rho(J))^q. \end{aligned}$$

Note that each net interval intersects at most two elements of $[x_i-\delta, x_i+\delta]_i$, by the above inequality we have

$$\sum_i \mu_\rho([x_i-\delta, x_i+\delta])^q \leq 2(\frac{2}{c\rho}+1)^q \cdot \sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q \quad \text{for } q\geq 0. \tag{11}$$

Since for each i, $[x_i-\delta, x_i+\delta]$ contains at least one m-th net intervals, it follows that

$$\sum_i \mu_\rho([x_i-\delta, x_i+\delta])^q \leq \sum_{J\in\mathcal{I}_{m,\rho}} (\mu_\rho(J))^q \quad \text{for } q<0. \tag{12}$$

Inequalities (11) and (12) imply (9)$\leq$(10). ■

Lemma 6 *For any $1/2 < \rho < 1$ and $m \in \mathbb{N}$, suppose that I and J are two adjoint m-th net intervals associated with ρ, then*

$$\frac{1}{m+1}N_{m,\rho}(J) \leq N_{m,\rho}(I) \leq (m+1)N_{m,\rho}(J) \tag{13}$$

Proof. We prove the statement by induction.

One may testify (13) directly for the case $m = 1$ since there are just three 1-th net intervals with the overlap times $1, 2, 1$ respectively. Now assume that (13) holds for $m \leq k$. In the following we will show that (13) holds for $m = k+1$.

Now suppose that $I,\ J$ are two adjoint $(k+1)$-th net intervals, where I lies on the left side of J. There are two possible cases:

(i) $I,\ J$ are contained in the same one k-th net interval U.

(ii) $I,\ J$ are contained in two adjoint k-th net interval I', J' respectively.

(Let us recall the property of overlap times for net interval: if Q is a n-th net interval, then

$$\begin{aligned} N_{n,\rho}(Q) &= \#\{\omega \in \{0,1\}^n :\ \phi_{\omega,\rho}([0,1]) \cap int(Q) \neq \emptyset\} \\ &= \#\{\omega \in \{0,1\}^n :\ \phi_{\omega,\rho}([0,1]) \supset Q\} \end{aligned} \tag{14}$$

)

In the case (i), it is clear that

$$N_{k,\rho}(U) \leq N_{k+1,\rho}(I) \leq 2N_{k,\rho}(U),\ \ N_{k,\rho}(U) \leq N_{k+1,\rho}(J) \leq 2N_{k,\rho}(U),$$

and therefore

$$\frac{1}{2}N_{k+1,\rho}(J) \leq N_{k+1,\rho}(I) \leq 2N_{k+1,\rho}(J).$$

In the case (ii), let us define

$$\begin{aligned} &\mathcal{A}_1 = \{\omega \in \{0,1\}^k :\ \phi_{\omega,\rho}([0,1]) \supset I' \text{ and they share the same right end-point}\} \\ &\mathcal{A}_2 = \{\omega \in \{0,1\}^k \backslash \mathcal{A}_1 :\ \phi_{\omega,\rho}([0,1]) \supset I'\}, \\ &\mathcal{A}_3 = \{\omega \in \{0,1\}^k :\ \phi_{\omega,\rho}([0,1]) \supset J' \text{ and they share the same left end-point}\}, \\ &\mathcal{A}_4 = \{\omega \in \{0,1\}^k \backslash \mathcal{A}_3 :\ \phi_{\omega,\rho}([0,1]) \supset J'\}. \end{aligned}$$

From the definition of net interval and the property (14), we have

$$\begin{aligned} &\mathcal{A}_2 = \mathcal{A}_4 \\ &N_{k,\rho}(I') = \#\mathcal{A}_1 + \#\mathcal{A}_2, \\ &N_{k,\rho}(J') = \#\mathcal{A}_3 + \#\mathcal{A}_4, \\ &\#\mathcal{A}_1 + \#\mathcal{A}_2 \leq N_{k+1,\rho}(I) \leq \#\mathcal{A}_1 + 2\#\mathcal{A}_2, \\ &\#\mathcal{A}_3 + \#\mathcal{A}_4 \leq N_{k+1,\rho}(J) \leq \#\mathcal{A}_3 + 2\#\mathcal{A}_4. \end{aligned}$$

According to the above relation, we can deduce

$$\frac{1}{k+2}N_{k+1,\rho}(J) \leq N_{k+1,\rho}(I) \leq (k+2)N_{k+1,\rho}(J)$$

from the assumption $\frac{1}{k+1}N_{k,\rho}(J') \leq N_{k,\rho}(I') \leq (k+1)N_{k,\rho}(J')$. ■

Combining Lemma 4 and Lemma 6, we have the following corollary:

Corollary 7 *If ρ $(\geq 1/2)$ is a reciprocal of a Pisot number, then there exists a positive constant c (only depending on ρ) such that for any $m \in \mathbb{N}$*

$$mc^{-1}\mu_\rho(J) \leq \mu_\rho(I) \leq mc\mu_\rho(J), \tag{15}$$

where I and J are any two adjoint m-th net intervals associated with ρ.

For $m \in \mathbb{N}$, let $a_{m,1} < a_{m,2} \cdots < a_{m,\sigma_m}$ be the different elements of $\{\phi_{\omega,\rho}(0) : \omega \in \{0,1\}^m\}$, and denote by $d_j^{(m)}$ $(1 \le j \le \sigma_m)$ the cardinal of $\{\omega \in \{0,1\}^m : \phi_{\omega,\rho}(0) = a_{m,j}\}$.

Lemma 8 *(i)* $\sum_{j=1}^{\sigma_m} (d_j^{(m)})^s \le \sum_{J \in \mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^s$ *for any* $s \ge 0$ *and* $m \in \mathbb{N}$.

(ii) If ρ^{-1} is a Pisot number, then there exists $D > 0$ (depending on ρ), such that

$$\sum_{j=1}^{\sigma_m} (d_j^{(m)})^s \ge D \sum_{J \in \mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^s$$

for any $0 \le s \le 1$ and $m \in \mathbb{N}$.

Proof. (i) By the definition of net intervals, each $a_{m,j}$ $(1 \le j \le \sigma_m)$ is the left endpoint of a m-th net interval which has the m-th overlap times not less than $d_j^{(m)}$, thus $\sum_{j=1}^{\sigma_m} (d_j^{(m)})^s \le \sum_{J \in \mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^s$ for any $s \ge 0$ and $m \in \mathbb{N}$.

(ii) Now suppose that ρ^{-1} is a Pisot number. Since both the sets

$$\bigcup_{m \ge 0} \left\{ \frac{\phi_{\omega,\rho}(0) - \phi_{v,\rho}(0)}{\rho^m} : \ |\phi_{\omega,\rho}(0) - \phi_{v,\rho}(0)| \le \rho^m, \ \omega, v \in \{0,1\}^m \right\}$$

and

$$\bigcup_{m \ge 0} \left\{ \frac{\phi_{\omega,\rho}(0) - \phi_{v,\rho}(1)}{\rho^m} : \ |\phi_{\omega,\rho}(0) - \phi_{v,\rho}(1)| \le \rho^m, \ \omega, v \in \{0,1\}^m \right\}$$

are finite, it follows that there exists $L \in \mathbb{N}$ such that for any $x \in \{\phi_{\omega,\rho}(0) : \ \omega \in \{0,1\}^m\}$, there are at most L different many $y \in P_{m,\rho} = \{\phi_{\omega,\rho}(0) : \ \omega \in \{0,1\}^m\} \bigcup \{\phi_{\omega,\rho}(1) : \ \omega \in \{0,1\}^m\}$ satisfying that

$$y - \rho^m \le x \le y.$$

Therefore, for any $a_{m,j}$, there are at most L different many m-th net intervals $J = [a, b]$, such that $a - \rho^m \le a_{m,j} \le a$.

On the contrary, for any m-th net interval $J = [a, b]$, suppose that $a_{m,l}, \cdots, a_{m,l+r}$ are the all points of $a_{m,j}(1 \le j \le \sigma_m)$ such that $a - \rho^m \le a_{m,j} \le a$, then $N_{m,\rho}(J) = d_l^{(m)} + \cdots + d_{l+r}^{(m)}$, thus

$$(N_{m,\rho}(J))^s = (d_l^{(m)} + \cdots + d_{l+r}^{(m)})^s \le (d_l^{(m)})^s + \cdots + (d_{l+r}^{(m)})^s$$

for $0 \le s \le 1$. Let J run over $\mathcal{I}_{m,\rho}$, since for each $a_{m,j}$ there are at most L's m-th net intervals $J = [a, b]$ satisfying $a - \rho^m \le a_{m,j} \le a$, it follows that

$$\sum_{J \in \mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^s \le L \sum_{j=1}^{\sigma_m} (d_j^{(m)})^s, \quad 0 \le s \le 1.$$

The desired result follows by letting $D = 1/L$. ■

Przytycki and Urbanski ([PU], p.184) have given a formal formula for the Hausdorff dimension of the graph of the limit Rademacher function f_ρ where ρ^{-1} is a Pisot number, that is

$$\dim_H(\text{graph } f_\rho) = \lim_{m\to\infty} \frac{\log \sum_{j=1}^{\sigma_m} (d_j^{(m)})^{\log\rho^{-1}/\log 2}}{m \log \rho^{-1}}. \tag{16}$$

For our necessity, we re-express it in the following form (using Lemma 8):

Lemma 9 *For $1/2 < \rho < 1$, if ρ^{-1} is a Pisot number, then*

$$\dim_H(\textit{graph } f_\rho) = \lim_{m\to\infty} \frac{\log \sum_{J\in\mathcal{I}_{m,\rho}} (N_{m,\rho}(J))^{\log\rho^{-1}/\log 2}}{m \log \rho^{-1}}.$$

3 Some elementary properties of the limit Rademacher functions

In this section, we give some elementary properties of the graphs and level sets of the limit Rademacher functions.

Denote by F the set $\{\frac{l}{2^n} : \quad n \in \mathbb{N}, l = 0, 1, \cdots, 2^n - 1\}$. One may check the following lemma directly.

Lemma 10 *For any $1/2 < \rho < 1$, (a) the function $f_\rho(x)$ is continuous on $(0,1)\backslash F$, (b) for any $x = l/2^n \in F$ (l is an odd integer), $f_\rho(x+) = f_\rho(x)$ and $f_\rho(x-) = f_\rho(x)+\rho^{n-1}(2\rho-1)$,(c) $f_\rho(0+) = 0$ and $f_\rho(1-) = 1$.*

Denote $\text{graph}(f_\rho) = \{(x, f_\rho(x)) \in \mathbb{R}^2 : \quad x \in [0,1)\}$, then the above lemma implies the following result at once:

Lemma 11 *For any $1/2 < \rho < 1$,*

$$\overline{\textit{graph}(f_\rho)} = \textit{graph}(f_\rho) \bigcup \{(\frac{l}{2^n}, f_\rho(\frac{l}{2^n}-)) : \quad n \in \mathbb{N}, 1 \le l \le 2^n - 1\}.$$

Define the mappings $\Phi_{0,\rho}, \Phi_{1,\rho} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi_{0,\rho}(x,y) = (\frac{1}{2}x, \rho y), \quad \Phi_{1,\rho}(x,y) = (\frac{1}{2}x + \frac{1}{2}, \rho y + (1-\rho)).$$

Then it is easy to check the following lemma:

Lemma 12 *For any $1/2 < \rho < 1$, $\textit{graph}(f_\rho), \overline{\textit{graph}(f_\rho)}$ are invariant under $\Phi_{0,\rho}, \Phi_{1,\rho}$. That is*

$$\textit{graph}(f_\rho) = \bigcup_{i=0}^{1} \Phi_{i,\rho}(\textit{graph}(f_\rho)), \quad \overline{\textit{graph}(f_\rho)} = \bigcup_{i=0}^{1} \Phi_{i,\rho}(\overline{\textit{graph}(f_\rho)}),$$

moreover,

$$\overline{\textit{graph}(f_\rho)} = \bigcap_{k\ge 1} \bigcup_{\mathbf{i}\in\{0,1\}^k} \Phi_{\mathbf{i},\rho}([0,1]\times[0,1]).$$

For any $t \in [0,1]$, the *t-level set* $L_{t,\rho}$ of f_ρ is defined as $\{x \in [0,1) : \quad f_\rho(x) = t\}$. From this definition, we have $L_{t,\rho} \times \{t\} = ([0,1] \times \{t\}) \bigcap \mathrm{graph}(f_\rho)$. For convenience, we define

$$\hat{L}_{t,\rho} = ([0,1] \times \{t\}) \bigcap \overline{\mathrm{graph}(f_\rho)}.$$

Then by lemma 11, $L_{t,\rho} \times \{t\} = \hat{L}_{t,\rho}$ for any $t \in [0,1] \backslash f_\rho(F-)$, where $f_\rho(F-)$ denotes the set $\{f_\rho(x-) : x \in F\}$.

Lemma 13 *(i) For any* $t \in [0,1]$, *and* $1/2 < \rho < 1$,

$$\overline{\dim}_B \hat{L}_{t,\rho} = \limsup_{m \to \infty} \frac{\log N_{m,\rho}(t)}{m \log 2}, \quad \underline{\dim}_B \hat{L}_{t,\rho} = \liminf_{m \to \infty} \frac{\log N_{m,\rho}(t)}{m \log 2}$$

(ii) For any $t \in [0,1] \backslash f_\rho(F-)$, *and* $1/2 < \rho < 1$,

$$\overline{\dim}_B L_{t,\rho} = \limsup_{m \to \infty} \frac{\log N_{m,\rho}(t)}{m \log 2}, \quad \underline{\dim}_B L_{t,\rho} = \liminf_{m \to \infty} \frac{\log N_{m,\rho}(t)}{m \log 2}$$

where $\overline{\dim}_B$, $\underline{\dim}_B$ *denote the upper and lower box-counting dimensions respectively.*

Proof. By Lemma 12,

$$\overline{\mathrm{graph}(f_\rho)} = \bigcap_{k \geq 1} \bigcup_{\mathbf{i} \in \{0,1\}^k} \Phi_{\mathbf{i},\rho}([0,1] \times [0,1]).$$

Notice that for any $\omega \in \{0,1\}^n$, $\Phi_{\omega,\rho}(x,y) = (\psi_{\omega,\rho}(x), \phi_{\omega,\rho}(y))$, where $\psi_{0,\rho}(x) = \frac{x}{2}, \psi_{1,\rho}(x) = \frac{x+1}{2}$. It follows that for any $m \in \mathbb{N}$, the number of different $\omega \in \{0,1\}^m$ for which $\Phi_{\omega,\rho}([0,1] \times [0,1])$ intersects $\hat{L}_{t,\rho}$ for fixed t, is the cardinality of $\{\nu \in \{0,1\}^m : \quad t \in \phi_{\nu,\rho}([0,1])\}$, i.e. $N_{m,\rho}(t)$, therefore the number of 2^{-m}-mesh cubes which intersect $\hat{L}_{t,\rho}$ is $N_{m,\rho}(t)$, hence the statement (i) follows from the definition of the upper and lower box-counting dimensions.

The statement (ii) follows from that $L_{t,\rho} \times \{t\} = \hat{L}_{t,\rho}$ for any $t \in [0,1] \backslash f_\rho(F-)$. ■

4 The case $\rho = \lambda_2$

In this section, we always assume $\rho = \lambda := \dfrac{\sqrt{5}-1}{2}$.

4.1 The generating relation of I-colors and the Markov codes for net intervals

Let J be any m-th net interval, and $J_1, \cdots, J_l$ be the adjoint (from left to right) $(m+1)$-th net subintervals of J. Denote by U, $U_i (1 \leq i \leq l)$ the I-colors of J, $J_i (1 \leq i \leq l)$ respectively, then we would like to express their relation by

$$U \longrightarrow U_1 + \cdots + U_l,$$

and say that U generates out U_i, $1 \leq i \leq l$.

Under this expression, by direct calculation in the way discussed in the proof of Lemma 1, we have

$$\begin{array}{lcl} (\{0\},\lambda) & \longrightarrow & (\{0\},\lambda)+(\{-\lambda,0\},1-\lambda) \\ (\{-\lambda,0\},1-\lambda) & \longrightarrow & (\{\lambda-1,0\},\lambda) \\ (\{\lambda-1\},\lambda) & \longrightarrow & (\{-\lambda,0\},1-\lambda)+(\{\lambda-1\},\lambda) \\ (\{\lambda-1,0\},\lambda) & \longrightarrow & (\{-\lambda,0\},1-\lambda)+(\{\lambda-1\},2\lambda-1)+(\{-\lambda,0\},1-\lambda) \\ (\{\lambda-1\},2\lambda-1) & \longrightarrow & (\{-\lambda,0\},1-\lambda) \end{array} \tag{17}$$

As we have seen, there are only five elements in the set $\mathcal{C}_\lambda$. In the following process, we will label the net intervals according to the above generating relations.

Let $\Xi=\{a,b,c,d,e,f,\overline{f}\}$ be an alphabet set. For any $m\in\mathbb{N}$, we will label every m-th net interval uniquely by a letter string of length m ("Markov code") in the following way. Let J be a m-th net interval, for convenience, we denote it also by $J^{(m)}$. For each $1\le i\le m-1$, there is only one i-th net interval that contains J, which we denote by $J^{(i)}$. Recall that $\Gamma_{i,\lambda}(\cdot)$ denotes the i-th I-color. Recall that $\Gamma_{i,\lambda}$ denotes the i-th I-color mapping. Then J is labelled as $(x_i)_{i=1}^m\in\Xi^m$, where

$$x_i=\begin{cases} a & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{0\},\lambda) \\ b & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{-\lambda,0\},1-\lambda), \text{ and} \\ & \text{either } i=1, \text{ or } i>1 \text{ with } \Gamma_{i-1,\lambda}(J^{(i-1)})=(\{\lambda-1\},2\lambda-1) \\ c & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{\lambda-1\},\lambda) \\ d & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{\lambda-1,0\},\lambda) \\ e & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{\lambda-1\},2\lambda-1) \\ f & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{-\lambda,0\},1-\lambda),\ i>1, \\ & \quad \Gamma_{i-1,\lambda}(J^{(i-1)})=(\{\lambda-1,0\},\lambda), \\ & \quad \text{and } J^{(i)} \text{ has the same left endpoint as } J^{(i-1)} \\ \overline{f} & \text{if } \Gamma_{i,\lambda}(J^{(i)})=(\{-\lambda,0\},1-\lambda), \\ & \quad \Gamma_{i-1,\lambda}(J^{(i-1)})=(\{\lambda-1,0\},\lambda), \\ & \quad \text{and } J^{(i)} \text{ has the same right endpoint as } J^{(i-1)} \end{cases} \tag{18}$$

For example, let us consider the Markov code for the 3-th net intervals $J=[1-\lambda,2\lambda^3]$ and $J'=[\lambda^2+\lambda^4,\lambda]$. By direct check, $[1-\lambda,\lambda]$ is the unique 1-th net interval (and also the 2-th net interval) which contains J (and also J'), the 1-th I-color for $[1-\lambda,\lambda]$, 2-th I-color for $[1-\lambda,\lambda]$ and 3-th I-color for J (or J') are

$$(\{-\lambda,0\},1-\lambda),\ (\{\lambda-1,0\},\lambda),\ (\{-\lambda,0\},1-\lambda),$$

by our labelling principle, the Markov codes for J , J' are bdf, $bd\overline{f}$ respectively.

By the above labelling principle, any two different m-th net intervals correspond to different relative Markov codes. A formal expression of the generating relation (17) can be given below:

$$\begin{cases} a & \longrightarrow & a+b \\ b & \longrightarrow & d \\ c & \longrightarrow & b+c \\ d & \longrightarrow & f+e+\overline{f} \\ e & \longrightarrow & b \\ f & \longrightarrow & d \\ \overline{f} & \longrightarrow & d \end{cases} \tag{19}$$

We will say that i generates out j if there is an arrow from i to j. The above relation determine a 0-1 matrix $H=(H_{i,j})_{i,j\in\Xi}$ by $H_{i,j}=1$ if i generates out j. That is

$$H = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ \overline{f} \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & \overline{f} \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{20}$$

For $m\geq 2$, it follows from (18) and (19) that each m-th net interval can be coded as an element in

$$S^m := \{(x_i)_{i=1}^m \in \Xi^m : \quad H_{x_i,x_{i+1}} = 1, 1\leq i\leq m-1, \quad x_1 = a, b \text{ or } c\}, \tag{21}$$

and each element of the above set corresponds to unique one m-th net interval. For any $\omega\in S^m$, we will use V_ω to denote the m-th net interval corresponding to ω.

We would like to know more about the possible forms of the elements in S^m. For this purpose, we write $X_0=f$ and $X_1=\overline{f}$, and define $\mathcal{B}_\lambda=\mathcal{B}$ to be a collection of letter strings as follows

$$\mathcal{B} := \{bde\}\bigcup\{bdX_{i_1}d\cdots X_{i_k}de : k\in\mathbb{N}, i_1,\cdots,i_k = 0 \text{ or } 1\}. \tag{22}$$

Then by the generating relation (19), each element in S^m is the prefix of a letter string of the form of the following three cases:

$$\begin{aligned} &\omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, \\ &\underbrace{a\cdots a}_{r\ a\text{'s}}\circ\omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, \\ &\underbrace{c\cdots c}_{r\ c\text{'s}}\circ\omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, \end{aligned} \tag{23}$$

where $r\in\mathbb{N}$ and $\omega_i\in\mathcal{B}$, $i\in\mathbb{N}$.

4.2 II-colors of net intervals

In this subsection, we give the generating relations of II-colors. And we show that the m-th overlap times of a m-th net interval is equal to the product of some matrixes, furthermore it can be decomposed into the product of integers.

Let J be any m-th net interval, suppose that $J_1, \cdots, J_l$ are the $(m+1)$-th net intervals (from left to right) which contained in J. Let Θ, $\Theta_i (1 \le i \le l)$ be the II-colors of J, $J_i (1 \le i \le l)$ respectively. We express this generating relation by

$$\Theta \Longrightarrow \Theta_1 + \cdots + \Theta_l.$$

Under this notion, we have

$$\begin{array}{lll}
(\{(0,r)\},\lambda) & \Longrightarrow & (\{(0,r)\},\lambda) + (\{(-\lambda,r),(0,r)\},1-\lambda) \\
(\{(-\lambda,p),(0,q)\},1-\lambda) & \Longrightarrow & (\{(\lambda-1,p),(0,q)\},\lambda) \\
(\{(\lambda-1,r)\},\lambda) & \Longrightarrow & (\{(-\lambda,r),(0,r)\},1-\lambda) + (\{(\lambda-1,r)\},\lambda) \\
(\{(\lambda-1,p),(0,q)\},\lambda) & \Longrightarrow & (\{(-\lambda,p),(0,p+q)\},1-\lambda) + (\{(\lambda-1,p+q)\},2\lambda-1) \\
 & & \quad + (\{(-\lambda,p+q),(0,q)\},1-\lambda) \\
(\{(\lambda-1,r)\},2\lambda-1) & \Longrightarrow & (\{(-\lambda,r)\},\{(0,r)\},1-\lambda)
\end{array}$$

where $p,q,r \in \mathbb{N}$.

Denote by

$$\begin{array}{ll}
A^{(r)} & := (\{(0,r)\},\lambda) \\
B^{(p,q)} & := (\{(-\lambda,p),(0,q)\},1-\lambda) \\
C^{(r)} & := (\{(\lambda-1,r)\},\lambda) \\
D^{(p,q)} & := (\{(\lambda-1,p),(0,q)\},\lambda) \\
E^{(r)} & := (\{(\lambda-1,r)\},2\lambda-1) \\
F^{(p,q)} & := (\{(-\lambda,p),(0,q)\},1-\lambda) \\
\overline{F}^{(p,q)} & := (\{(-\lambda,p),(0,q)\},1-\lambda)
\end{array}$$

then the generating relations of II-colors can be written as

$$\left\{\begin{array}{lll}
A^{(r)} & \Longrightarrow & A^{(r)} + B^{(r,r)} \\
B^{(p,q)} & \Longrightarrow & D^{(p,q)} \\
C^{(r)} & \Longrightarrow & B^{(r,r)} + C^{(r)} \\
D^{(p,q)} & \Longrightarrow & F^{(p,p+q)} + E^{(p+q)} + \overline{F}^{(p+q,q)} \\
E^{(r)} & \Longrightarrow & B^{(r,r)} \\
F^{(p,q)} & \Longrightarrow & D^{(p,q)} \\
\overline{F}^{(p,q)} & \Longrightarrow & D^{(p,q)}
\end{array}\right. \tag{24}$$

Now according to the above generating relations, we define a family of matrixes $T_{i,j}$ for each pair $(i,j) \in \Xi \times \Xi$ with i generating out j in the sense of (19):

$$
\begin{cases}
T_{a,a} = 1, & T_{a,b} = (1,1), \\
T_{b,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\
T_{c,b} = (1,1), & T_{c,c} = 1 \\
T_{d,f} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & T_{d,e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad T_{d,\overline{f}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
T_{e,b} = (1,1) & \\
T_{f,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\
T_{\overline{f},d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, &
\end{cases}
\tag{25}
$$

With the above definition, the generating relation (24) can be re-written as:

$$
\begin{cases}
A^{(r)} & \Longrightarrow & A^{(r)T_{a,a}} + B^{(r)T_{a,b}} \\
B^{(p,q)} & \Longrightarrow & D^{(p,q)T_{b,d}} \\
C^{(r)} & \Longrightarrow & B^{(r)T_{c,b}} + C^{(r)T_{c,c}} \\
D^{(p,q)} & \Longrightarrow & F^{(p,q)T_{d,f}} + E^{(p,q)T_{d,e}} + \overline{F}^{(p,q)T_{d,\overline{f}}} \\
E^{(r)} & \Longrightarrow & B^{(r)T_{e,b}} \\
F^{(p,q)} & \Longrightarrow & D^{(p,q)T_{f,d}} \\
\overline{F}^{(p,q)} & \Longrightarrow & D^{(p,q)T_{\overline{f},d}}
\end{cases}
$$

This is, if $i \longrightarrow i_1 + \cdots + i_l$, then we have

$$
I^{(n_1,\cdots,n_r)} \Longrightarrow I_1^{(n_1,\cdots,n_r)\cdot T_{i,i_1}} + \cdots I_l^{(n_1,\cdots,n_r)\cdot T_{i,i_l}}. \tag{26}
$$

For any matrix M, denote by $||M||$ the absolute value sum of all the entries of M. Then according to the formula (26), we obtain the following lemma at once:

Lemma 14 *Let J be a m-th net interval ($m \geq 2$) corresponding to $\omega = (x_i)_{i=1}^m \in S^m$, suppose its II-color is $(\{(t_1, n_1), \cdots, (t_r, n_r)\}, \gamma)$, then*

$$
(n_1, \cdots, n_r) = \begin{cases} 1 \cdot T_{x_1,x_2} \cdot T_{x_2,x_3} \cdots T_{x_{m-1},x_m} & \text{if } x_1 = a \text{ or } c \\ (1,1) \cdot T_{x_1,x_2} \cdot T_{x_2,x_3} \cdots T_{x_{m-1},x_m} & \text{if } x_1 = b \end{cases}
$$

and

$$
N_{m,\lambda}(J) := \sum_{i=1}^{r} n_i = ||T_{x_1,x_2} \cdot T_{x_2,x_3} \cdots T_{x_{m-1},x_m}|| \tag{27}
$$

Let us consider a little more about the value of the right hand of Formula (27). For convenience, we write

$$
T_{x_1 x_2 \cdots x_m} := T_{x_1,x_2} \cdots T_{x_{m-1},x_m}. \tag{28}
$$

Lemma 15 *Suppose $\omega \in S^m$ can be written as the concatenation $\omega_1 \circ \omega_2$, where the end-letter of ω_1 is e. Then*

$$N_{m,\lambda}(V_\omega) = ||T_\omega|| = ||T_{\omega_1}|| \times ||T_{\omega_2}||, \tag{29}$$

where V_ω is the m-th net interval corresponding to ω.

Proof. Suppose that the II-color of V_ω is $(\{(t_1, n_1), \cdots, (t_r, n_r)\}, \gamma)$. Noting that the II-color of V_{ω_1} is of the form $E^{(p)}$ and the first letter of ω_2 is b, by Lemma 14 we have

$$(n_1, \cdots, n_r) = p \cdot (1,1)T_{\omega_2}, \;\; p = ||T_{\omega_1}||,$$

which proves the lemma. ∎

4.3 The exponential sums of matrix products

In this subsection, we consider about the limit $\lim_{m\to\infty}(\sum_{\omega\in S^m} ||T_\omega||^q)^{\frac{1}{m}}$ for any real number q, where S^m is defined as in (21) and T_ω's are defined by (25),(28). We show that this limit value is either the positive root of a transversal equation or equal to 1, and as a function of q it is differentiable except for one point $q_0 < -2$.

Let the matrixes M_0, M_1 be defined as in (3). For $\mathbf{j} = j_1 \cdots j_n \in \{0,1\}^n$, denote $M_{\mathbf{j}} = M_{j_1} \circ \cdots \circ M_{j_n}$. For any $q \in \mathbb{R}$, define

$$u_{0,q} = 2^q, \;\; u_{n,q} = \sum_{\mathbf{j}\in\{0,1\}^n} ||M_{\mathbf{j}}||^q \;\; (n \geq 1). \tag{30}$$

Then we can formulate the main result of this subsection as follows:

Theorem 16 *For any real number q, the limit $\lim_{m\to\infty}(\sum_{\omega\in S^m} ||T_\omega||^q)^{\frac{1}{m}}$ exists and is equal to $\mathbf{x}(q)^{-1}$, where $\mathbf{x}(q)$ is defined by*

$$\mathbf{x}(q) := \sup\{x \geq 0 : \sum_{n\geq 0} u_{n,q} x^{2n+3} \leq 1\}. \tag{31}$$

Moreover, let $q = q_0$ be the real root of $\sum_{n\geq 0} u_{n,q} = 1$. then $q_0 \in (-\infty, -2)$. And when $q > q_0$, $\mathbf{x}(q)$ is the root of $\sum_{n=0}^{\infty}(\sum_{|J|=n} ||M_J||^q)x^{2n+3} = 1$, and it is infinitely differentiable on $(q_0, +\infty)$; When $q \leq q_0$, $\mathbf{x}(q) = 1$. Furthermore, $\mathbf{x}(q)$ is not differentiable at $q = q_0$,

$$x'(q_0-) = 0, \; x'(q_0+) = -\frac{\sum_{n\geq 0}(\sum_{|J|=n} ||M_J||^{q_0} \log ||M_J||)}{\sum_{n\geq 0} u_{n,q_0} \cdot (2n+3)} \in (-\infty, 0).$$

We will prove the above theorem by a series of lemmas. At first, we define

$$S_b^m = \{(x_i)_{i=1}^m \in S^m : \;\; x_1 = b\},$$

and

$$v_{m,q} = \sum_{\omega\in S_b^m} ||T_\omega||^q,$$

for any positive integer m and real number q.

Lemma 17 $v_{1,q}=2^q=u_{0,q}, v_{2,q}=2^q=u_{0,q}, v_{3,q}=u_{0,q}+u_{1,q}$, *and for* $k\geq 2$

$$\begin{aligned} v_{2k,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k-(2i+3),q}) + u_{k-1,q} \\ v_{2k+1,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q}v_{2k+1-(2i+3),q}) + u_{k-1,q} + u_{k,q} \end{aligned}$$

Proof. Since $S_b^1=\{b\}$, $S_b^2=\{bd\}$, $S_b^3=\{bde, bdf, bd\overline{f}\}$, we can calculate $v_{1,q}, v_{2,q}$ and $v_{3,q}$ directly. Denote $X_0=f$ and $X_1=\overline{f}$. For any $k\geq 2$, by (23) each element $\omega\in S_b^{2k}$ can be written as one of the following two cases:

(i) $\omega=bdX_{i_1}d\cdots X_{i_{k-1}}d$, $i_1,\cdots,i_{k-1}\in\{0,1\}$.

(ii) $\omega=bdX_{i_1}d\cdots X_{i_l}de\circ\omega_2$, $0\leq l\leq k-2$, $i_1,\cdots,i_l\in\{0,1\}$ and $\omega_2\in S_b^{2k-(3+2l)}$.

For the case (i), by the definition of T_ω (see (25),(28)), $||T_\omega||=||M_{i_1}\cdots M_{i_{k-1}}||$. For the case (ii), by the formula (29), $||T_\omega||=||M_{i_1}\cdots M_{i_l}||\cdot||T_{\omega_2}||$. Thus

$$\begin{aligned} v_{2k,q} &= \sum_{\omega\in S_b^{2k}}||T_\omega||^q \\ &= \sum_{i_1,\cdots,i_{k-1}\in\{0,1\}}||M_{i_1\cdots i_{k-1}}||^q + \sum_{0\leq l\leq k-2}\Big(\sum_{i_1,\cdots,i_l\in\{0,1\}}||M_{i_1\cdots i_l}||^q\cdot\sum_{\omega\in S_b^{2k-(3+2l)}}||T_\omega||^q\Big) \\ &= u_{k-1,q}+\Big(\sum_{l=0}^{k-2}u_{l,q}\cdot v_{2k-(2l+3),q}\Big) \end{aligned}$$

In the other hand, by (23) each element $\omega\in S_b^{2k+1}$ can be written as one of the following three cases:

(iii) $\omega=bdX_{i_1}d\cdots X_{i_{k-1}}dX_{i_k}$, $i_1,\cdots,i_k\in\{0,1\}$.

(iv) $\omega=bdX_{i_1}d\cdots X_{i_{k-1}}de$, $i_1,\cdots,i_{k-1}\in\{0,1\}$.

(v) $\omega=bdX_{i_1}d\cdots X_{i_l}de\circ\omega_2$, $0\leq l\leq k-2$, $i_1,\cdots,i_l\in\{0,1\}$ and $\omega_2\in S_b^{2k+1-(3+2l)}$.

For the case (iii), $||T_\omega||=||M_{i_1}\cdots M_{i_k}||$. For the case (iv), $||T_\omega||=||M_{i_1}\cdots M_{i_{k-1}}||$. And for the case (v), by the formula (29), $||T_\omega||=||M_{i_1}\cdots M_{i_l}||\cdot||T_{\omega_2}||$. Thus by a discussion similar to that for $v_{2k,q}$,we have

$$v_{2k+1,q}=\Big(\sum_{l=0}^{k-2}u_{l,q}v_{2k+1-(2l+3),q}\Big)+u_{k-1,q}+u_{k,q}.$$

■

Lemma 18 $\lim_{m\to\infty}\Big(\sum_{\omega\in S_b^m}||T_\omega||^q\Big)^{\frac{1}{m}}=\mathbf{x}(q)^{-1}$, *where* $\mathbf{x}(q)$ *is given by (31).*

Proof. We will prove the statement in two steps.

(i) $\overline{\lim}_{m\to\infty}(v_{m,q})^{\frac{1}{m}}\leq\mathbf{x}(q)^{-1}$

Since $\sum_{n\geq 0}u_{n,q}\mathbf{x}(q)^{3+2n}\leq 1$ it follows that

$$\begin{cases} \mathbf{x}(q)^{-2k} \geq \sum_{i=0}^{k}u_{i,q}\mathbf{x}(q)^{(3+2i)-2k} \geq \sum_{i=0}^{k-2}u_{i,q}\mathbf{x}(q)^{(3+2i)-2k}+u_{k-1,q}\mathbf{x}(q) \\ \mathbf{x}(q)^{-2k-1} \geq \sum_{i=0}^{k}u_{i,q}\mathbf{x}(q)^{(3+2i)-2k-1} \geq \sum_{i=0}^{k-2}u_{i,q}\mathbf{x}(q)^{(3+2i)-2k-1}+u_{k-1,q}+u_{k,q}\mathbf{x}(q)^2 \end{cases} \tag{32}$$

Select a positive number $C > \max\{1, \mathbf{x}(q)^{-2}, \mathbf{x}(q)^{-1}\}$ such that

$$v_{i,q} < C \cdot \mathbf{x}(q)^{-i}, \quad i = 1, 2, 3$$

Now we will prove by induction that

$$v_{i,q} < C \cdot \mathbf{x}(q)^{-i} \tag{33}$$

for all $i \in \mathbb{N}$. Suppose that this inequality holds for any $i < 2k$, then by Lemma 17 and Inequality (32), we have

$$\begin{aligned}
v_{2k,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q} v_{2k-(2i+3),q}) + u_{k-1,q} \\
&\leq C(\textstyle\sum_{i=0}^{k-2} u_{i,q} \mathbf{x}(q)^{(2i+3)-2k}) + u_{k-1,q} \\
&\leq C(\textstyle\sum_{i=0}^{k-2} u_{i,q} \mathbf{x}(q)^{(2i+3)-2k}) + C\mathbf{x}(q) u_{k-1,q} \\
&\leq C\mathbf{x}(q)^{-2k}, \\
v_{2k+1,q} &= (\textstyle\sum_{i=0}^{k-2} u_{i,q} v_{2k+1-(2i+3),q}) + u_{k-1,q} + u_{k,q} \\
&\leq C(\textstyle\sum_{i=0}^{k-2} u_{i,q} \mathbf{x}(q)^{(2i+3)-2k-1}) + u_{k-1,q} + u_{k,q} \\
&\leq C(\textstyle\sum_{i=0}^{k-2} u_{i,q} \mathbf{x}(q)^{(2i+3)-2k-1}) + C u_{k-1,q} + C\mathbf{x}(q)^2 u_{k,q} \\
&\leq C\mathbf{x}(q)^{-2k-1}.
\end{aligned}$$

Thus the inequality (33) holds also for $i = 2k, 2k+1$. By induction, Inequality (33) holds for all $i \in \mathbb{N}$, which proves the statement (i).

(ii) $\underline{\lim}_{m\to\infty} (v_{m,q})^{\frac{1}{m}} \geq \mathbf{x}(q)^{-1}$

Given any $0 < y < \mathbf{x}(q)^{-1}$, then there exists positive integer N such that

$$1 < \sum_{i=0}^{N-2} u_{i,q} y^{-3-2i}.$$

Thus when $k \geq N$, we have

$$\begin{cases} y^{2k} & \leq \sum\limits_{i=0}^{k-2} u_{i,q} y^{2k-(3+2i)}, \\ y^{2k+1} & \leq \sum\limits_{i=0}^{k-2} u_{i,q} y^{2k+1-(3+2i)}, \end{cases} \tag{34}$$

Select a positive number $D < \min\{1, \mathbf{x}(q)^{-1}, \mathbf{x}(q)^{-2}\}$ such that

$$v_{i,q} > D y^i, \quad i = 1, \cdots, 2N-1.$$

Then by Lemma (17), Formula (34) and a discussion similar to that in the part (i), we have

$$v_{i,q} > D y^i, \quad \forall i \in \mathbb{N},$$

which yields $\lim\limits_{m\to\infty} (v_{m,q})^{\frac{1}{m}} \geq y$ $(0 < y < \mathbf{x}(q)^{-1})$. Thus $\lim\limits_{m\to\infty} (v_{m,q})^{\frac{1}{m}} \geq \mathbf{x}(q)^{-1}$. ■

Lemma 19 $\lim\limits_{m\to\infty} (\sum\limits_{\omega \in S^m} ||T_\omega||^q)^{\frac{1}{m}} = \mathbf{x}(q)^{-1}$, *where* $\mathbf{x}(q)$ *is given by (31).*

Proof. By (23) each element in S^m can be written as $\underbrace{a\cdots a}_{m_1}\circ\omega$, or $\underbrace{c\cdots c}_{m_1}\circ\omega$, where $0\le m_1\le m$ and $\omega\in S_b^{m-m_1}$, thus we have

$$\sum_{\omega\in S^m}||T_\omega||^q=\sum_{\omega\in S_b^m}||T_\omega||^q+2\sum_{j=1}^{m-1}\sum_{\omega\in S_b^j}||T_\omega||^q+2. \tag{35}$$

Since $u_{n,q}>||M_0^n||^q=(n+2)^q$, it follows that the series $\sum_{n\ge0}u_{n,q}x^{2n+3}$ diverges for $x>1$. By the definition of $\mathbf{x}(q)$, we have $\mathbf{x}(q)\le1$ and thus $\mathbf{x}(q)^{-1}\ge1$. By (35) and Lemma 18, we have

$$\lim_{m\to\infty}(\sum_{\omega\in S^m}||T_\omega||^q)^{\frac{1}{m}}=\mathbf{x}(q)^{-1}.$$

■

The following lemmas consider about the differentiability of $\mathbf{x}(q)$.

Lemma 20 *(i) If $q\ge0$, then for any $m,n\in\mathbb{N}$,*

$$u_{m,q}u_{n,q}\ge u_{m+n,q}.$$

(ii) If $q<0$, then for any $m,n\in\mathbb{N}$,

$$u_{m,q}u_{n,q}\le u_{m+n,q}.$$

Proof. The above statement follows immediately from the observation that for any integer $m,n\ge1$,

$$\begin{aligned}
u_{m,q}u_{n,q} &= \sum_{\mathbf{i}\in\{0,1\}^m}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q\\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1\\1\end{bmatrix}(1,1)M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q,\\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1&1\\1&1\end{bmatrix}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q,\\
u_{m+n,q} &= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q,\\
&= \sum_{\mathbf{i}\in\{0,1\}^m}\sum_{\mathbf{j}\in\{0,1\}^n}\left((1,1)M_{\mathbf{i}}\begin{bmatrix}1&0\\0&1\end{bmatrix}M_{\mathbf{j}}\begin{bmatrix}1\\1\end{bmatrix}\right)^q,
\end{aligned}$$

■

Lemma 21 *Let θ_0 be the positive root of $x^2+2x-\frac{9}{8}=0$, i.e., $\theta_0\approx0.45774$. And let ζ be the Riemann-Zeta function, that is $\zeta(x)=\sum_{n\ge1}n^{-x}$ $(x>1)$. Then for any $q\in(-\zeta^{-1}(\frac{16}{11}),-\zeta^{-1}(1+\theta_0))\approx(-2.2599,-2.2543)$, we have*

$$1<\sum_{n\ge0}u_{n,q}<+\infty.$$

Proof. Denote $U = (-\zeta^{-1}(\frac{16}{11}), -\zeta^{-1}(1+\theta_0))$. By direct check, we have $\zeta(3) \approx 1.2021 < \frac{16}{11}$ and $\zeta(2) \approx 1.6449 > 1+\theta_0$, therefore $(\zeta(3), \zeta(2)) \supset (\frac{16}{11}, 1+\theta_0)$, it follows $U \subset (-3, -2)$. Furthermore by computation, $U \approx (-2.2599, -2.2543)$.

Since that any element in $\{0,1\}^n$ can be written as $0^{n_1}1^{n_2}\cdots$, or $1^{n_1}0^{n_2}\cdots$, it follows that

$$\begin{array}{rcl}\sum_{n\geq 0} u_{n,q} & = & \sum_{n\geq 0}\sum_{|J|=n}||M_J||^q \\ & = & 2^q + 2\sum_{n\geq 1}||M_0^n||^q + 2\sum_{l\geq 1}\sum_{n_1,\cdots,n_{2l}\geq 1}||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}||^q \\ & & +2\sum_{l\geq 1}\sum_{n_1,\cdots,n_{2l+1}\geq 1}||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}M_0^{n_{2l+1}}||^q.\end{array} \tag{36}$$

Since

$$\begin{cases} ||(M_0^{n_1}M_1^{n_2})\cdots(M_0^{n_{2l-1}}M_1^{n_{2l}})|| & = ||\begin{pmatrix} 1+n_1n_2 & n_1 \\ n_2 & 1\end{pmatrix}\cdots\begin{pmatrix} 1+n_{2l-1}n_{2l} & n_{2l-1} \\ n_{2l} & 1\end{pmatrix}|| \\ & \geq (1+n_1n_2)\cdots(1+n_{2l-1}n_{2l}) \\ ||M_0^{n_1}M_1^{n_2}\cdots M_1^{n_{2l}}M_0^{n_{2l+1}}|| & \geq ||\begin{pmatrix} (1+n_1n_2)\cdots(1+n_{2l-1}n_{2l}) & * \\ * & *\end{pmatrix}\begin{pmatrix} 1 & n_{2l+1} \\ 0 & 1\end{pmatrix}|| \\ & \geq (1+n_1n_2)\cdots(1+n_{2l-1}n_{2l})(1+n_{2l+1}) \end{cases} \tag{37}$$

and

$$\begin{cases} ||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l-1}}M_1^{n_{2l}}|| & \leq (1+n_1)(1+n_2)\cdots(1+n_{2l-1})(2+n_{2l}) \\ ||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2l}}M_1^{n_{2l+1}}|| & \leq (1+n_1)(1+n_2)\cdots(1+n_{2l})(2+n_{2l+1}) \end{cases}$$

(which follows from that $(1,1)M_i^n \leq (n+1, n+1) = (n+1)(1,1)$ for any $i \in \{0,1\},\ n \geq 0.$), by(36), when $q < 0$ we have

$$\sum_{n\geq 0} u_{n,q} \leq 2^q + 2\sum_{n\geq 1}(2+n)^q + 2\cdot(1+\sum_{n\geq 1}n^q)\cdot(\sum_{l\geq 1}(\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q)^l) \tag{38}$$

and

$$\sum_{n\geq 0} u_{n,q} \geq 2^q + 2\sum_{n\geq 1}(2+n)^q\cdot(1+\sum_{l\geq 1}(\sum_{n\geq 1}(1+n)^q)^l) \tag{39}$$

From now on, we assume that $q \in U$. As we have proved, $-3 < q < -2$.

At first, we have

$$\begin{array}{rl}\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q & = 2\sum_{n\geq 1}(1+n)^q - 2^q + \sum_{n_1,n_2\geq 2}(1+n_1n_2)^q \\ & < 2\sum_{n\geq 1}(1+n)^q - 2^q + (\sum_{n\geq 2}n^q)^2 \\ & = 2(\zeta(-q)-1) - 2^q + (\zeta(-q)-1)^2 \\ & < 2\theta_0 - \frac{1}{8} + \theta_0^2 = 1,\end{array} \tag{40}$$

by Inequality (38) we have $\sum_{n\geq 0} u_{n,q} < +\infty$.

On the other hand,

$$\begin{array}{l} 2^q + 2\sum_{n\geq 1}(2+n)^q\cdot(1+\sum_{l\geq 1}(\sum_{n\geq 1}(1+n)^q)^l) \\ = 2^q + 2\cdot\dfrac{\zeta(-q)-1-2^q}{2-\zeta(-q)} = 1 + \dfrac{(3-2^q)\zeta(-q)-4}{2-\zeta(-q)} \\ > 1 + \dfrac{(3-2^{-2})\zeta(-q)-4}{2-\zeta(-q)} > 1 + \dfrac{(3-2^{-2})\cdot\frac{16}{11}-4}{2-\zeta(-q)} = 1,\end{array}$$

by Inequality (39) we have $\sum_{n\geq 0} u_{n,q} > 1$. ■

Corollary 22 *(i) $\sum_{n\geq 0} u_{n,q}$ tends to 0 when q tends to $-\infty$.*

(ii) There exists unique $q_0 < -2.25$ such that $\sum_{n\geq 0} u_{n,q_0} = 1$.

Proof. By Lemma 21, there exists real number $q_1 < -2.25$ such that $1 < \sum_{n\geq 0} u_{n,q_1} < +\infty$. Thus from the definition of $u_{n,q}$, the sum $\sum_{n\geq 0} u_{n,q}$ (as a function of q) is increasing and continuous on $(-\infty, q_1)$. On the other hand, note that

$$\frac{u_{n,q}}{u_{n,q'}} \leq \max_{|J|=n} ||M_J||^{q-q'} \leq 2^{q-q'}$$

for any integer $n > 0$ and real numbers $q < q' \leq q_1$, therefore

$$\frac{\sum_{n\geq 0} u_{n,q}}{\sum_{n\geq 0} u_{n,q'}} < 2^{q-q'}$$

holds for any $q < q' \leq q_1$, which implies (i). The statement (ii) follows from the continuity of $\sum_{n\geq 0} u_{n,q}$ on $(-\infty, q_1)$. ■

Lemma 23 *Let θ_0 be the positive root of $x^2 + 2x - \frac{9}{8} = 0$, then $\sum_{n\geq 1} n \cdot u_{n,q} < +\infty$ if $q < -\zeta^{-1}(1+\theta_0) \approx -2.2544$, where ζ is the Riemann-Zeta function.*

Proof. By Inequality (37), for $q < 0$ we have

$$\begin{aligned}
\sum_{n\geq 0} n \cdot u_{n,q} &= 2\sum_{n\geq 1} n||M_0^n||^q + 2\sum_{k\geq 1}\sum_{n_1,\cdots n_{2k}\geq 1} (n_1 + \cdots + n_{2k})||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2k-1}}M_1^{n_{2k}}||^q \\
&\quad +2\sum_{k\geq 1}\sum_{n_1,\cdots n_{2k+1}\geq 1} (n_1 + \cdots + n_{2k+1})||M_0^{n_1}M_1^{n_2}\cdots M_0^{n_{2k-1}}M_1^{n_{2k}}M_0^{n_{2k+1}}||^q \\
&\leq 2\sum_{n\geq 1} n(2+n)^q + 2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k}\geq 1} (n_1 + \cdots + n_{2k})(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q \\
&\quad + 2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1} (n_1 + \cdots + n_{2k+1})(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&= 2\sum_{n\geq 1} n(2+n)^q + 2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k}\geq 1} 2kn_1(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q \\
&\quad +2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1} 2kn_1(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&\quad +2\sum_{k\geq 1}\sum_{n_1,n_2,\cdots,n_{2k},n_{2k+1}\geq 1} n_{2k+1}(1+n_1n_2)^q\cdots(1+n_{2k-1}n_{2k})^q n_{2k+1}^q \\
&= 2\sum_{n\geq 1} n(2+n)^q + \sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q \times \sum_{k\geq 1} 4k(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^{k-1} \\
&\quad + \sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q \times \sum_{n\geq 1} n^q \times \sum_{k\geq 1} 4k(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^{k-1} \\
&\quad +2\sum_{n\geq 1} n^{q+1} \times \sum_{k\geq 1}(\sum_{m_1,m_2\geq 1}(1+m_1m_2)^q)^k
\end{aligned} \tag{41}$$

Now suppose $q < -\zeta^{-1}(1+\theta_0)$. By Inequality (40) , we have

$$\sum_{n_1,n_2\geq 1}(1+n_1n_2)^q < 1.$$

On the other hand, since $q < -2$, it follows that the series $\sum_{n\geq 1} n^{q+1}$ and $\sum_{n_1,n_2\geq 1} n_1(1+n_1n_2)^q$ converge. Therefore by Inequality (41), $\sum_{n\geq 1} n\cdot u_{n,q} < +\infty$. ■

Lemma 24 *Suppose that $q\in\mathbb{R}$ satisfies $\sum_{n\geq 0} u_{n,q} = +\infty$, then for any integer L there exists $0<y<1$ such that*

$$L < \sum_{n\geq 0} u_{n,q} y^n < +\infty.$$

Proof. Case 1: $q > 0$.

In this case, $u_{n,q} > 1$ for $n\geq 0$, therefore $\sum_{n\geq 0} u_{n,q} = +\infty$. By Lemma 20, $\{u_{n,q}\}_n$ is submultiplicative, therefore

$$\lim_{n\to+\infty} u_{n,q}^{1/n} = \inf_{n\geq 1} u_{n,q}^{1/n}.$$

Denote by r_q the value of above limit, then $1\leq r_q < \infty$ and $u_{n,q}\geq r_q^n$ for $n\geq 1$. Hence $\lim\limits_{x\to r_q^{-1}} \sum\limits_{n\geq 0} u_{n,q}x^n = +\infty$, which implies the desired result since the series $\sum_{n\geq 0} u_{n,q}y^n$ converges on $(0, r_q^{-1})$.

Case 2: $q<0$, and $\sum_{n\geq 0} u_{n,q} = \infty$.

For any integer $l\geq 0$ and positive integers $n_1, n_2, \cdots, n_l$, define

$$\begin{aligned} a(n_1,n_2,\cdots,n_l) &= (1,0)M_0^{n_1}M_1^{n_2}\cdots M_{l(\bmod 2)}^{n_l}\begin{pmatrix}1\\0\end{pmatrix}, \\ b(n_1,n_2,\cdots,n_l) &= (1,1)M_0^{n_1}M_1^{n_2}\cdots M_{l(\bmod 2)}^{n_l}\begin{pmatrix}1\\1\end{pmatrix}. \end{aligned}$$

It is clear that

$$a(n_1,n_2,\cdots,n_l)\leq b(n_1,n_2,\cdots,n_l)$$

and

$$a(n_1,n_2,\cdots,n_l)a(m_1,m_2,\cdots,m_s)\leq a(n_1,n_2,\cdots,n_l,m_1,m_2,\cdots,m_s), \tag{42}$$

where $m_1,m_2,\cdots,m_s$ are positive integers. It is not hard to show that

$$a(n_1,n_2,\cdots,n_l)\geq \frac{1}{4}b(n_1,n_2,\cdots,n_l), \text{ if } l \text{ is even.} \tag{43}$$

(To see this, denote

$$\begin{pmatrix} x_1 & x_2\\ x_3 & x_4\end{pmatrix} = (M_0^{n_1}M_1^{n_2})\cdots(M_0^{n_{l-1}}M_1^{n_l})$$

for even integer l. Then by induction on l, one can verify that among the x_i's, x_1 is the greatest and x_4 the smallest .)

For any integer $L \geq 1$, take an integer $y(L) \geq L \cdot 4^{-q}$, and define $p = 2^{y(L)}$. Now for any $0 < x < 1$,

$$\begin{aligned}
\sum_{n\geq 0} u_{n,q}x^n &= 2^q + 2\cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j\geq 1} b(n_1,n_2,\cdots,n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad +2\cdot \sum_{j=0}^{2p-1}\sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j}\geq 1} b(n_1,\cdots,n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}} \\
&\leq 2^q + 2\cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j\geq 1} b(n_1,n_2,\cdots,n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad +2\cdot \sum_{j=0}^{2p-1}\sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j}\geq 1} a(n_1,\cdots,n_{2kp+j})^q \cdot x^{n_1+\cdots+n_{2kp+j}} \\
&\leq 2^q + 2\cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j\geq 1} b(n_1,n_2,\cdots,n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad +2\cdot \sum_{j=0}^{2p-1}\sum_{k=1}^{+\infty} \sum_{n_1,\cdots,n_{2kp+j}\geq 1} a(n_1,\cdots,n_{2kp})^q a(n_{2kp+1},\cdots,n_{2kp+j})^q x^{n_1+\cdots+n_{2kp+j}} \\
&\leq 2^q + 2\cdot \sum_{j=1}^{2p-1} \sum_{n_1,\cdots,n_j\geq 1} b(n_1,n_2,\cdots,n_j)^q \cdot x^{n_1+\cdots+n_j} \\
&\quad +2\cdot (\sum_{j=0}^{2p-1} \sum_{n_1,\cdots,n_j\geq 1} a(n_1,\cdots,n_j)^q\ x^{n_1+\cdots+n_j}) \\
&\quad\ (\sum_{k=1}^{+\infty}(\sum_{n_1,\cdots,n_{2p}\geq 1} a(n_1,\cdots,n_{2p})^q\ x^{n_1+\cdots+n_{2p}})^k). \qquad (44)
\end{aligned}$$

Since $a(n_1,n_2,\cdots,n_l), b(n_1,n_2,\cdots,n_l)$ are polynomials about $n_1,n_2,\cdots,n_l$ and $0<x<1$, it follows

$$\sum_{n_1,\cdots,n_l\geq 1} a(n_1,\cdots,n_l)^q\ x^{n_1+\cdots+n_l} < \infty$$
$$\sum_{n_1,\cdots,n_l\geq 1} b(n_1,\cdots,n_l)^q\ x^{n_1+\cdots+n_l} < \infty$$

for any positive integer l. Thus by (44), $\sum_{n\geq 0} u_{n,q}x^n < \infty$ if $\sum_{n_1,\cdots,n_{2p}\geq 1} a(n_1,\cdots,n_{2p})^q\ x^{n_1+\cdots+n_{2p}} < 1$.

Since $\sum_{n\geq 0} u_{n,q} = \infty$, it follows from (44) that $\sum_{n_1,\cdots,n_{2p}\geq 1} a(n_1,\cdots,n_{2p})^q \geq 1$ (or $= +\infty$). Therefore there exists $0 < z \leq 1$ such that $\sum_{n_1,\cdots,n_{2p}\geq 1} a(n_1,\cdots,n_{2p})^q z^{n_1+\cdots+n_{2p}} = 1$. Moreover,

$$\sum_{n\geq 0} u_{n,q}x^n < \infty \quad \text{for } x \in (0,z). \qquad (45)$$

For $l = 2, 2^2, \cdots, p$, by Inequality (42), we obtain that

$$\sum_{n_1,\cdots,n_{2p}\geq 1} a(n_1,\cdots,n_{2p})^q z^{n_1+\cdots+n_{2p}} \leq (\sum_{n_1,\cdots,n_l\geq 1} a(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l})^{2p/l},$$

which implies that $\sum_{n_1,\cdots,n_l\geq 1} a(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l} \geq 1$. Thus by (43), we have

$$\sum_{n_1,\cdots,n_l\geq 1} b(n_1,\cdots,n_l)^q z^{n_1+\cdots+n_l} \geq 4^q, \quad l=2,2^2,\cdots,p.$$

Therefore

$$\begin{aligned} \lim_{x\to z-}\sum_{n\geq 0} u_{n,q}x^n &\geq 2^q + 2\cdot\sum_{j=1}^{2p-1}\sum_{n_1,\cdots,n_j\geq 1} b(n_1,\cdots,n_j)^q\cdot z^{n_1+\cdots+n_j} \\ &\geq 2^q + 2\cdot y(L)\cdot 4^q \\ &\geq 2^q + 2L, \end{aligned}$$

this and (45) yield the desired result. ■

Proposition 25 *Let* $\mathbf{x}(q)$ *be defined by (31) and* q_0 *be given as in Corollary 22 (ii), then*

(i) $\mathbf{x}(q)=1$ *for* $q\leq q_0$;

(ii) if $q>q_0$, *then* $\mathbf{x}(q)$ *is the positive root of* $\sum_{n\geq 0} u_{n,q}x^{2n+3}=1$, *and it is infinitely differentiable on* $(q_0,+\infty)$, *and*

$$x'(q) = -\frac{\sum_{n\geq 0}(\sum_{|J|=n}||M_J||^q\log||M_J||)\cdot \mathbf{x}(q)^{2n+3}}{\sum_{n\geq 0} u_{n,q}\cdot(2n+3)\cdot\mathbf{x}(q)^{2n+2}},$$

(iii) $\mathbf{x}(q)$ *is not differentiable at* $q=q_0$, *moreover,*

$$x'(q_0-)=0,\ x'(q_0+) = -\frac{\sum_{n\geq 0}(\sum_{|J|=n}||M_J||^{q_0}\log||M_J||)}{\sum_{n\geq 0} u_{n,q_0}\cdot(2n+3)} < 0.$$

Proof. Fix $q\leq q_0$. Since $\sum_{n\geq 0} u_{n,q}\leq 1$, it follows $\mathbf{x}(q)\geq 1$ by the definition (31). On the other hand, $u_{n,q}>||M_0^n||^q=(n+1)^q$, therefore $\sum_{n\geq 0} u_{n,q}x^{2n+3}=\infty$ if $x>1$, thus $\mathbf{x}(q)\leq 1$ by (31). The statement (i) follows.

To show (ii), let $q>q_0$. We have either $1<\sum_{n\geq 0} u_{n,q}<\infty$ or $\sum_{n\geq 0} u_{n,q}=\infty$. In the former case, $\sum_{n\geq 0} u_{n,q}x^{2n+3}$ is continuous on $(0,1)$ and thus there exists x_0 satisfying $\sum_{n\geq 0} u_{n,q}x_0^{2n+3}=1$. By (31) $\mathbf{x}(q)=x_0$. Now we assume $\sum_{n\geq 0} u_{n,q}=\infty$. By Lemma 24, there exists $0<t_1<t_2<1$ such that $1<\sum_{n\geq 0} u_{n,q}t_1^{2n}<+\infty$ and $t_1^{-3}<\sum_{n\geq 0} u_{n,q}t_2^{2n}<\infty$. Thus $1<\sum_{n\geq 0} u_{n,q}t_2^{2n+3}<\infty$, similarly we can show that $\mathbf{x}(q)$ satisfies $\sum_{n\geq 0} u_{n,q}\mathbf{x}(q)^{2n+3}=1$. Now we show below that $\mathbf{x}(q)$ is infinitely differentiable on $(q_0,+\infty)$. Define

$$G(q,x)=\sum_{n\geq 0} u_{n,q}x^{2n+3}.$$

Fix $q_1\in(q_0,+\infty)$. As we have shown, there exists real number $y>\mathbf{x}(q_1)$ such that $1<G(q_1,y)<+\infty$. Take a real number z so that $\mathbf{x}(q_1)<z<y$, and take q_2 such that

$$q_2>q_1, \quad 4^{q_2-q_1}<\frac{y}{z}.$$

Note that for any integer $n\geq 0$,

$$\frac{u_{n,q_2}}{u_{n,q_1}} \leq \max_{|J|=n}||M_J||^{q_2-q_1} \leq 4^{n(q_2-q_1)}.$$

Therefore for any $q < q_2$ and $0 < x < z$, we have

$$\begin{aligned} G(q,x) &\leq \sum_{n\geq 0} u_{n,q_2} z^{2n+3} \\ &= \sum_{n\geq 0} u_{n,q_1} y^{2n+3} 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty, \end{aligned}$$

$$\begin{aligned} \sum_{n\geq 0} \frac{du_{n,q}}{dq} x^{2n+3} &= \sum_{n\geq 0}\sum_{|J|=n} ||M_J||^q \log ||M_J|| x^{2n+3} \leq \sum_{n\geq 0} u_{n,q} (\log 4^n) x^{2n+3} \\ &\leq \sum_{n\geq 0} u_{n,q_1} y^{2n+3} (\log 4^n) 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty, \end{aligned}$$

and

$$\begin{aligned} \sum_{n\geq 0} u_{n,q}(3+2n) x^{2n+2} &< \sum_{n\geq 0} u_{n,q_2}(3+2n) z^{2n+3} \\ &= \sum_{n\geq 0} u_{n,q_1} y^{2n+3} (3+2n) 4^{n(q_2-q_1)} (\frac{z}{y})^{2n+3} < +\infty. \end{aligned}$$

The above three inequality imply that $G(q,x)$ is well defined and differentiable on $(-\infty, q_2) \times (0,z)$. A similar more discussion shows that $G(q,x)$ is infinitely differentiable on $(-\infty, q_2) \times (0,z)$. Thus by the Implicit Function Theorem, $\mathbf{x}(q)$ is infinitely differentiable on a neighborhood of q_1. Since q_1 is taken arbitrarily on $(q_0, +\infty)$, $\mathbf{x}(q)$ is infinitely differentiable on $(q_0, +\infty)$ and (ii) follows.

To show the statement (iii), we only need to calculate $x'(q_0+)$. For $q > q_0$, starting from the fact that

$$\sum_{n\geq 0} u_{n,q} \mathbf{x}(q)^{2n+3} - \sum_{n\geq 0} u_{n,q_0} \mathbf{x}(q_0)^{2n+3} = 0,$$

we have

$$\begin{aligned} \frac{\mathbf{x}(q)-\mathbf{x}(q_0)}{q-q_0} &= -\frac{\sum_{n\geq 0} \frac{u_{n,q}-u_{n,q_0}}{q-q_0} \cdot \mathbf{x}(q_0)^{2n+3}}{\sum_{n\geq 0} u_{n,q}(\mathbf{x}(q)^{2n+2} + \mathbf{x}(q)^{2n+1}\mathbf{x}(q_0) + \cdots + \mathbf{x}(q_0)^{2n+2})} \\ &= -\frac{\sum_{n\geq 0} \frac{u_{n,q}-u_{n,q_0}}{q-q_0}}{\sum_{n\geq 0} u_{n,q}(\mathbf{x}(q)^{2n+2} + \mathbf{x}(q)^{2n+2} + \cdots + \mathbf{x}(q) + 1)}. \end{aligned}$$

Since $\sum_{n\geq 0} u_{n,q}(2n+3) < +\infty$ on a neighborhood of q_0 (by Lemma 23 and 21), taking $q \downarrow q_0$ we get the desired result. ∎

Proof of Theorem 16: it follows immediately from Lemma 19 and Proposition 25.

4.4 The Hausdorff dimension of graph(f_λ)

Theorem 26 *Let* $\alpha = \frac{\log \lambda^{-1}}{\log 2}$, *then*

$$\dim_H graph(f_\lambda) = \frac{\log \mathbf{x}(\alpha)}{\log \lambda},$$

where $\mathbf{x}(\alpha)$ *is the unique positive root of the transcendental equation*

$$\sum_{n=0}^{\infty} u_{n,\alpha} x^{3+2n} = 1.$$

Here $u_{0,q} = 2^q$ *and* $u_{n,\alpha} = \sum_{\mathbf{i}\in\{0,1\}^n} ||M_{\mathbf{i}}||^{\alpha}$, $n \geq 1$.

Proof. The theorem follows from Lemma 9, Lemma 14 and Theorem 16. ■

4.5 The box-counting dimension and Hausdorff dimension of level sets of f_λ

4.5.1 Symbolic space

Recall that for any positive integer m, there is an one-to-one correspondence between the collection of m-th net intervals associated with λ and the string set S^m which is defined by

$$S^m := \{(x_i)_{i=1}^m \in \Xi^m : \quad H_{x_i,x_{i+1}} = 1, 1 \leq i \leq m-1, \quad x_1 = a, b \text{ or } c\},$$

where $\Xi = \{a, b, c, d, e, f, \overline{f}\}$, H is a 0-1 matrix defined by (20).

For any $\omega \in \cup_{m\geq 1} S^m$, we use V_ω to denote the net interval corresponding to ω. Define

$$S^{\mathbb{N}} := \{(x_i)_{i=1}^{\infty} \in \Xi^{\mathbb{N}} : H_{x_i,x_{i+1}} = 1, i \geq 1, x_1 = a, b \text{ or } c\},$$

and consider the mapping $\Pi : S^{\mathbb{N}} \longrightarrow [0,1]$ defined by

$$\omega = (x_i)_{i=1}^{\infty} \mapsto \bigcap_{m\geq 1} V_{\omega|m}, \tag{46}$$

where $\omega|n$ denotes $(x_i)_{i=1}^m$. Clearly Π is surjective. And it is also one-to-one except for a set of countable points; more precisely, denote by L the set of all left and right endpoints of net intervals associated with λ, that is, $L = \cup_{m\geq 1} P_{m,\lambda}$, where $P_{m,\lambda}$ is defined as in Section 2, then Π is injective on $[0,1]\backslash L$ and two-to-one on L.

Consider the set

$$S_b^{\mathbb{N}} := \{(x_i)_{i=1}^{\infty} \in \Xi^{\mathbb{N}} : H_{x_i,x_{i+1}} = 1, i \geq 1, x_1 = b\},$$

its image under the mapping Π is the 1-th net interval $V_b = [1-\lambda, \lambda]$. By the generating relations (19), any element in $\Sigma := \{b, d, e, f, \overline{f}\}$ generates out neither a nor c, it follows that

$$S_b^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty} \in \Sigma^{\mathbb{N}} : \quad \hat{H}_{x_i,x_{i+1}} = 1, i \geq 1, x_1 = b\},$$

where $\hat{H} = (\hat{H}_{i,j})_{i,j\in\Sigma}$ is the restriction of H to the index set Σ, i.e.,

$$\hat{H} = \begin{array}{c} \\ b \\ d \\ e \\ f \\ \overline{f} \end{array} \begin{array}{c} \begin{array}{ccccc} b & d & e & f & \overline{f} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{47}$$

The matrix $\hat{H}$ is primitive, that is, there exists a positive integer m such that all the entries of $\hat{H}^m$ are positive. In our case, $\hat{H}^6 > 0$.

Now let us consider the subshift space of finite type, $(\Sigma_{\hat{H}}^{\mathbb{N}}, \sigma)$, where

$$\Sigma_{\hat{H}}^{\mathbb{N}} := \{(x_i)_{i=1}^{\infty} \in \Sigma^{\mathbb{N}} : \quad \hat{H}_{x_i, x_{i+1}} = 1, i \geq 1\},$$

and the shift mapping $\sigma : \Sigma_{\hat{H}}^{\mathbb{N}} \longrightarrow \Sigma_{\hat{H}}^{\mathbb{N}}$ is defined by

$$(x_i)_{i=1}^{\infty} \mapsto (x_{i+1})_{i=1}^{\infty}.$$

In what follows we define a Markov measure on $(\Sigma_{\hat{H}}^{\mathbb{N}}, \sigma)$.

Recall that we have used the letters $b, f, \overline{f}$ to denote the I-color $(\{-\lambda, 0\}, 1-\lambda)$, d to denote the I-color $(\{\lambda-1, 0\}, \lambda)$, and e to denote $(\{\lambda-1\}, 2\lambda-1)$. Now we define

$$\gamma_b = \gamma_f = \gamma_{\overline{f}} = 1-\lambda,\ \gamma_d = \lambda,\ \gamma_e = 2\lambda - 1.$$

Define a matrix $P = (P_{i,j})_{i,j\in\Sigma}$ by

$$P_{i,j} = \begin{cases} \lambda\frac{\gamma_j}{\gamma_i} & \text{if } i \text{ generates out } j \text{ in the sense of (19)} \\ 0 & \text{otherwise.} \end{cases}$$

That is

$$P = \begin{array}{c} \\ b \\ d \\ e \\ f \\ \overline{f} \end{array} \begin{array}{c} \begin{array}{ccccc} b & d & e & f & \overline{f} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda-1 & 1-\lambda & 1-\lambda \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

Suppose $\mathbf{p} = (p_b, p_d, p_e, p_f, p_{\overline{f}})$ is the probability vector satisfying that $\mathbf{p}P = \mathbf{p}$. By direct calculation,

$$\begin{aligned} \mathbf{p} &= (\frac{1}{7+4\lambda}, \frac{1}{1+2\lambda}, \frac{1}{7+4\lambda}, \frac{1}{4+3\lambda}, \frac{1}{4+3\lambda}) \\ &= (\frac{5-2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, \frac{5-2\sqrt{5}}{5}, \frac{3\sqrt{5}-5}{10}, \frac{3\sqrt{5}-5}{10}). \end{aligned}$$

Since P is primitive, there exists a σ-invariant ergodic measure ξ (which is often called the $(\mathbf{p}, P)$ Markov measure) on $\Sigma_{\hat{H}}^{\mathbb{N}}$, such that for any n-th cylinder set $[x_1 x_2 \cdots x_n] \subset \Sigma_{\hat{H}}^{\mathbb{N}}$,

$$\xi([x_1 x_2 \cdots x_n]) = p_{x_1} P_{x_1 x_2} P_{x_2 x_3} \cdots P_{x_{n-1} x_n}. \tag{48}$$

(one may refer to [Wal] for further information about the $(\mathbf{p}, P)$ measure).

Now we consider the projection of the measure ξ under the mapping $\Pi|_{S_b^{\mathbb{N}}} : S_b^{\mathbb{N}} \to [1-\lambda, \lambda]$, which is written as $(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi$ and defined by

$$(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi(A) = \xi(\Pi|_{S_b^{\mathbb{N}}}^{-1}(A)), \text{ for } A \subset [1-\lambda, \lambda].$$

It is clear that $(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi$ is a Borel measure on $[1-\lambda,\lambda]$. The following lemma shows that $(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi$ is equivalent to the Lebesgue measure on $[1-\lambda,\lambda]$.

Lemma 27 *For any Borel set $A\subset[1-\lambda,\lambda]$, we have*

$$(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi(A)=\frac{p_b}{2\lambda-1}\mathcal{L}(A).$$

Proof. For any n-th cylinder set $[bx_2\cdots x_n]\subset S_b^{\mathbb{N}}$, $\Pi|_{S_b^{\mathbb{N}}}([bx_2\cdots x_n])$ is just the n-th net interval $V_{bx_2\cdots x_n}$. Note that the length of $V_{bx_2\cdots x_n}$ is equal to $\lambda^n\gamma_{x_n}$, we have

$$\begin{aligned}
\mathcal{L}(V_{bx_2\cdots x_n}) &= \mathcal{L}(V_b)\times\frac{\mathcal{L}(V_{bx_2})}{\mathcal{L}(V_b)}\times\cdots\times\frac{\mathcal{L}(V_{bx_2\cdots x_n})}{\mathcal{L}(V_{b\cdots x_{n-1}})}\\
&= \lambda\gamma_b\times\frac{\lambda\gamma_{x_2}}{\gamma_b}\times\cdots\times\frac{\lambda\gamma_{x_n}}{\gamma_{x_{n-1}}}\\
&= \lambda\gamma_b\cdot P_{b,x_2}\cdots P_{x_{n-1},x_n}\\
&= \frac{\lambda\gamma_b}{p_b}p_bP_{b,x_2}\cdots P_{x_{n-1},x_n}\\
&= \frac{\lambda\gamma_b}{p_b}\xi\left([bx_2\cdots x_n]\right)\\
&= \frac{\lambda\gamma_b}{p_b}(\Pi|_{S_b^{\mathbb{N}}})_{\#}\xi\left(V_{bx_2\cdots x_n}\right),
\end{aligned}$$

thus the lemma holds for all the n-th net intervals which contained in V_b. By a standard argument, the lemma holds for every Borel subset of V_b. $\blacksquare$

Lemma 28 *For ξ almost all $\omega=(x_i)_{i=1}^{\infty}\in S_b^{\mathbb{N}}$,*

$$\lim_{n\to\infty}\frac{\log||T_{\omega|n}||}{n}=(\log 2)\xi([bde])+\sum_{k\geq 1}\sum_{i_1,\cdots,i_k=0\ or\ 1}(\log||M_{i_1\cdots i_k}||)\xi([bdX_{i_1}d\cdots X_{i_k}de])$$

where $\omega|n=(x_i)_{i=1}^n$, $X_0=f$, $X_1=\overline{f}$, and $T_{x_1\cdots x_n}$ is defined by (25) and (28).

Proof. Let E denote the set

$$\{(x_i)_{i=1}^{\infty}\in\Sigma_{\hat{H}}^{\mathbb{N}}:\ \exists\text{ integer sequence }m_j\uparrow\infty,\lim_{j\to\infty}\frac{m_j}{m_{j+1}}=1,x_{m_j}=e\}.$$

Since ξ is ergodic and $\xi([e])>0$, it follows that $\xi(E)=1$.

For any $\omega=(x_i)_{i=1}^{\infty}\in E\bigcap S_b^{\mathbb{N}}$, define the integer sequence $m_j(\omega)$ ($j\in\mathbb{N}$) such that

$$\begin{cases}1<m_1(\omega)<m_2(\omega)<\cdots<m_n(\omega)<\cdots\\ x_{m_j(\omega)}=e,\ j=1,2,\cdots\\ x_i\neq e,\text{ if }i\in\mathbb{N}\backslash\{m_j(\omega):\ j\in\mathbb{N}\}.\end{cases}$$

We can write ω as

$$\omega=\omega_1\circ\omega_2\circ\cdots\circ\omega_n\circ\cdots,$$

where $\omega_1 = (x_i)_{i=1}^{m_1(\omega)}$, $\omega_2 = (x_i)_{i=m_1(\omega)+1}^{m_2(\omega)}$, $\cdots$, $\omega_n = (x_i)_{i=m_{n-1}(\omega)+1}^{m_n(\omega)}$, $\cdots$. By the structure of $S_b^{\mathbb{N}}$, one can see that

$$\omega_j \in \mathcal{B} = \{bde\} \bigcup \{bdX_{i_1}d\cdots X_{i_k}de: \ k \in \mathbb{N},\ i_1, \cdots, i_k = 0 \text{ or } 1\}$$

for each integer j. Thus by Lemma 15, we have

$$||T_{\omega|m_k(\omega)}|| = ||T_{\omega_1}|| \cdot ||T_{\omega_2}|| \cdots ||T_{\omega_k}||. \tag{49}$$

For any $\nu \in \mathcal{B}$, denote by $[\nu]$ the cylinder set in $S_b^{\mathbb{N}}$ associated with ν. Then the formula (49) means that

$$||T_{\omega|m_k(\omega)}|| = \prod_{\nu\in\mathcal{B}, |\nu|\le m_k(\omega)} ||T_\nu||^{\sum_{j=0}^{m_k(\omega)-|\nu|} \mathcal{X}_{[\nu]}(\sigma^j\omega)} \tag{50}$$

where $\mathcal{X}_{[\nu]}(\cdot)$ is the characteristic function on $[\nu]$, that is,

$$\mathcal{X}_{[\nu]}(y) = \begin{cases} 1 & \text{if } y \in [\nu] \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\frac{\log ||T_{\omega|m_k(\omega)}||}{m_k(\omega)} = \sum_{\nu\in\mathcal{B}, |\nu|\le m_k(\omega)} \frac{1}{m_k(\omega)} \sum_{j=0}^{m_k(\omega)-|\nu|} \mathcal{X}_{[\nu]}(\sigma^j\omega) \log ||T_\nu||. \tag{51}$$

Since ξ is ergodic, by the Birkhoff ergodic theorem, for each $\nu \in \mathcal{B}$,

$$\lim_{k\to\infty} \frac{1}{m_k(\omega)} \sum_{j=0}^{m_k(\omega)-|\nu|} \mathcal{X}_{[\nu]}(\sigma^j\omega) = \xi([\nu]) \ \text{ for } \xi \text{ almost all } \omega \in E \bigcap S_b^{\mathbb{N}}. \tag{52}$$

Combine (51) and (52) to obtain that

$$\underline{\lim}_{k\to\infty} \frac{\log ||T_{\omega|m_k(\omega)}||}{m_k(\omega)} \ge \sum_{\nu\in\mathcal{B}} \xi([\nu]) \log ||T_\nu|| \quad \text{for } \xi \text{ almost all } \omega \in E \bigcap S_b^{\mathbb{N}}. \tag{53}$$

Since $||T_{\omega|m_k(\omega)}|| \le 4^{m_k(\omega)}$, the right-hand side of the above inequality is convergent and bounded by $\log 4$.

Now fix $l \in \mathbb{N}$. For any $m > l$ and $\nu = bdX_{i_1}d\cdots X_{i_m}de \in \mathcal{B}$, we have

$$||T_\nu|| \le ||T_{dX_{i_1}d\cdots X_{i_l}d}|| \cdot ||T_{dX_{i_2}d\cdots X_{i_{l+1}}d}|| \cdots ||T_{dX_{i_{m-l+1}}d\cdots X_{i_m}d}||.$$

Thus by the formula (50), for $m_k(\omega) > 2l+3$,

$$\begin{aligned} \frac{\log ||T_{\omega|m_k(\omega)}||}{m_k(\omega)} \quad &\le \quad \sum_{\nu\in\mathcal{B}, |\nu|\le 2l+3} \frac{1}{m_k(\omega)} \sum_{j=0}^{m_k(\omega)-|\nu|} \mathcal{X}_{[\nu]}(\sigma^j\omega) \log ||T_\nu|| \\ &+ \sum_{\nu'\in\{dX_{i_1}\cdots dX_{i_l}:\ i_1,\cdots,i_l=0 \text{ or } 1\}} \frac{1}{m_k(\omega)} \sum_{j=0}^{m_k(\omega)-2l} \mathcal{X}_{[\nu']}(\sigma^j\omega) \log ||T_{\nu'}||. \end{aligned}$$

By the above inequality and Birkhoff ergodic theorem, for ξ almost all $\omega \in E\bigcap S_b^{\mathbb{N}}$,

$$\begin{aligned}\overline{\lim_{k\to\infty}}\frac{\log||T_{\omega|m_k(\omega)}||}{m_k(\omega)} \quad &\leq \quad \sum_{\nu\in\mathcal{B},|\nu|\leq 2l+3}\xi([\nu])\log||T_\nu|| \\ &\quad + \sum_{\nu'\in\{dX_{i_1}\cdots dX_{i_l}:\, i_1,\cdots,i_l=0 \text{ or } 1\}}\xi([\nu'])\log||T_{\nu'}||. \qquad (54)\end{aligned}$$

Note that for any $\nu' = dX_{i_1}d\cdots dX_{i_l}$ we have

$$\begin{aligned}\xi([\nu']) &= p_dP_{dX_{i_1}}P_{X_{i_1}d}\cdots P_{dX_{i_l}} = \frac{p_d}{p_bP_{bd}P_{X_{i_l}d}P_{de}}\cdot p_bP_{bd}P_{dX_{i_1}}P_{X_{i_1}d}\cdots P_{dX_{i_l}}P_{X_{i_l}d}P_{de} \\ &= \frac{p_d}{p_bP_{bd}P_{X_{i_l}d}P_{de}}\xi([bdX_{i_1}d\cdots dX_{i_l}e]) \\ &\leq \max\{\frac{p_d}{p_bP_{bd}P_{fd}P_{de}},\frac{p_d}{p_bP_{bd}P_{\overline{f}d}P_{de}}\}\times\xi([bdX_{i_1}d\cdots dX_{i_l}e]),\end{aligned}$$

and

$$\log||T_{\nu'}|| \leq \log||T_{bdX_{i_1}d\cdots dX_{i_l}e}||,$$

thus

$$\begin{aligned}\sum_{\nu'\in\{dX_{i_1}\cdots dX_{i_l}:\, i_1,\cdots,i_k=0 \text{ or } 1\}}\xi([\nu'])\log||T_{\nu'}|| \quad &\leq \quad \max\{\frac{p_d}{p_bP_{bd}P_{fd}P_{de}},\frac{p_d}{p_bP_{bd}P_{\overline{f}d}P_{de}}\} \\ &\qquad \times\sum_{\nu\in\mathcal{B},|\nu|=2l+3}\xi([\nu])\log||T_\nu||\end{aligned}$$

Since $\sum_{\nu\in\mathcal{B}}\xi([\nu])\log||T_\nu||$ converges, the right-hand side of the above inequality tends to 0 when $l\to+\infty$. Thus by (53) and (54), we have

$$\lim_{k\to\infty}\frac{\log||T_{\omega|m_k(\omega)}||}{m_k(\omega)} = \sum_{\nu\in\mathcal{B}}\xi([\nu])\log||T_\nu|| \text{ for } \xi \text{ a.e. } \omega\in E\bigcap S_b^{\mathbb{N}}.$$

Therefore we obtain the desired result by the facts $\lim_{k\to\infty}\frac{m_{k+1}(\omega)}{m_k(\omega)} = 1$ for $\omega\in E\bigcap S_b^{\mathbb{N}}$ and $\xi(E)=1$. ■

4.5.2 Box-counting dimension

Lemma 29 *For $\mathcal{L}$ almost all $t\in[1-\lambda,\lambda]$, the box-counting dimension of the t-level set $L_{t,\lambda}$ of f_λ exists and is given by the following formula*

$$\dim_B(L_{t,\lambda}) = \frac{1}{\log 2}\sum_{\nu\in\mathcal{B}}\xi([\nu])\log||T_\nu|| = \frac{7\sqrt{5}-15}{10\log 2}\cdot\sum_{n=0}^{\infty}\left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\sum_{|J|=n}\log||M_J||. \quad (55)$$

Proof. The first equality follows from Lemma 13, Formula (27), Lemma 27 and Lemma 28. To see the second equality, note that

$$\begin{aligned}
||T_{bde}|| &= 2, \\
\xi([bde]) &= \frac{p_b}{2\lambda - 1} \cdot \mathcal{L}(V_{bde}) = \frac{p_b}{2\lambda - 1} \cdot \lambda^3 \cdot (2\lambda - 1) = p_b \lambda^3 \\
||T_{bdX_{i_1}d\cdots X_{i_k}de}|| &= ||M_{i_1\cdots i_k}||, \\
\xi([bdX_{i_1}d \cdots X_{i_k}de]) &= \frac{p_b}{2\lambda - 1} \cdot \mathcal{L}(V_{bdX_{i_1}d\cdots X_{i_k}de}) = p_b \lambda^{2k+3},
\end{aligned}$$

thus,

$$\begin{aligned}
\sum_{\nu \in \mathcal{B}} \xi([\nu]) \log ||T_\nu|| &= p_b \sum_{n=0}^{\infty} \lambda^{2n+3} \cdot \sum_{|J|=n} \log ||M_J|| \\
&= p_b \lambda \sum_{n=0}^{\infty} \lambda^{2n+2} \cdot \sum_{|J|=n} \log ||M_J|| \\
&= \frac{\lambda}{7 + 4\lambda} \sum_{n=0}^{\infty} \lambda^{2n+2} \cdot \sum_{|J|=n} \log ||M_J|| \\
&= \frac{7\sqrt{5} - 15}{10} \cdot \sum_{n=0}^{\infty} \left(\frac{3 - \sqrt{5}}{2} \right)^{n+1} \cdot \sum_{|J|=n} \log ||M_J||.
\end{aligned}$$

■

Theorem 30 *For $\mathcal{L}$ almost all $t \in [0, 1]$, the box-counting dimension of the t-level set $L_{t,\lambda}$ of f_λ exists and is given by the following formula*

$$\dim_B(L_{t,\lambda}) = \frac{7\sqrt{5} - 15}{10 \log 2} \cdot \sum_{n=0}^{\infty} \left(\frac{3 - \sqrt{5}}{2} \right)^{n+1} \cdot \sum_{|J|=n} \log ||M_J||.$$

Proof. Note that

$$(0, 1) = V_b \bigcup (\cup_{i=1}^{\infty} V_{a^i b}) \bigcup (\cup_{j=1}^{\infty} V_{c^j b}).$$

For any $i \in \mathbb{N}$, denote by F_i the affine mapping from $V_{a^i b}$ onto V_b with the ratio λ^{-i}. For each $x \in V_{a^i b}$, the infinite Markov code $\Pi^{-1}(x)$ of x is of the form $a^i b \circ \omega$ (where Π is defined as in (46)) and the infinite Markov code of $F_i(x)$ is of the form $b \circ \omega$. Thus the mapping F_i preserves the box-counting dimension of x-level set, therefore the formula (55) holds for $\mathcal{L}$ almost all $t \in V_{a^i b}$. The same result holds for $V_{c^j b}$, $j \in \mathbb{N}$. Thus we have proved the theorem. ■

4.5.3 Hausdorff dimension

As we have seen in the proof of Lemma 28, for $\mathcal{L}$ almost all $t \in [1 - \lambda, \lambda]$ there are infinitely many e's which appear in the code $\Pi^{-1}(t)$; We will show that for such t, the t-level set $L_{t,\lambda}$ is a kind of Moran set–*homogeneous Moran set* and its Hausdorff dimension can be estimated from below rigorously.

Let us recall the definition of homogeneous Moran set. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $\{c_k\}_{k\geq 1}$ be a sequence of positive numbers satisfying $n_k \geq 2$, $0 < c_k < 1$, $n_1 c_1 \leq \delta$ and $n_k c_k \leq 1$ $(k \geq 2)$, where δ is some positive number. Let

$$D = \bigcup_{k\geq 0} D_k \quad \text{with} \quad D_0 = \{\emptyset\}, D_k = \{(i_1, \cdots, i_k);\ \ 1 \leq i_j \leq n_j,\ \ 1 \leq j \leq k\}.$$

If $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$, $\tau = (\tau_1, \cdots, \tau_m) \in D_m$, we define $\sigma * \tau = (\sigma_1, \cdots, \sigma_k, \tau_1, \cdots, \tau_m)$.

Suppose J be a closed interval of length δ. A collection $\mathcal{F} = \{J_\sigma : \ \sigma \in D\}$ of closed subintervals of J is said to have a *homogeneous Moran structure* if it satisfies

(1) $J_\emptyset = J$;

(2) For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma*1}, J_{\sigma*2}, \cdots, J_{\sigma*n_{k+1}}$ are subintervals of J_σ and $\overset{o}{J}_{\sigma*i} \bigcap \overset{o}{J}_{\sigma*j} = \emptyset$ $(i \neq j)$ where $\overset{o}{A}$ denotes the interior of A;

(3) For any $k \geq 1$ and any $\sigma \in D_{k-1}$, $1 \leq j \leq n_k$, We have

$$\frac{|J_{\sigma*j}|}{|J_{\sigma}|} = c_k$$

where $|A|$ denotes the diameter of A.

Suppose that $\mathcal{F}$ is a collection of closed subintervals of J having homogeneous Moran structure, $E(\mathcal{F}) := \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} J_\sigma$ is called a *homogeneous Moran set* determined by $\mathcal{F}$ and the intervals in $\mathcal{F}_k = \{J_\sigma;\ \ \sigma \in D_k\}$ are called the k-order fundamental intervals of $E(\mathcal{F})$ and J is called the original interval of $E(\mathcal{F})$. It can be seen from above definition that for any fixed J, $\{n_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$, if the positions of k-order fundamental intervals are changed, we get different homogeneous Moran sets. We use $\mathcal{M}(J, \{n_k\}, \{c_k\})$ to denote the collection of all such homogeneous Moran sets determined by J, $\{n_k\}_{k\geq 1}$, $\{c_k\}_{k\geq 1}$. One may refer to [FWW, FRW] for more informations about homogeneous Moran sets. For the purpose of the present paper, we only need a simplified version of a result contained in [FWW], whose simpler proof will be given here for the convenience of the reader.

Proposition 31 *For any $F \in \mathcal{M}(J, \{n_k\}, \{c_k\})$, we have*

$$\dim_H F \geq \liminf_{n\to\infty} \frac{\log n_1 n_2 \cdots n_k}{-\log c_1 c_2 \cdots c_{k+1} n_{k+1}}.$$

Proof. Denote by t the right hand side of the above inequality. Suppose $t > 0$. Let μ be the probability measure concentrated on F such that $\mu(A) = (n_1 n_2 \cdots n_k)^{-1}$ for any $A \in \mathcal{F}_k$. Let $0 < s < t$. By the definition of t, there exists $c > 0$ such that

$$n_1 n_2 \cdots n_k (c_1 c_2 \cdots c_{k+1} n_{k+1})^s \geq c \qquad (\forall k \geq 1).$$

Let $U \subset [0, 1]$ be an arbitrary closed interval with $|U| \leq c_1$. There exists a positive integer k such that $c_1 c_2 \cdots c_{k+1} \leq |U| < c_1 c_2 \cdots c_k$. It follows that

i) U intersects at most $\dfrac{3|U|}{c_1 c_2 \cdots c_{k+1}}$ $(k+1)$-order fundamental intervals;

ii) U intersects at most 2 k-order fundamental intervals.

By using the inequality $\min(a,b) \leq a^{1-s} b^s$ ($0 \leq s \leq 1$), we have

$$\begin{aligned}
\mu(U) &\leq \min\left(\frac{2}{n_1 n_2 \cdots n_k}, \frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}} \times \frac{1}{n_1 n_2 \cdots n_k}\right) \\
&\leq \frac{1}{n_1 n_2 \cdots n_k} \left(\frac{3|U|}{c_1 c_2 \cdots c_{k+1} n_{k+1}}\right)^s 2^{1-s} \\
&\leq \frac{1}{c} 3^s 2^{1-s} |U|^s \leq \frac{6}{c} |U|^s.
\end{aligned}$$

This implies $\dim_H F \geq s$ then $\dim_H F \geq t$. ■

Lemma 32 *For $\mathcal{L}$ almost all $t \in [1-\lambda, \lambda]$,*

$$\dim_H(L_{t,\lambda}) = \frac{7\sqrt{5}-15}{10\log 2} \cdot \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} \sum_{|J|=n} \log ||M_J||.$$

Proof. We define the set E and the integer sequence $m_j(\omega)$ ($j \in \mathbb{N}$) for every $\omega \in E \bigcap S_b^{\mathbb{N}}$ in the same way as in the proof of lemma 28. Let Π defined as in (46). If $\omega \in E \bigcap S_b^{\mathbb{N}}$, we claim that

$$L_{\Pi(\omega),\lambda} \in \mathcal{M}([0,1], \{||T_{\omega_k}||\}, \{2^{-|\omega_k|}\}), \tag{56}$$

where $\omega = \omega_1 \circ \omega_2 \circ \cdots \circ \omega_n \circ \cdots$. Using this claim and Proposition 31, we have

$$\begin{aligned}
\dim_H L_{\Pi(\omega)} &\geq \liminf_{k\to\infty} \frac{\log(||T_{\omega_1}|| \times \cdots \times ||T_{\omega_k}||)}{\log(2^{|\omega_1|+\cdots+|\omega_{k+1}|} \times ||T_{\omega_{k+1}}||^{-1})} \\
&\geq \liminf_{k\to\infty} \frac{\log(||T_{\omega_1}|| \times \cdots \times ||T_{\omega_k}||)}{\log(2^{|\omega_1|+\cdots+|\omega_{k+1}|})} \\
&= \liminf_{k\to\infty} \frac{\log(||T_{\omega_{|m_k(\omega)}}||)}{m_{k+1}(\omega)\log 2} \\
&= \liminf_{k\to\infty} \frac{\log(||T_{\omega_{|m_k(\omega)}}||)}{m_k(\omega)\log 2},
\end{aligned}$$

thus according to Lemmas 27-29 and the fact that $\dim_H(\cdot) \leq \dim_B(\cdot)$, we have obtained the result of this lemma.

Now we begin to prove the claim (56). Fix $\omega \in E \bigcap S_b^{\mathbb{N}}$. For any positive integer k the II-color of the $m_k(\omega)$-th net interval $V_{\omega|m_k(\omega)}$ is given by

$$E^{(||T_{\omega|m_k(\omega)}||)} := (\{(\lambda-1, ||T_{\omega|m_k(\omega)}||)\}, 2\lambda-1),$$

thus from the definition of II-color, we know that the collection

$$\mathcal{D}_k := \left\{\mathbf{i} \in \{0,1\}^{m_k(\omega)} : \ \Pi(\omega) \in \phi_{\mathbf{i},\lambda}([0,1])\right\}$$

has the cardinality $||T_{\omega|m_k(\omega)}||$, and any two elements $\mathbf{i},\mathbf{j}$ of $\mathcal{D}_k$ satisfy that $\phi_{\mathbf{i},\lambda}([0,1]) = \phi_{\mathbf{j},\lambda}([0,1])$. Define $J=[0,1]$. For any $\mathbf{i}\in\mathcal{D}_k$, define

$$J_{\mathbf{i}} = \psi_{\mathbf{i},\lambda}\left([0,1]\right). \quad \text{(recall that } \psi_0(x)=x/2,\ \psi_1(x)=(x+1)/2.)$$

Then $L_{\Pi(\omega),\lambda} = \bigcap_{k=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{D}_k} J_{\mathbf{i}}$, and thus $L_{\Pi(\omega),\lambda}\in\mathcal{M}([0,1],\{||T_{\omega_k}||\},\{2^{-|\omega_k|}\})$, which proves the claim. ■

Theorem 33 *For $\mathcal{L}$ almost all $t\in[0,1]$,*

$$\dim_H(L_{t,\lambda}) = \frac{7\sqrt{5}-15}{10\log 2}\cdot\sum_{n=0}^{\infty}\left(\frac{3-\sqrt{5}}{2}\right)^{n+1}\sum_{|J|=n}\log||M_J||.$$

Proof. It follows from Lemma 32 and a discussion similar to that in the proof of Theorem 30. ■

□

4.6 The L^q-spectrum of μ_λ

Theorem 34 *For any $q\in\mathbb{R}$, the L^q-spectrum $\tau_{\mu_\lambda}(q)$ of μ_λ is equal to*

$$\frac{q\log 2}{\log\lambda^{-1}}+\frac{\log\mathbf{x}(q)}{\log\lambda^{-1}},$$

where

$$\mathbf{x}(q)=\sup\{x\in\mathbb{R}:\ \sum_{n=0}^{\infty}u_{n,q}x^{3+2n}\le 1\},$$

and $u_{n,q}=\sum_{\mathbf{i}\in\{0,1\}^n}||M_{\mathbf{i}}||^q$. There exists unique $q_0<0$ such that $\sum\limits_{n=0}^{\infty}(\sum_{|J|=n}||M_J||^{q_0})=1$. When $q>q_0$, $\mathbf{x}(q)$ is the positive root of $\sum\limits_{n=0}^{\infty}(\sum_{|J|=n}||M_J||^q)x^{2n+3}=1$, and it is infinitely differentiable on $(q_0,+\infty)$. When $q\le q_0$, $\mathbf{x}(q)=1$. Moreover $\mathbf{x}(2,q)$ is not differentiable at $q=q_0$.

Proof. The theorem follows from Corollary 5, Lemma 14, Lemma 19 and Theorem 16. ■

Remark 1 *Using a different method, Lau and Ngai [LN2] obtained the same formula of $\tau_{\mu_\lambda}(q)$ only for $q>0$. And they proved the differentiability of $\tau_{\mu_\lambda}(q)$ on $(0,+\infty)$. Porzio [Po] extended the range of differentiability to $(-\frac{1}{2},+\infty)$. The above theorem give the complete answer to the question posed in [LN2] that how to get the formula of $\tau_{\mu_\lambda}(q)$ for $q<0$.*

4.7 The dimensions of μ_λ

We first consider the local dimension of μ_λ. As we have shown in section 4.5., there is a natural projection Π from the symbolic space $S^{\mathbb{N}}$ onto the interval $[0,1]$; see (46). Denote by L the set of all left and right endpoints of net intervals associated with λ, that is, $L=\cup_{m\ge1}P_{m,\lambda}$, where $P_{m,\lambda}$ is defined as in Section 2, then Π is injective on $[0,1]\backslash L$ and two-to-one on L. For $x \in [0,1]$, each point in $\Pi^{-1}(x)$ is called *the infinite Markov code* of x. We will give the formula of the local dimension of μ_λ at x in terms of matrix products by the infinite Markov code of x.

Theorem 35 *For each $\omega = (x_i)_{i=1}^{\infty} \in S^{\mathbb{N}}$, the upper and lower local dimension of μ_λ at $\Pi(\omega)$ are given by*

$$\overline{d}(\mu_\lambda, \Pi(\omega)) = \frac{\log 2}{\log \lambda^{-1}} + \lim_{n\to\infty} \sup \frac{\log ||T_{\omega|n}||}{n \log \lambda},$$

$$\underline{d}(\mu_\lambda, \Pi(\omega)) = \frac{\log 2}{\log \lambda^{-1}} + \lim_{n\to\infty} \inf \frac{\log ||T_{\omega|n}||}{n \log \lambda},$$

where $\omega|n = x_1 \cdots x_n$ and $T_{\omega|n}$ is defined as in (25) and (28).

Proof. By the definition of upper local dimension (see Section 1),

$$\begin{aligned}
\overline{d}(\mu_\lambda, \Pi(\omega)) &= \limsup_{r\downarrow 0} \frac{\log \mu_\lambda(\Pi(\omega) - r, \Pi(\omega) + r)}{\log r} \\
&= \lim_{n\to+\infty} \sup \frac{\log \mu_\lambda(\Pi(\omega) - \lambda^n, \Pi(\omega) + \lambda^n)}{\log \lambda^n}.
\end{aligned}$$

By the above equality, Lemma 4(ii) and Lemma 6,

$$\overline{d}(\mu_\lambda, \Pi(\omega)) = \lim_{n\to+\infty} \sup \frac{\log 2^{-n} N_{m,\lambda}(V_{\omega|n})}{\log \lambda^n}, \tag{57}$$

where $V_{\omega|n}$ is the net interval corresponding to $\omega|n$, and $N_{m,\lambda}(V_{\omega|n})$ is the overlap times of $V_{\omega|n}$.

Similarly, we have

$$\underline{d}(\mu_\lambda, \Pi(\omega)) = \lim_{n\to+\infty} \inf \frac{\log 2^{-n} N_{m,\lambda}(V_{\omega|n})}{\log \lambda^n}. \tag{58}$$

By Lemma 14, $N_{m,\lambda}(V_{\omega|n}) = ||T_{\omega|n}||$, this and (57)-(58) yield the desired formulas. ■

Theorem 36 *For $\mathcal{L}$ almost all $x \in [0,1]$, the upper and lower local dimension of μ_λ at x coincide, and the common value is equal to*

$$d(\mu_\lambda, x) = \frac{\log 2}{\log \lambda^{-1}} + \frac{7\sqrt{5} - 15}{10 \log \lambda} \cdot \sum_{n=0}^{\infty} \left(\frac{3-\sqrt{5}}{2}\right)^{n+1} \sum_{|J|=n} \log ||M_J||.$$

Proof. There is a natural connection between the local dimension of μ_λ at a given point x and the box-counting dimension of the x-level set $L_{x,\lambda}$ of the limit Rademacher function f_λ. That is,

$$\overline{d}(\mu_\lambda, x) = \frac{\log 2}{\log \lambda^{-1}} + \frac{\log 2}{\log \lambda} \cdot \underline{\dim}_B L_{x,\lambda}$$

$$\underline{d}(\mu_\lambda, x) = \frac{\log 2}{\log \lambda^{-1}} + \frac{\log 2}{\log \lambda} \cdot \overline{\dim}_B L_{x,\lambda}$$

for $x \in [0,1]\backslash U$, where U is a set of countable many points. This is the direct corollary of Lemma 13(ii) and Formulas (57)-(58). By Theorem 30 we obtain the desired result. ■

Using Theorem 35, we can determine the set

$$\mathcal{R}(\mu_\lambda) := \{y : \ \exists x \in [0,1], \ d(\mu_\lambda, x) = y\}.$$

The following Theorem was first proved in [Hu] by using some combinatorial techniques.. Here we provide a different proof, the method used in which is valid to determine the sets $\mathcal{R}(\mu_{\lambda_k})$ for $k \geq 3$; see Theorem 51, where we correct the false result given in [Hu] about $\mathcal{R}(\mu_{\lambda_k})$ for $k \geq 3$.

Theorem 37 $\mathcal{R}(\mu_\lambda) = [\frac{\log 2}{\log \lambda^{-1}} - \frac{1}{2}, \frac{\log 2}{\log \lambda^{-1}}]$.

Proof. Let $\mathcal{B}$ be the collection of letter strings defined as in (22). Since for any letter strings $\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_n, \cdots \in \mathcal{B}$,

$$\mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_n \circ \cdots \in S^{\mathbb{N}},$$

and

$$||T_{\mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_n}|| = ||T_{\mathbf{i}_1}|| \times ||T_{\mathbf{i}_2}|| \times \cdots \times ||T_{\mathbf{i}_n}||,$$

by Theorem 35 we have

$$\mathcal{R}(\mu_\lambda) \supset [\frac{\log 2}{\log \lambda^{-1}} - \frac{1}{\log \lambda^{-1}} \cdot x_0, \ \ \frac{\log 2}{\log \lambda^{-1}} - \frac{1}{\log \lambda^{-1}} \cdot y_0], \tag{59}$$

where

$$x_0 = \sup_{\mathbf{i} \in \mathcal{B}} \frac{\log ||T_{\mathbf{i}}||}{|\mathbf{i}|}, \quad y_0 = \inf_{\mathbf{i} \in \mathcal{B}} \frac{\log ||T_{\mathbf{i}}||}{|\mathbf{i}|},$$

here $|\mathbf{i}|$ denotes the length of the letter string $\mathbf{i}$. In the end of our proof, we will show that $x_0 = \frac{1}{2} \log \lambda^{-1}$ and $y_0 = 0$.

On the other hand, for any $\omega \in S^{\mathbb{N}}$, $\omega|n$ is the prefix of an infinite letter string of the following possible forms:

$$a^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots, \ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots, \ c^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots,$$

where $l \in \mathbb{N}$ and $\mathbf{i}_1, \mathbf{i}_2, \cdots \mathbf{i}_m, \cdots \in \mathcal{B}$; furthermore, by the generating relation (19), there exists a letter string $\mathbf{j}$ of length less than 4 such that the concatenation $(\omega|n) \circ \mathbf{j}$ is of the following possible forms

$$a^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m, \ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m, \ c^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m,$$

where $l \in \mathbb{N}$ and $\mathbf{i}_1, \mathbf{i}_2, \cdots \mathbf{i}_m \in \mathcal{B}$. Since $||T_{a^l}|| = ||T_{c^l}|| = 1$, therefore the above analysis implies that

$$\{y : \ \exists x \in [0,1], \ \overline{d}(\mu_\lambda, x) = y\} \subset [\frac{\log 2}{\log \lambda^{-1}} - \frac{1}{\log \lambda^{-1}} \cdot x_0, \ \ \frac{\log 2}{\log \lambda^{-1}}] \tag{60}$$

and

$$\{y:\ \exists x\in[0,1],\ \underline{d}(\mu_\lambda,x)=y\}\subset[\frac{\log 2}{\log\lambda^{-1}}-\frac{1}{\log\lambda^{-1}}\cdot x_0,\ \ \frac{\log 2}{\log\lambda^{-1}}]. \tag{61}$$

Therefore by (59)-(61) and $y_0=0$(what we will prove afterwards),

$$\mathcal{R}(\mu_\lambda)=[\frac{\log 2}{\log\lambda^{-1}}-\frac{1}{\log\lambda^{-1}}\cdot x_0,\ \ \frac{\log 2}{\log\lambda^{-1}}-\frac{1}{\log\lambda^{-1}}\cdot y_0].$$

In what follows, we determine the exact values of x_0 and y_0.

Note that for any integer $n>0$,

$$bd(fd)^n e\in\mathcal{B},\quad T_{bd(fd)^n e}=(M_0)^n=\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix},$$

it follows $y_0=0$ since $(\log||\,T_{bd(fd)^n e}||)/(2n+3)$ tends to 0 as n tends to infinity.

To determine the value of x_0, let us consider the possible form of the element $\mathcal{B}$. Recall that each element of $\mathcal{B}$ is of the form

$$bdX_{i_1}d\cdots X_{i_n}de,\ \ n\ge 0,\ \ i_1,\cdots,i_n=0 \text{ or } 1, \tag{62}$$

where $X_0=f$ and $X_1=\overline{f}$; and $||T_{bdX_{i_1}d\cdots X_{i_n}de}||=||M_{i_1}\cdots M_{i_n}||$. For fixed integer n, one can verify that the maximal value of $||M_{i_1}\cdots M_{i_n}||$ is equal to

$$||M_0M_1M_0\cdots M_{n(\mathrm{mod}\,2)}||=\frac{2+\lambda+(-1)^n\lambda^{2n+4}}{1+\lambda^2}\cdot\lambda^{-n}.$$

Therefore

$$\begin{aligned} x_0 &= \sup_{n\ge 0}\frac{\log(\dfrac{2+\lambda+(-1)^n\lambda^{2n+4}}{1+\lambda^2}\cdot\lambda^{-n})}{2n+3}\\ &= \sup_{n\ge 0}\frac{\log(\dfrac{2+\lambda+(-1)^n\lambda^{2n+4}}{1+\lambda^2})+\log\lambda^{-n}}{3+2n}. \end{aligned}$$

Note that

$$\frac{1}{3}\log(\frac{2+\lambda+(-1)^n\lambda^{2n+4}}{1+\lambda^2})\le\frac{1}{3}\log(\frac{2+\lambda+\lambda^4}{1+\lambda^2})=\frac{1}{3}\log 2,$$

$$\frac{1}{2n}\log\lambda^{-n}=\frac{1}{2}\log\lambda^{-1},$$

we have

$$x_0=\max\{\frac{1}{3}\log 2,\ \ \frac{1}{2}\log\lambda^{-1}\}=\frac{1}{2}\log\lambda^{-1}.$$

■

The Hausdorff dimension and information dimension of μ_λ have been considered by many people, e.g. see ([AY], [AZ], [LP1], [Ng], [SV]). For completeness, we present here Ngai's results:

Lemma 38 *([Ng]) Suppose that ν is a Borel probability measure on $\mathbb{R}$ with bounded support and its L^q-spectrum $\tau_\nu(q)$ is differentiable at $q=1$. Then the local dimension $d(\nu,x)$ is equal to $\tau_\nu'(1)$ for ν almost all $x\in\mathbb{R}$. As a result, the Hausdorff dimension and information dimension of ν coincide, the common value is equal to $\tau_\nu'(1)$.*

The following theorem follows by a direct calculation of $\tau'_{\mu_\lambda}(1)$.

Theorem 39 *([Ng]) The Hausdorff dimension and information dimension of μ_λ satisfy*

$$\dim_H \mu_\lambda = \dim_{info}\mu_\lambda = -\frac{\log 2}{\log\lambda} + \frac{\sum\limits_{n=0}^{\infty} 2^{-2n-3} \sum\limits_{|J|=n} ||M_J|| \log ||M_J||}{9\log\lambda}.$$

5 The case $\rho=\lambda_k$ $(k\geq 3)$.

In this section, we prove our main results for the case $\rho=\lambda_k$ $(k\geq 3)$. The main ideas and proofs are similar to that for the case $\rho=\lambda$. Certainly, the generating relations of I-colors and II-colors for this case are much more complicated.

For simplicity, in this section we would use some symbols and notions the same as in Section 4, however the meaning of which are changed here(e.g. Ξ, Σ, S^m, $S^{\mathbb{N}}$, Π, $v_{m,q}$, $T_{i,j}$).

5.1 The generating relations of I-colors and II- colors.

At first we give the generating relations of I-colors associated with $\rho=\lambda_k$:

$$\begin{cases}
(\{0\},1-\rho^k) & \longrightarrow\ (\{0\},1-\rho^k)+(\{\rho^k-1,0\},\rho^k) \\
& \quad +(\{-\rho^k\},1-\rho^k-\rho^{k-1}) \\
(\{-\rho^k\},1-\rho^k) & \longrightarrow\ (\{-\rho^{k-1}\},1-\rho^k-\rho^{k-1})+(\{\rho^k-1,0\},\rho^k) \\
& \quad +(\{-\rho^k\},1-\rho^k) \\
(\{\rho^i-1,0\},\rho^i) & \longrightarrow\ (\{-\rho^{i-1},0\},\rho^{i-1}) \quad \text{for} \ \ 2\leq i\leq k, \\
(\{\rho-1\},\rho) & \longrightarrow\ (\{\rho^k-1,0\},\rho^k)+(\{-\rho^k\},1-2\rho^k) \\
& \quad +(\{\rho^k-1,0\},\rho^k) \\
(\{-\rho^k\},1-2\rho^k) & \longrightarrow\ (\{-\rho^{k-1}\},1-\rho^k-\rho^{k-1})+(\{\rho^k-1,0\},\rho^k) \\
& \quad +(\{-\rho^k\},1-\rho^k-\rho^{k-1}) \\
(\{-\rho^{k-j}\},1-\rho^k-\rho^{k-j}) & \longrightarrow\ (\{-\rho^{k-j-1}\},1-\rho^k-\rho^{k-j-1})+(\{\rho^k-1,0\},\rho^k) \\
& \quad +(\{-\rho^k\},1-\rho^k-\rho^{k-1}) \ \text{ for } \ 1\leq j\leq k-2 \\
(\{-\rho\},1-\rho^k-\rho) & \longrightarrow\ (\{-\rho^k\},1-\rho^k-\rho^{k-1}) \\
(\{-\rho^k\},1-\rho^k-\rho^{k-l}) & \longrightarrow\ (\{-\rho^{k-1}\},1-\rho^k-\rho^{k-1})+(\{\rho^k-1,0\},\rho^k) \\
& \quad +(\{-\rho^k\},1-\rho^k-\rho^{k-l-1}) \ \text{ for } \ 1\leq l\leq k-2 \\
(\{-\rho^k\},1-\rho^k-\rho) & \longrightarrow\ (\{-\rho^{k-1}\},1-\rho^k-\rho^{k-1})
\end{cases}$$

Let $\Xi=\{a,b,c,d_1,\cdots,d_{k-1},e,f,\overline{f},g_1,\cdots,g_{k-1},h_1,\cdots,h_{k-1}\}$ be an alphabet set. For any $m\in\mathbb{N}$, we label every m-th net interval uniquely by a letter string of length m in the following way: let $J = J_m$ be a m-th net interval. For $1 \le i \le m-1$, denote by J_i the i-th net interval that contains J. Then J is labeled as $(x_i)_{i=1}^m \in \Xi^m$, where

$$x_i = \begin{cases} a & \text{if } \Gamma_{i,\rho}(J_i) = (\{0\}, 1-\rho^k) \\ b & \text{if } \Gamma_{i,\rho}(J_i) = (\{\rho^k - 1, 0\}, \rho^k), \text{ and either} \\ & \ i = 1, \text{ or} \\ & \ i > 1 \text{ with } \Gamma_{i,\rho}(J_{i-1}) \neq (\{\rho - 1, 0\}, \rho) \\ c & \text{if } \Gamma_{i,\rho}(J_i) = (\{-\rho^k\}, 1-\rho^k) \\ d_m \ (m = 1, \cdots, k-1) & \text{if } \Gamma_{i,\rho}(J_i) = (\{\rho^{k-m} - 1, 0\}, \rho^{k-m}) \\ e & \text{if } \Gamma_{i,\rho}(J_i) = (\{-\rho^k\}, 1-2\rho^k) \\ f & \text{if } \Gamma_{i,\rho}(J_i) = (\{\rho^k - 1, 0\}, \rho^k), \ i > 1, \\ & \ \Gamma_{i,\rho}(J_{i-1}) = (\{\rho - 1, 0\}, \rho), \\ & \ \text{and } J_i \text{ has the same left endpoint as } J_{i-1} \\ \overline{f} & \text{if } \Gamma_{i,\rho}(J_i) = (\{\rho^k - 1, 0\}, \rho^k), \ i > 1, \\ & \ \Gamma_{i,\rho}(J_{i-1}) = (\{\rho - 1, 0\}, \rho), \\ & \ \text{and } J_i \text{ has the same right endpoint as } J_{i-1} \\ g_m \ (m = 1, \cdots, k-1) & \text{if } \Gamma_{i,\rho}(J_i) = (\{-\rho^{k-m}\}, 1-\rho^k-\rho^{k-m}) \\ h_m \ (m = 1, \cdots, k-1) & \text{if } \Gamma_{i,\rho}(J_i) = (\{-\rho^k\}, 1-\rho^k-\rho^{k-m}) \end{cases}$$

The above generating relations of I-colors can be given formally as below:

$$\begin{cases} a & \longrightarrow & a + b + h_1 \\ b & \longrightarrow & d_1 \\ c & \longrightarrow & g_1 + b + c \\ d_m \ (1 \le m \le k-2) & \longrightarrow & d_{m+1} \\ d_{k-1} & \longrightarrow & f + e + \overline{f} \\ e & \longrightarrow & g_1 + b + h_1 \\ f & \longrightarrow & d_1 \\ \overline{f} & \longrightarrow & d_1 \\ g_m \ (1 \le m \le k-2) & \longrightarrow & g_{m+1} + b + h_1 \\ g_{k-1} & \longrightarrow & h_1 \\ h_m \ (1 \le m \le k-2) & \longrightarrow & g_1 + b + h_{m+1} \\ h_{k-1} & \longrightarrow & g_1 \end{cases} \tag{63}$$

The above relations determine a 0-1 matrix $Q = (Q_{i,j})_{i,j \in \Xi}$, so that $Q_{i,j} = 1$ if i generates out j.

For $m \ge 2$, set

$$S^m := \{(x_i)_{i=1}^m \in \Xi^m : Q_{x_i, x_{i+1}} = 1, 1 \le i \le m-1, \quad x_1 = a, b \text{ or } c\}, \tag{64}$$

then there is a one-to-one correspondence between S^m and the collection of all m-th net intervals. For any $\omega \in S^m$, we will use V_ω to denote the m-th net interval corresponding to ω.

We would like to know more about the possible forms of the elements in S^m. For this purpose, write

$$\begin{array}{ll} b(k) & := bd_1\cdots d_{k-1} \\ X_0(k) & := fd_1\cdots d_{k-1} \\ X_1(k) & := \overline{f}d_1\cdots d_{k-1} \\ Y_0(j) & := g_1\cdots g_j,\ \ j=1,\cdots,k-1 \\ Y_1(j) & := h_1\cdots h_j,\ \ j=1,\cdots,k-1 \end{array} \tag{65}$$

and define a collection $\mathcal{B}_k$ of letter strings as

$$\begin{array}{ll} \mathcal{B}_k := & \{b(k)e\} \\ & \bigcup\{b(k)X_{j_1}(k)\cdots X_{j_l}(k)e : l\in\mathbb{N},\ j_1,\cdots,j_l=0 \text{ or } 1\} \\ & \bigcup\{Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)e : m\in\mathbb{N},\ 1\le p_1,\cdots p_{m-1}\le k-1, \\ & \ 1\le p_m\le k-2,\ \ i_j=\frac{1-(-1)^j}{2}\ (1\le j\le m)\ or\ i_j=\frac{1+(-1)^j}{2}\ (1\le j\le m)\} \\ & \bigcup\{Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)X_{j_1}(k)\cdots X_{j_l}(k)e :\ l,m\in\mathbb{N}, \\ & \ j_1,\cdots,j_l=0 \text{ or } 1,\ 1\le p_1,\cdots p_{m-1}\le k-1,\ 1\le p_m\le k-2, \\ & \ i_s=\frac{1-(-1)^s}{2}\ (1\le s\le m)\ or\ i_s=\frac{1+(-1)^s}{2}\ (1\le s\le m)\} \end{array} \tag{66}$$

then the generating relation (63) implies that each element in S^m is the prefix of a letter string of the following three forms:

$$\begin{array}{ll} \omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, & \text{the first letter of } \omega_1 \text{ is } b \\ \underbrace{a\cdots a}_{r\ a\text{'s}}\circ\omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, & \text{the first letter of } \omega_1 \text{ is } b \text{ or } h_1 \\ \underbrace{c\cdots c}_{r\ c\text{'s}}\circ\omega_1\circ\omega_2\cdots\circ\omega_n\circ\cdots, & \text{the first letter of } \omega_1 \text{ is } b \text{ or } g_1 \end{array} \tag{67}$$

where $r\in\mathbb{N}$ and $\omega_i\in\mathcal{B}_k$, $i\in\mathbb{N}$.

Let us consider the generating relations of II-colors associated with λ_k. Denote by

$$\begin{array}{ll} A^{(1)} & := (\{(0,1)\},1-\rho^k) \\ B^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\},\rho^k) \\ C^{(1)} & := (\{(-\rho^k,1)\},1-\rho^k) \\ D_m^{(p,q)}\ (1\le m\le k-1) & := (\{(\rho^{k-m}-1,p),(0,q)\},\rho^{k-m}) \\ E^{(r)} & := (\{(-\rho^k,r)\},1-2\rho^k) \\ F^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\},\rho^k) \\ \overline{F}^{(p,q)} & := (\{(\rho^k-1,p),(0,q)\},\rho^k) \\ G_m^{(r)}\ (1\le m\le k-1) & := (\{(-\rho^{k-m},r)\},1-\rho^k-\rho^{k-m}) \\ H_m^{(r)}\ (1\le m\le k-1) & := (\{(-\rho^k,r)\},1-\rho^k-\rho^{k-m}) \end{array}$$

then the generating relations of II-colors associated with λ_k can be written as:

$$\begin{cases}
A^{(1)} & \Longrightarrow & A^{(1)} + B^{(1,1)} + H_1^{(1)} \\
B^{(p,q)} & \Longrightarrow & D_1^{(p,q)} \\
C^{(1)} & \Longrightarrow & G_1^{(1)} + B^{(1,1)} + C^{(1)} \\
D_m^{(p,q)} \ (1 \le m \le k-2) & \Longrightarrow & D_{m+1}^{(p,q)} \\
D_{k-1}^{(p,q)} & \Longrightarrow & F^{(p,p+q)} + E^{(p+q)} + \overline{F}^{(p+q,q)} \\
E^{(r)} & \Longrightarrow & G_1^{(r)} + B^{(r,r)} + H_1^{(r)} \\
F^{(p,q)} & \Longrightarrow & D_1^{(p,q)} \\
\overline{F}^{(p,q)} & \Longrightarrow & D_1^{(p,q)} \\
G_m^{(r)} \ (1 \le m \le k-2) & \Longrightarrow & G_{m+1}^{(r)} + B^{(r,r)} + H_1^{(r)} \\
G_{k-1}^{(r)} & \Longrightarrow & H_1^{(r)} \\
H_m^{(r)} \ (1 \le m \le k-2) & \Longrightarrow & G_1^{(r)} + B^{(r,r)} + H_{m+1}^{(r)} \\
H_{k-1}^{(r)} & \Longrightarrow & G_1^{(1)}
\end{cases}$$

We can define a family of matrixes $T_{i,j}$ for each pair $(\ i, j) \in \Xi \times\ \Xi$ with i generating out j in (63), such that if an I-color generating relation is given by

$$i \longrightarrow i_1 + \cdots + i_l,$$

then the associated II-color generating relation will be given by

$$I^{(n_1,\cdots,n_r)} \Longrightarrow I_1^{(n_1,\cdots,n_r)\cdot T_{i,i_1}} + \cdots I_l^{(n_1,\cdots,n_r)\cdot T_{i,i_l}}. \tag{68}$$

That is,

$$\begin{cases}
T_{a,a} = 1, & T_{a,b} = (1,1), & \\
T_{b,d_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & & \\
T_{c,g_1} = (1,1), & T_{c,b} = (1,1), & T_{c,c} = 1, \\
T_{d_m,d_{m+1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & (1 \le m \le k-2) & \\
T_{d_{k-1},f} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & T_{d_{k-1},e} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & T_{d_{k-1},\overline{f}} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\
T_{e,g_1} = 1, & T_{e,b} = (1,1), & T_{e,h_1} = 1, \\
T_{f,d_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & T_{\overline{f},d_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\
T_{g_m,g_{m+1}} = T_{g_m,h_1} = 1, & T_{g_m,b} = (1,1), & (1 \le m \le k-2) \\
T_{g_{k-1},h_1} = 1, & & \\
T_{h_m,h_{m+1}} = T_{h_m,g_1} = 1, & T_{h_m,b} = (1,1), & (1 \le m \le k-2) \\
T_{h_{k-1},g_1} = 1. & &
\end{cases} \tag{69}$$

For any $\omega = (x_i)_{i=1}^m \in S^m$, define $T_\omega = T_{x_1,x_2} \cdot T_{x_2,x_3} \cdots T_{x_{m-1},x_m}$. Then we have

Lemma 40 *For $m \geq 2$, let J be a m-th net interval associated with λ_k corresponding to $\omega = (x_i)_{i=1}^m \in S^m$. Suppose its II-color is $(\{(t_1, n_1), \cdots, (t_r, n_r)\}, \gamma)$, then*

$$N_{m,\lambda_k}(J) := \sum_{i=1}^{r} n_i = ||T_{x_1,x_2} \cdot T_{x_2,x_3} \cdots T_{x_{m-1},x_m}||.$$

Furthermore if ω can be written as the concatenation $\omega_1 \circ \omega_2$, where the end-letter of ω_1 is e, then

$$N_{m,\lambda_k}(V_\omega) = ||T_\omega|| = ||T_{\omega_1}|| \times ||T_{\omega_2}||.$$

Define a sequence of integers $\{t_{r,k}\}_{r=0}^{\infty}$ such that $t_{0,k} = 1$, and $\frac{1}{2}t_{r,k}$ $(r > 0)$ is the number of different integral solutions of the following conditional Diophantine equation:

$$p_1 + \cdots + p_m = r \quad \text{such that } m \in \mathbb{N},\ 1 \leq p_1, \cdots, p_{m-1} \leq k-1,\ 1 \leq p_m \leq k-2.$$

Lemma 41 *For any positive integer r,*

$$t_{r,k} = 2 \cdot (\underbrace{1, 1, \cdots, 1}_{k-2}, 0) \cdot \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}^{r-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proof. Denote by $\frac{1}{2}t_{r,k}^{(i)}$ $(1 \leq i \leq k-1)$ the number of integral solutions of the equation

$$p_1 + \cdots + p_m = r, \text{ such that } m \in \mathbb{N},\ 1 \leq p_1, \cdots, p_{m-1} \leq k-1,\ p_m = i,$$

then it is clear that

$$\begin{cases} t_{r+1,k}^{(1)} = \sum_{i=1}^{k-1} t_{r,k}^{(i)} \\ t_{r+1,k}^{(i)} = t_{r,k}^{(i-1)} \quad (2 \leq i \leq k-1) \end{cases} \tag{70}$$

Note $t_{r,k} = \sum_{i=1}^{k-2} t_{r,k}^{(i)}$, by (70) we can get the desired results. ■

For any integer $m \geq 1$ and real number q, define

$$S_{b,g_1,h_1}^m = \{(x_i)_{i=1}^m \in S^m,\ x_1 = b, g_1, \text{ or } h_1\},$$

$$v_{m,q} = \sum_{\omega \in S_{b,g_1,h_1}^m} ||T_\omega||^q,\ q \in \mathbb{R},$$

where S^m is defined by (64) and $T_{i,j}$'s are defined by (69).

By a discussion similar to that in the proofs of Lemma 17 and Lemma 18, we can show the following two lemmas.

Lemma 42 *Let $u_{n,q}$ be defined by (30). For $m \geq k+2$,*

$$v_{m,q} = (\sum_{l=k+1}^{m-1} c_{l,q} v_{m-l,q}) + d_{m,q},$$

where

$$c_{l,q} = \sum_{r,i \geq 0, r+ki+k+1=l} t_{r,k} u_{i,q},\ l = k+1, k+2, \cdots \tag{71}$$

and $c_{m,q} \leq d_{m,q} \leq c_{m,q} + c_{m+1,q} + \cdots + c_{m+k,q}$.

Lemma 43 $\lim_{m\to\infty} (v_{m,q})^{\frac{1}{m}} = \mathbf{x}(k,q)^{-1}$, *where*

$$\mathbf{x}(k,q) = \sup\{x : \sum_{l=k+1}^{\infty} c_{l,q} x^l \leq 1\}. \tag{72}$$

Now let us consider the differentiability of $\mathbf{x}(k,q)$.

Lemma 44 *For any real number q, let $\mathbf{x}(k,q)$ be defined by (72). Then $\mathbf{x}(k,q)$ is the positive root of $\frac{1-2x^{k-1}+x^k}{1-2x+x^k} \cdot \sum_{n\geq 0} u_{n,q} x^{kn+k+1} = 1$. Moreover, $0 < \mathbf{x}(k,q) \leq \lambda_{k-1}$ and it is an infinitely differentiable function of q on the whole line.*

Proof. Note from (71) that

$$\sum_{l=k+1}^{\infty} c_{l,q} x^l = (1 + \sum_{r\geq 1} t_{r,k} x^r)(\sum_{n\geq 0} u_{n,q} x^{kn+k+1}). \tag{73}$$

Using Lemma 40 and a direct calculation, we have

$$1 + \sum_{r\geq 1} t_{r,k} x^r = \begin{cases} \dfrac{1-2x^{k-1}+x^k}{1-2x+x^k} & \text{if} \quad 0 < x < \lambda_{k-1} \\ +\infty & \text{if} \quad x \geq \lambda_{k-1}. \end{cases} \tag{74}$$

Thus to prove Lemma 44, by a reason similar to that in the proof of Proposition 25, we only need to show that for any q there exists $0 < y < 1$ such that $1 < \sum_{l=k+1}^{\infty} c_{l,q} y^l < \infty$.

Assume that this statement is not true, then there exist two real numbers q' and $0 < x' < 1$ such that

$$\sum_{l=k+1}^{\infty} c_{l,q'} (x')^l \leq 1$$

and

$$\sum_{l=k+1}^{\infty} c_{l,q'} t^l = +\infty \quad \text{if} \quad t > x'. \tag{75}$$

Since $\sum_{l=k+1}^{\infty} c_{l,q'} (x')^l \leq 1$, it follows from (73)-(74) that

$$x' < \lambda_{k-1},\ \sum_{n\geq 0} u_{n,q} \cdot (x')^{kn+k+1} < +\infty. \tag{76}$$

Therefore by (75), we obtain

$$\sum_{n\geq 0} u_{n,q} t^{kn+k+1} = +\infty \quad \text{for} \;\; t > x', \tag{77}$$

which means $\sum_{n\geq 0} u_{n,q} = +\infty$ since $x' < 1$. According to Lemma 24, we can find $0 < y < 1$ such that

$$\sum_{n\geq 0} u_{n,q} \cdot (x')^{kn} < \sum_{n\geq 0} u_{n,q} \cdot y^{kn} < +\infty,$$

which leads to a contradiction with (77). ■

Lemma 45 *Let S^m and $T_{i,j}$'s be defined as in (64), (69), then*

$$\lim_{m\to\infty} \Big(\sum_{\omega\in S^m} ||T_\omega||^q \Big)^{\frac{1}{m}} = \mathbf{x}(k,q)^{-1},$$

where $\mathbf{x}(k,q)$ is given as in (72).

Proof. Similar to the proof of Lemma 19. ■

5.2 The theorems

Theorem 46 *For $k \geq 3$, let $\alpha_k = \log \lambda_k^{-1} / \log 2$, then*

$$\dim_H \mathit{graph}(f_{\lambda_k}) = \frac{\log \mathbf{x}(k,\alpha_k)}{\log \lambda_k},$$

where $0 < \mathbf{x}(k,\alpha_k) < \lambda_{k-1}$, and it is the positive root of

$$\frac{1-2x^{k-1}+x^k}{1-2x+x^k} \cdot \sum_{n\geq 0} u_{n,\alpha_k} x^{kn+k+1} = 1.$$

Proof. The theorem follows from Lemma 9, Lemma 40, Lemma 45 and Lemma 44. ■

Theorem 47 *For $k \geq 3$, denote by L_{t,λ_k} the t-level set of f_{λ_k}. Then for $\mathcal{L}$ almost all $t \in [0,1]$,*

$$\dim_H(L_{t,\lambda_k}) = \dim_B(L_{t,\lambda_k}) = \frac{(\lambda_k)^k (1-2(\lambda_k)^k)^2}{(2-(k+1)(\lambda_k)^k)\log 2} \sum_{n=0}^{\infty} ((\lambda_k)^{kn} \sum_{|J|=n} \log ||M_J||).$$

Proof. Denote $\Sigma = \{b, d_1, \cdots, d_{k-1}, e, f, \overline{f}, g_1, \cdots, g_{k-1}, h_1, \cdots, h_{k-1}\}$. Define a 0-1 matrix $\hat{Q} = (\hat{Q}_{i,j})_{i,j\in\Sigma}$ such that

$$\hat{Q}_{i,j} = \begin{cases} 1, & \text{if } i \text{ generates out } j \text{ in the formula (63)} \\ 0, & \text{otherwise} \end{cases}$$

Denote by $\gamma_i (i \in \Sigma)$ be the relative length of the color i, that is,

$$\gamma_b = \gamma_f = \gamma_{\overline{f}} = (\lambda_k)^k,\ \gamma_e = 1 - 2(\lambda_k)^k,$$
$$\gamma_{d_s} = (\lambda_k)^{k-s},\ (s = 1, \cdots, k-1)$$
$$\gamma_{g_s} = \gamma_{h_s} = 1 - (\lambda_k)^k - (\lambda_k)^{k-s},\ (s = 1, \cdots, k-1)$$

Define a probability matrix $P = (P_{i,j})_{i,j\in\Sigma}$ by

$$P_{i,j} = \begin{cases} \lambda_k \cdot \frac{\gamma_j}{\gamma_i}, & \text{if } i \text{ generates out } j \text{ in (63)} \\ 0, & \text{otherwise} \end{cases}$$

Let $\mathbf{p}$ $=(p_i)_{i\in\Sigma}$ be the probability vector such that $\mathbf{p} = \mathbf{p}P$. By direct calculation, we have

$$p_b = \frac{1-2(\lambda_k)^k}{2-(k+1)(\lambda_k)^k} \cdot (\lambda_k)^k,\ p_{d_1} = \cdots = p_{d_{k-1}} = \frac{(\lambda_k)^k}{2-(k+1)(\lambda_k)^k},$$
$$p_f = p_{\overline{f}} = \frac{(\lambda_k)^{2k}}{2-(k+1)(\lambda_k)^k},\ p_e = \frac{(\lambda_k)^k(1-2(\lambda_k)^k)}{2-(k+1)(\lambda_k)^k},$$
$$p_{g_s} = p_{h_s} = \frac{\lambda_k}{2-(k+1)(\lambda_k)^k} \cdot (1-(\lambda_k)^k-(\lambda_k)^{k-s}),\ s = 1, \cdots, k-1.$$

Let ξ be the $\mathbf{p}$-P Markov measure on the subshift space $\Sigma_{\hat{Q}}^{\mathbb{N}}$ which is defined as

$$\Sigma_{\hat{Q}}^{\mathbb{N}} = \{\omega = (x_i)_{i=1}^\infty \in \Sigma^{\mathbb{N}} : \hat{Q}_{x_i,x_{i+1}} = 1 \text{ for all } i \geq 1\}.$$

Then by a discussion similar to the proof of Lemma 28, we obtain that for ξ almost all $\omega = (x_i)_{i=1}^\infty \in \Sigma_{\hat{Q}}^{\mathbb{N}}$,

$$\lim_{n\to\infty} \frac{\log ||T_{\omega|n}||}{n} = \sum_{\nu\in\mathcal{B}_k} \xi([e\nu]) \log ||T_\nu||,$$

where $T_{i,j}$'s and $\mathcal{B}_k$ are defined by (69), (66). Since each element ν of $\mathcal{B}_k$ can be written as one of the following four forms:

$$\begin{array}{ll} (i). & b(k)e, \\ (ii). & b(k)X_{j_1}(k)\cdots X_{j_l}(k)e, \\ (iii). & Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)e, \\ (iv). & Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)X_{j_1}(k)\cdots X_{j_l}(k)e, \end{array}$$

where $j_1, \cdots, j_l = 0$ or 1, $1 \leq p_1, \cdots p_{m-1} \leq k-1$, $1 \leq p_m \leq k-2$, and $i_s = \frac{1-(-1)^s}{2}$ $(1 \leq s \leq m)$ or $i_s = \frac{1+(-1)^s}{2}$ $(1 \leq s \leq m)$. And the values of $\xi([e\nu])$ and $||T_\nu||$ for these four cases are equal to

$$\begin{array}{lcc} & \xi([e\nu]) & ||T_\nu|| \\ (i). & \frac{p_e}{\lambda_k} \cdot (\lambda_k)^{k+2} & 2 \\ (ii). & \frac{p_e}{\lambda_k} \cdot (\lambda_k)^{kl+k+2} & ||M_{j_1}\cdots M_{j_l}|| \\ (iii). & \frac{p_e}{\lambda_k} \cdot (\lambda_k)^{p_1+\cdots p_m+k+2} & 2 \\ (iv). & \frac{p_e}{\lambda_k} \cdot (\lambda_k)^{p_1+\cdots p_m+kl+k+2} & ||M_{j_1}\cdots M_{j_l}|| \end{array}$$

Therefore

$$\begin{aligned}
\sum_{\nu\in\mathcal{B}}\xi([e\nu])\log||T_\nu|| &= \frac{p_e}{\lambda_k}\cdot\sum_{l=0}^{+\infty}(\sum_{|J|=l}\log||M_J||)(\lambda_k)^{lk+k+2}\\
&\quad+\frac{p_e}{\lambda_k}(\sum_{r\geq 1}t_{r,k}\cdot(\lambda_k)^r)\cdot\sum_{l=0}^{+\infty}(\sum_{|J|=l}\log||M_J||)(\lambda_k)^{lk+k+2}\\
&= \frac{p_e}{\lambda_k}\cdot(1+\sum_{r\geq 1}t_{r,k}\cdot(\lambda_k)^r)\cdot\sum_{l=0}^{+\infty}(\sum_{|J|=l}\log||M_J||)(\lambda_k)^{lk+k+2}\\
&= \frac{(\lambda_k)^k(1-2(\lambda_k)^k)}{2-(k+1)(\lambda_k)^k}\cdot(\lambda_k)^{k+1}\cdot\frac{1-2(\lambda_k)^{k-1}+(\lambda_k)^k}{1-2\lambda_k+(\lambda_k)^k}\\
&\quad\cdot\sum_{l=0}^{+\infty}(\sum_{|J|=l}\log||M_J||)(\lambda_k)^{lk}\\
&= \frac{(\lambda_k)^k(1-2(\lambda_k)^k)^2}{(2-(k+1)(\lambda_k)^k)}\sum_{n=0}^{\infty}((\lambda_k)^{kn}\sum_{|J|=n}\log||M_J||).
\end{aligned}$$

By a discussion similar to that in the proof of Theorem 33, we have for $\mathcal{L}$ almost all $t\in[0,1]$,

$$\begin{aligned}
\dim_H(L_{t,\lambda_k}) &= \dim_B(L_{t,\lambda_k})=\frac{1}{\log 2}\sum_{\nu\in\mathcal{B}}\xi([e\nu])\log||T_\nu||\\
&= \frac{(\lambda_k)^k(1-2(\lambda_k)^k)^2}{(2-(k+1)(\lambda_k)^k)\log 2}\sum_{n=0}^{\infty}((\lambda_k)^{kn}\sum_{|J|=n}\log||M_J||).
\end{aligned}$$

■

Theorem 48 *Let $k\geq 3$. For any $q\in\mathbb{R}$, the L^q-spectrum $\tau_{\lambda_k}(q)$ of μ_{λ_k} is equal to*

$$\frac{q\log 2}{\log\lambda_k^{-1}}+\frac{\log\mathbf{x}(k,q)}{\log\lambda_k^{-1}},$$

where $0<\mathbf{x}(k,q)<\lambda_{k-1}$, and it satisfies that $\frac{1-2x^{k-1}+x^k}{1-2x+x^k}\cdot\sum_{n\geq 0}u_{n,q}x^{kn+k+1}=1$. Moreover, $\tau_{\lambda_k}(q)$ is an infinitely differentiable function of q on the whole line.

Proof. The theorem follows from Corollary 5, Lemma 40, Lemma 45 and Lemma 44. ■

Define

$$S^{\mathbb{N}}=\{(x_i)_{i=1}^{\infty}\in\Xi^{\mathbb{N}}:\ Q_{x_i,x_{i+1}}=1 \text{ for all } i\geq 1,\ x_1=a,b \text{ or } c\},\tag{78}$$

there is a natural projection Π from the symbolic space $S^{\mathbb{N}}$ onto the interval $[0,1]$, which is defined by

$$(x_i)_{i=1}^{\infty}\mapsto\cap_{i=1}^{n}V_{x_1\cdots x_n},$$

here $V_{x_1\cdots x_n}$ is the n-th net interval associated with λ_k corresponding to $(x_i)_{i=1}^{n}$. Then by a discussion similar to the proofs of Theorem 35 and Theorem 36, we obtain the following two theorems.

Theorem 49 *Let $\Pi : S^{\mathbb{N}} \to [0,1]$ defined as above. Then for each $\omega = (x_i)_{i=1}^{\infty} \in S^{\mathbb{N}}$, the upper and lower local dimension of μ_{λ_k} at $\Pi(\omega)$ are given by*

$$\overline{d}(\mu_{\lambda_k}, \Pi(\omega)) = \frac{\log 2}{\log \lambda_k^{-1}} + \lim_{n\to\infty} \sup \frac{\log ||T_{\omega|n}||}{n \log \lambda_k},$$

$$\underline{d}(\mu_{\lambda_k}, \Pi(\omega)) = \frac{\log 2}{\log \lambda_k^{-1}} + \lim_{n\to\infty} \inf \frac{\log ||T_{\omega|n}||}{n \log \lambda_k}.$$

Theorem 50 *For $\mathcal{L}$ almost all $x \in [0,1]$, the upper and lower local dimension of μ_{λ_k} at x coincide, and the common value is equal to*

$$d(\mu_{\lambda_k}, x) = \frac{\log 2}{\log \lambda_k^{-1}} + \frac{1}{\log \lambda_k} \cdot \frac{(\lambda_k)^k (1 - 2(\lambda_k)^k)^2}{(2 - (k+1)(\lambda_k)^k)} \sum_{n=0}^{\infty} ((\lambda_k)^{kn} \sum_{|J|=n} \log ||M_J||).$$

Let consider the set

$$\mathcal{R}(\mu_{\lambda_k}) := \{y : \ \exists x \in [0,1], \ d(\mu_{\lambda_k}, x) = y\}$$

for $k \geq 3$. Hu gave a result that $\mathcal{R}(\mu_{\lambda_k}) = [\frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{k}\frac{\log \lambda}{\log \lambda_k}, \ \frac{\log 2}{\log \lambda_k^{-1}}]$; see Theorem 1.19 of [Hu]. However, this result is false. In the following we give the correct result.

Theorem 51 *For $k \geq 3$, $\mathcal{R}(\mu_{\lambda_k}) = [\frac{k}{k+1} \cdot \frac{\log 2}{\log \lambda_k^{-1}}, \ \frac{\log 2}{\log \lambda_k^{-1}}]$.*

Proof. The proof given here is similar to that of Theorem 37. Let $\mathcal{B}_k$ be the collection of letter strings defined as in (66) and $T_{i,j}$'s defined by (69). Since for any letter strings $\mathbf{i}_1, \mathbf{i}_2, \cdots, \mathbf{i}_n, \cdots \in \mathcal{B}_k$

$$\mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_n \circ \cdots \in S^{\mathbb{N}},$$

and

$$||T_{\mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_n}|| = ||T_{\mathbf{i}_1}|| \times ||T_{\mathbf{i}_2}|| \times \cdots \times ||T_{\mathbf{i}_n}||,$$

by Theorem 49 we have

$$\mathcal{R}(\mu_\lambda) \supset [\frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot x_1, \ \frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot y_1], \tag{79}$$

where

$$x_1 = \sup_{\mathbf{i} \in \mathcal{B}_k} \frac{\log ||T_{\mathbf{i}}||}{|\mathbf{i}|}, \ \ y_1 = \inf_{\mathbf{i} \in \mathcal{B}_k} \frac{\log ||T_{\mathbf{i}}||}{|\mathbf{i}|},$$

here $|\mathbf{i}|$ denotes the length of the letter string $\mathbf{i}$. In the end of our proof, we will show that $x_1 = \frac{1}{k+1} \log 2$ and $y_1 = 0$.

On the other hand, for any $\omega \in S^{\mathbb{N}}$ ($S^{\mathbb{N}}$ is defined by (78)), $\omega|n$ is the prefix of an infinite letter string of the following possible forms:

$\mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots,$	the first letter of $\mathbf{i}_1$ is b
$a^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots,$	the first letter of $\mathbf{i}_1$ is b or h_1
$c^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m \circ \cdots,$	the first letter of $\mathbf{i}_1$ is b or g_1

where $l \in \mathbb{N}$ and $\mathbf{i}_1, \mathbf{i}_2, \cdots \mathbf{i}_m, \cdots \in \mathcal{B}_k$; furthermore, by the generating relation (63), there exists a letter string $\mathbf{j}$ of length less than $k+2$ such that the concatenation $(\omega|n)\circ \mathbf{j}$ is of the following possible forms

$$\begin{array}{ll} \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m, & \text{the first letter of } \mathbf{i}_1 \text{ is } b \\ a^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m, & \text{the first letter of } \mathbf{i}_1 \text{ is } b \text{ or } h_1 \\ c^l \circ \mathbf{i}_1 \circ \mathbf{i}_2 \circ \cdots \circ \mathbf{i}_m, & \text{the first letter of } \mathbf{i}_1 \text{ is } b \text{ or } g_1 \end{array}$$

where $l \in \mathbb{N}$ and $\mathbf{i}_1, \mathbf{i}_2, \cdots \mathbf{i}_m \in \mathcal{B}_k$. Since $||T_{a^l}|| = ||T_{c^l}|| = 1$, therefore the above analysis implies that

$$\{y:\ \exists x \in [0,1],\ \overline{d}(\mu_\lambda, x) = y\} \subset [\frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot x_1,\ \ \frac{\log 2}{\log \lambda_k^{-1}}] \tag{80}$$

and

$$\{y:\ \exists x \in [0,1],\ \underline{d}(\mu_\lambda, x) = y\} \subset [\frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot x_1,\ \ \frac{\log 2}{\log \lambda_k^{-1}}]. \tag{81}$$

Therefore by (79)-(81) and $y_1 = 0$(what we will prove afterwards),

$$\mathcal{R}(\mu_{\lambda_k}) = [\frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot x_1,\ \ \frac{\log 2}{\log \lambda_k^{-1}} - \frac{1}{\log \lambda_k^{-1}} \cdot y_1].$$

In what follows, we determine the exact values of x_1 and y_1.

Note that for any integer $n > 0$,

$$b(k)(X_0(k))^n e \in \mathcal{B}_k, \quad T_{b(k)(X_0(k))^n e} = (M_0)^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

it follows $y_1 = 0$ since $(\log ||\, T_{b(k)(X_0(k))^n e}||)/(kn+k+1)$ tends to 0 as n tends to infinity.

To determine the value of x_1, let us consider the possible form of the element $\mathcal{B}_k$. Recall that each element of $\mathcal{B}_k$ is of the forms

$$\begin{array}{ll} (i). & b(k)e, \\ (ii). & b(k)X_{j_1}(k)\cdots X_{j_l}(k)e, \\ (iii). & Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)e, \\ (iv). & Y_{i_1}(p_1)\cdots Y_{i_m}(p_m)b(k)X_{j_1}(k)\cdots X_{j_l}(k)e, \end{array}$$

where $j_1, \cdots, j_l = 0$ or 1, $1 \le p_1, \cdots p_{m-1} \le k-1$, $1 \le p_m \le k-2$, and $i_s = \frac{1-(-1)^s}{2}$ $(1 \le s \le m)$ or $i_s = \frac{1+(-1)^s}{2}$ $(1 \le s \le m)$. Since $T_{Y_0(p)} = T_{Y_1(p)} = 1$ for each $1 \le p \le k-1$, it follows that

$$\begin{aligned} x_1 &= \sup_{l\ge 0,\ j_1,\cdots,j_l=0 \text{ or } 1} \frac{\log ||T_{b(k)X_{j_1}(k)\cdots X_{j_l}(k)e}||}{|b(k)X_{j_1}(k)\cdots X_{j_l}(k)e|} \\ &= \sup_{l\ge 0,\ j_1,\cdots,j_l=0 \text{ or } 1} \frac{\log ||M_{j_1\cdots j_l}||}{kl+k+1} \\ &= \sup_{l\ge 0} \frac{\log ||M_0 M_1 \cdots M_{l(\mathrm{mod}\, 2)}||}{kl+k+1} \end{aligned}$$

$$= \sup_{l\geq 0} \frac{\log(\frac{2+\lambda+(-1)^l\lambda^{2l+4}}{1+\lambda^2}\cdot\lambda^{-l})}{kl+k+1}$$

$$= \sup_{l\geq 0} \frac{\log(\dfrac{2+\lambda+(-1)^l\lambda^{2l+4}}{1+\lambda^2})+\log\lambda^{-l}}{(k+1)+kl}.$$

Note that

$$\frac{1}{k+1}\log(\frac{2+\lambda+(-1)^l\lambda^{2l+4}}{1+\lambda^2}) \leq \frac{1}{k+1}\log(\frac{2+\lambda+\lambda^4}{1+\lambda^2}) = \frac{1}{k+1}\log 2,$$

$$\frac{1}{kl}\log\lambda^{-l} = \frac{1}{k}\log\lambda^{-1},$$

we have

$$x_1 = \max\{\frac{1}{k+1}\log 2,\ \ \frac{1}{k}\log\lambda^{-1}\} = \frac{1}{k+1}\log 2.$$

■

Theorem 52 *The Hausdorff dimension and information dimension of μ_{λ_k} $(k \geq 3)$ satisfy*

$$\dim_H \mu_{\lambda_k} = \dim_{info}\mu_{\lambda_k} = -\frac{\log 2}{\log\lambda_k} + \left(\frac{2^k-3}{2^k-1}\right)^2 \cdot \frac{\sum_{n=0}^{\infty} 2^{-kn-k-1} \sum_{|J|=n} ||M_J||\log||M_J||}{\log\lambda_k}.$$

Proof. By Theorem 48, $\tau_{\lambda_k}(q)$ is a differentiable function of q. Thus by Lemma 38,

$$\dim_H \mu_{\lambda_k} = \dim_{\text{info}}\mu_{\lambda_k} = \tau'_{\lambda_k}(1),$$

the desired result follows from the direct calculation of $\tau'_{\lambda_k}(1)$ by the formula of $\tau_{\lambda_k}(q)$. We omit the detail of this calculation. ■

6 The case for other Pisot numbers.

In this section we will summarize some results for the reciprocals of general Pisot numbers.

Suppose ρ $(> 1/2)$ is the reciprocal of a Pisot number. By Lemma 2, $\#C_\rho < +\infty$, that is, the number of different possible I-colors associated with ρ is finite. Thus according to the generating relations of I-colors associated with ρ, we can find a finite alphabet set $\Omega = \{a, b, c, \cdots\}$ (each letter in Ω represents an element of C_ρ, sometimes several letters represent the same one element of C_ρ. We always use a to represent $(\{0\}, \frac{1-\rho}{\rho})$, b to $(\{\frac{\rho-1}{\rho}, 0\}, \frac{2\rho-1}{\rho})$ and c to $(\{\frac{1-2\rho}{\rho}\}, \frac{1-\rho}{\rho})$.), and obtain a 0-1 matrix $H = (H_{i,j})_{i,j\in\Omega}$ such that for any positive integer n there exists a one-to-one correspondence between the collection of all the n-th net intervals associated with ρ and the set

$$S^n := \{(x_i)_{i=1}^n \in \Omega^n :\ H_{x_i,x_{i+1}} = 1 \text{ for } 1 \leq i < n,\ \text{and } x_1 = a, b \text{ or } c\}.$$

Moreover, according to the generating relations of II-colors, we can define a family of matrixes $T_{i,j}$ for each pair $(i,j) \in \Omega\times\Omega$ with $H_{i,j}=1$, such that if the m-th net interval J corresponding to $\omega=(x_i)_{i=1}^m \in S^m$ has the II-color $(\{(t_1,n_1),\cdots,(t_r,n_r)\},\gamma)$, then

$$(n_1,\cdots,n_r)=\begin{cases} 1\cdot T_{x_1,x_2}\cdot T_{x_2,x_3}\cdots T_{x_{m-1},x_m} & \text{if } x_1=a \text{ or } c \\ (1,1)\cdot T_{x_1,x_2}\cdot T_{x_2,x_3}\cdots T_{x_{m-1},x_m} & \text{if } x_1=b \end{cases} \tag{82}$$

and

$$N_{m,\rho}(J):=\sum_{i=1}^r n_i=||T_\omega||, \tag{83}$$

where $T_\omega := T_{x_1,x_2}\cdots T_{x_{m-1},x_m}$.

Before we give the explicit μ_ρ measure of a given net interval, we define at first two families of vectors p_i, q_i $(i\in\Omega)$. Suppose $i\in\Omega$ represents the I-color $(\{(t_1,n_1),\cdots,(t_r,n_r)\},\gamma)$, then define

$$p_i=(\underbrace{1,\cdots,1}_{r\ 1\text{'s}})$$

and

$$q_i=\begin{pmatrix} \mu_\rho([-t_1,-t_1+\gamma]) \\ \vdots \\ \mu_\rho([-t_r,-t_r+\gamma]) \end{pmatrix}.$$

By (82) and Lemma 4(i), we obtain

Theorem 53 *If J is a m-th net interval corresponding to $\omega=(x_i)_{i=1}^m\in S^m$, then*

$$\mu_\rho(J)=2^{-m}p_{x_1}T_{x_1,x_2}\cdots T_{x_{m-1},x_m}q_{x_m}.$$

Moreover $\{q_i\}_{i\in\Omega}$ satisfies

$$\begin{cases} q_a+(1,1)q_b+q_c=2 \\ q_i=2^{-1}\sum_{j,\ H_{i,j}=1} T_{i,j}q_j,\ \textit{for } i\in\Omega \end{cases} \tag{84}$$

One may get the exact values of $\{q_i\}_{i\in\Omega}$ by using (84). For example in the case $\rho=\frac{\sqrt{5}-1}{2}$, $\Omega=\{a,b,c,d,e,f,\overline{f}\}$, $T_{i,j}$'s are given by (25), and

$$\begin{cases} q_a=q_c=\dfrac{2}{3}, \\ q_b=q_f=q_{\overline{f}}=\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}, \\ q_d=\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \\ q_e=\dfrac{1}{3}. \end{cases}$$

Now define

$$S^{\mathbb{N}}=\{(x_i)_{i=1}^\infty\in\Omega^{\mathbb{N}}:\ H_{x_i,x_{i+1}}=1 \text{ for } 1\le i<\infty, \text{ and } x_1=a,b \text{ or } c\}.$$

And consider the mapping $\Pi : S^{\mathbb{N}} \to [0,1]$ which is defined by

$$\omega = (x_i)_{i=1}^{\infty} \mapsto \bigcap_{m\geq 1} V_{\omega|m},$$

where $\omega|n$ denotes $(x_i)_{i=1}^{m}$. Clearly Π is surjective, and it is one to one except at countable many points.

Theorem 54 *For each $\omega = (x_i)_{i=1}^{\infty} \in S^{\mathbb{N}}$, the upper and lower dimension of μ_ρ at $\Pi(\omega)$ are given by*

$$\overline{d}(\mu_\rho, \Pi(\omega)) = \frac{\log 2}{\log \rho^{-1}} + \lim_{n\to\infty} \sup \frac{\log ||T_{\omega|n}||}{\log \rho}$$

$$\underline{d}(\mu_\rho, \Pi(\omega)) = \frac{\log 2}{\log \rho^{-1}} + \lim_{n\to\infty} \inf \frac{\log ||T_{\omega|n}||}{\log \rho}$$

Proof. It follows from Theorem 53, Corollary 3, Lemma 4(ii) and Lemma 6. ■

Lemma 55 *For any $q \geq 0$, the following limit*

$$\lim_{m\to\infty} \frac{1}{m} \log \sum_{\omega\in S^m} ||T_\omega||^q \tag{85}$$

exists.

Proof. Let L be the minimal positive integer such that for any $i \in \Omega$, there is a string $W \in \cup_{m=1}^{L} S^m$ such that the end letter of W is i. Define

$$U^m = \{(x_i)_{i=1}^n \in \Omega^n : \ H_{x_i,x_{i+1}} = 1 \text{ for } 1 \leq i < n\}, \ m = 1, 2, \cdots.$$

Then for any positive integer m and real number $q \geq 0$,

$$\begin{aligned}
\sum_{\omega\in S^m} ||T_\omega||^q &\leq \sum_{\omega\in U^m} ||T_\omega||^q \leq \sum_{j=0}^{L} \sum_{\omega\in S^{m+j}} ||T_\omega||^q \\
&\leq \sum_{j=0}^{L} (\sum_{\omega\in S^m} ||T_\omega||^q \sum_{\omega'\in U^j} ||T_{\omega'}||^q) \\
&= \sum_{\omega\in S^m} ||T_\omega||^q . (\sum_{j=0}^{L} \sum_{\omega'\in U^j} ||T_{\omega'}||^q).
\end{aligned} \tag{86}$$

Since for any $q \geq 0$, $\sum_{\omega\in U^m} ||T_\omega||^q$ is a submultiplicative sequence of m, that is

$$\sum_{\omega\in U^{m+n}} ||T_\omega||^q \leq \sum_{\omega\in U^m} ||T_\omega||^q \times \sum_{\omega\in U^n} ||T_\omega||^q$$

for any m, n. Therefore the limit $\lim_{m\to\infty} \frac{1}{m} \log \sum_{\omega\in U^m} ||T_\omega||^q$ exists. This and (86) yield the desired result. ■

By Lemma 9 and (83) we obtain

Theorem 56 *The Hausdorff dimension of f_ρ is equal to*

$$\lim_{m\to\infty} \frac{1}{m\log\rho^{-1}} \log \sum_{\omega\in S^m} ||T_\omega||^{-\log\rho\ /\ \log 2}. \tag{87}$$

By Corollary 5 and (83), we have

Theorem 57 *For any real number q, the L^q-spectrum $\tau_{\mu_\rho}(q)$ is equal to*

$$\frac{q\log 2}{\log\rho^{-1}} + \lim_{m\to\infty}\inf \frac{1}{m\log\rho^{-1}} \log \sum_{\omega\in S^m} ||T_\omega||^q. \tag{88}$$

As we have shown, the formulas (87) and (88) can be simplified furthermore for the cases $\rho = \lambda_k$ $(k \geq 2)$, the essential reason for which is that in these cases,

$$\begin{aligned}&\text{any I-color associated with } \rho \text{ can generate out ultimately}\\ &\text{an I-color of the form } (\{t_1\}, \gamma)\end{aligned} \tag{89}$$

and therefore $||T_\omega||$ can be decomposed into some product of $||T_{\omega_i}||$'s.

We call ρ^{-1} the Pisot number **of the first class** if the property (89) holds for ρ, or we call it the Pisot number of **the second class**.

As we have mentioned in Section 1, there are only four algebraic integers of degree 3 to be Pisot numbers which are less than 2. They are respectively the positive roots of (i) $x^3 - x^2 - x - 1 = 0$, (ii) $x^3 - x^2 - 1 = 0$, (iii)$x^3 - x - 1 = 0$, and (iv) $x^3 - 2x^2 + x - 1 = 0$; In Section 5, we have discussed the case when ρ is the reciprocal of the positive root of (i). However what about the other three cases? By direct computation, (89) holds only if ρ is the reciprocal of the positive root of (iv). In section 8, as an appendix, we will give the generating relations of I-colors associated with the reciprocal of positive root of (iv). One can simplify the formulas (87) and (88) for this case in a manner similar to that for the case $\rho = \lambda_k$ $(k \geq 2)$.

We end this section by two questions:

(**i**) Is the limit (85) always differentiable on $q > 0$ for any reciprocal of Pisot number?

(**ii**) Beside the generating relations of I-colors, is it possible to find a simpler method to determine whether a given Pisot number is of the first class?

7 Final remarks

In this section, we would like to point out that with some additive work our method is valid to analyze the local properties of some other self-similar measures.

(i) **Biased Bernoulli convolutions associated with Pisot numbers:** let ρ $(> 1/2)$ be the reciprocal of a Pisot number, for fixed $0 < p < 1$, the Biased Bernoulli convolution $\mu_\rho^{(p)}$ is defined as the self-similar measure satisfying that

$$\mu_\rho^{(p)} = p\mu_\rho^{(p)} \circ \phi_{0,\rho}^{-1} + (1-p)\mu_\rho^{(p)} \circ \phi_{1,\rho}^{-1},$$

where $\phi_{0,\rho}(x) = \rho x$ and $\phi_{1,\rho}(x) = \rho x + (1-\rho)$.

To consider $\mu_\rho^{(p)}$, we define the net intervals and their I-colors the same as those in Section 2. In place of II-colors, we will introduce the notion of $\mathrm{II}^{(p)}$-colors. Let $\Lambda = \{0,1\}^{\mathbb{N}}$ be the sequence space with the Bernoulli measure $v_p = (p, 1-p)^{\mathbb{N}}$. It is not hard to see that

$$\mu_\rho^{(p)} = v_p \circ \Pi_\rho^{-1} \quad \text{where } \Pi_\rho(x) = (1-\rho)\sum_{i=0}^{\infty} x_n \rho^n.$$

Now suppose $J = [a,b]$ is a m-th net interval associated with ρ, the $\mathrm{II}^{(p)}$-color of J is defined as an element of $2^{\mathbb{R}\times\mathbb{R}} \times \mathbb{R}$:

$$(\{(\frac{\phi_{\omega,\rho}(0)-a}{\rho^m}, u(\omega)) : \omega \in \{0,1\}^m \text{ such that } \quad -\rho^m < \phi_{\omega,\rho}(0) - a \le 0\}, \frac{b-a}{\rho^m}),$$

where $u(\omega) = v_p\{x = (x_i)_{i=1}^{\infty} \in \Lambda : \ \Pi_\rho(x) \in [a,b],\ \phi_{x_1\cdots x_m,\rho}(0) = \phi_{\omega,\rho}(0)\}$.

By considering the generated relations of $\mathrm{II}^{(p)}$-colors, we can show similarly that the $\mu_\rho^{(p)}$ measure on a m-th net interval is still given by the product of m matrixes. For simplicity, here we only give some results for $\rho = \frac{\sqrt{5}-1}{2}$.

Proposition 58 *Let $\rho = \frac{\sqrt{5}-1}{2}$. Suppose that J is a m-th net interval ($m \ge 2$) corresponding to $\omega = (x_i)_{i=1}^m \in S^m$ (S^m defined as in (21)), and the $II^{(p)}$-color of J is $(\{(t_1,u_1),\cdots,(t_r,u_r)\},\gamma)$, then*

$$(u_1,\cdots,u_r) = \begin{cases} \dfrac{p^2}{1-p+p^2} \cdot T^{(p)}_{x_1,x_2} \cdot T^{(p)}_{x_2,x_3} \cdots T^{(p)}_{x_{m-1},x_m} & \text{if } x_1 = a \\ (\dfrac{p(1-p)^2}{1-p+p^2}, \dfrac{p^2(1-p)}{1-p+p^2}) \cdot T^{(p)}_{x_1,x_2} \cdot T^{(p)}_{x_2,x_3} \cdots T^{(p)}_{x_{m-1},x_m} & \text{if } x_1 = b \\ \dfrac{(1-p)^2}{1-p+p^2} \cdot T^{(p)}_{x_1,x_2} \cdot T^{(p)}_{x_2,x_3} \cdots T^{(p)}_{x_{m-1},x_m} & \text{if } x_1 = c \end{cases}$$

and $\mu_\rho^{(p)}(J) = u_1 + \cdots + u_r$, where the matrixes $T^{(p)}_{i,j}$ are given by

$$\begin{cases} T^{(p)}_{a,a} = p, & T^{(p)}_{a,b} = ((1-p)^2, (1-p)p), \\ T^{(p)}_{b,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\ T^{(p)}_{c,b} = ((1-p)p, p^2), & T^{(p)}_{c,c} = 1-p \\ T^{(p)}_{d,f} = \begin{pmatrix} p(1-p) & p^2 \\ 0 & p^2 \end{pmatrix}, & T^{(p)}_{d,e} = \begin{pmatrix} (1-p)p \\ (1-p)p \end{pmatrix}, \qquad T^{(p)}_{d,\overline{f}} = \begin{pmatrix} (1-p)^2 & 0 \\ (1-p)^2 & (1-p)p \end{pmatrix}, \\ T^{(p)}_{e,b} = (1-p, p) & \\ T^{(p)}_{f,d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \\ T^{(p)}_{\overline{f},d} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \end{cases}$$

Using the above proposition and similar proofs as in Section 4, we can give the explicit formulas for the Hausdorff dimension, information dimension and L^q-spectrum of $\mu^{(p)}_{(\sqrt{5}-1)/2}$. For example, we have

$$\dim_H \mu^{(p)}_{(\sqrt{5}-1)/2} = \dim_{\text{info}} \mu^{(p)}_{(\sqrt{5}-1)/2} = \frac{p-p^2}{2+p-p^2} \cdot \frac{\sum_{n\geq 0}\sum_{|J|=n} c_J \log c_J}{\log((\sqrt{5}-1)/2)}$$

where

$$c_J = (1-p,p)M^{(p)}_{j_1}\cdots M^{(p)}_{j_n}\begin{pmatrix} p-p^2 \\ p-p^2 \end{pmatrix}$$

and $M^{(p)}_0 = T^{(p)}_{d,f}$, $M^{(p)}_1 = T^{(p)}_{d,\overline{f}}$.

More generally our method can be used to consider the following measures:

(ii) **Self-similar measures generated by a family of similitudes** with *weak separate condition. Let* $\{S_i\}_{i=1}^m$ be similitudes with the same contraction ratio ρ, $\{S_i\}_{i=1}^m$ is said to satisfy *weak separate condition* if there exists a positive constant c such that

$$|S_{i_1}\circ S_{i_2}\circ\cdots\circ S_{i_n}(0) - S_{j_1}\circ S_{j_2}\circ\cdots\circ S_{j_n}(0)| = 0 \quad \text{or} \quad \geq c\rho^n$$

for integers $n \geq 1$, $1 \leq i_{1,}\cdots, i_n \leq m$, and $1 \leq j_{1,}\cdots, j_n \leq m$.(the weak separate condition was first introduced in [LN1], the present setting is the same as in [FLN]). Let ν be the self-similar measure generated by $\{S_i\}_{i=1}^m$ with the probability weights $(p_{1,}\cdots, p_m)$. If $\{S_i\}_{i=1}^m$ satisfies the weak separate condition, then one can still show that the ν measure on any net interval is given by the matrix product(letting I denote the minimal bounded interval so that $S_i(I) \subset (I)$ for $1 \leq i \leq m$, then m-th net intervals are obtained by the partition of I by all the endpoints of $S_J(I)$, $|J| = m$.)

8 Appendix: the generating relations of I-colors associated with the positive root of $x^3 - x^2 + 2x - 1 = 0$.

In this case the number of different I-colors is 29, which can be written as:

$V_1 := (\{0\}, \rho^2 - \rho + 1),$
$V_2 := (\{-\rho^2 + \rho - 1, 0\}, -\rho^2 + \rho),$
$V_3 := (\{\rho^2 - \rho\}, \rho^2 - \rho + 1),$
$V_4 := (\{\rho^2 - \rho\}, \rho^2),$
$V_5 := (\{-\rho, 0\}, -\rho + 1),$
$V_6 := (\{\rho - 1\}, \rho^2),$
$V_7 := (\{\rho^2 - \rho, 0\}, \rho^2 - \rho + 1),$
$V_8 := (\{\rho - 1, 0\}, \rho^2),$
$V_9 := (\{-\rho^2 + \rho - 1, -\rho^2, 0\}, -\rho^2 + \rho),$
$V_{10} := (\{-\rho, \rho^2 - \rho\}, \rho^2 - 2\rho + 1),$
$V_{11} := (\{-\rho^2 + \rho - 1, \rho - 1, 0\}, -\rho^2 + \rho),$
$V_{12} := (\{\rho^2 - 1, \rho^2 - \rho\}, \rho^2),$
$V_{13} := (\{-\rho, 0\}, \rho^2 - 2\rho + 1),$
$V_{14} := (\{-\rho^2 + \rho - 1, -\rho^2 + 2\rho - 1, 0\}, -\rho^2 + \rho),$
$V_{15} := (\{\rho - 1, \rho^2 - \rho\}, \rho^2),$
$V_{16} := (\{-\rho^2 + \rho - 1, -\rho, 0\}, -\rho^2 + \rho + 0),$
$V_{17} := (\{\rho^2 - 2\rho, \rho^2 - \rho\}, \rho^2 - 2\rho + 1),$
$V_{18} := (\{\rho^2 - \rho, 0\}, \rho^2),$
$V_{19} := (\{-\rho, -\rho^2, 0\}, -\rho + 1),$
$V_{20} := (\{\rho^2 - 1, \rho^2 - \rho\}, 2\rho^2 - \rho),$
$V_{21} := (\{-\rho, -\rho^2 - \rho, 0\}, -\rho + 1),$
$V_{22} := (\{\rho^2 - 1, \rho - 1\}, \rho^2),$
$V_{23} := (\{-\rho, \rho^2 - \rho, 0\}, \rho^2 - 2\rho + 1),$
$V_{24} := (\{-\rho^2 + \rho - 1, \rho - 1, -\rho^2 + 2\rho - 1, 0\}, -\rho^2 + \rho),$
$V_{25} := (\{\rho^2 - 1, \rho - 1, \rho^2 - \rho\}, \rho^2),$
$V_{26} := (\{\rho - 1\}, 2\rho - 1),$
$V_{27} := (\{\rho - 1, \rho^2 - \rho, 0\}, \rho^2),$
$V_{28} := (\{-\rho^2 + \rho - 1, -\rho, -\rho^2, 0\}, -\rho^2 + \rho),$
$V_{29} := (\{\rho^2 - 2\rho, -\rho, \rho^2 - \rho\}, \rho^2 - 2\rho + 1),$

And the generating relations of I-colors can be written as:

$$
\begin{array}{ll}
V_1 \to V_1 + V_2 + V_4; & V_2 \to V_5; \\
V_3 \to V_6 + V_2 + V_3; & V_4 \to V_6 + V_2; \\
V_5 \to V_7; & V_6 \to V_2 + V_4; \\
V_7 \to V_8 + V_9 + V_{10} + V_{11} + V_{12}; & V_8 \to V_2 + V_4; \\
V_9 \to V_{13} + V_{14}; & V_{10} \to V_{15}; \\
V_{11} \to V_{16} + V_{17}; & V_{12} \to V_6 + V_2; \\
V_{13} \to V_{18}; & V_{14} \to V_{19}; \\
V_{15} \to V_{11} + V_{20} + V_9; & V_{16} \to V_{21}; \\
V_{17} \to V_{22}; & V_{18} \to V_8 + V_9; \\
V_{19} \to V_{23} + V_{24} + V_{25}; & V_{20} \to V_{26}; \\
V_{21} \to V_{27} + V_{28} + V_{29}; & V_{22} \to V_{11} + V_{12}; \\
V_{23} \to V_{27}; & V_{24} \to V_{28} + V_{29}; \\
V_{25} \to V_{11} + V_{20} + V_9; & V_{26} \to V_2; \\
V_{27} \to V_{11} + V_{20} + V_9; & V_{28} \to V_{23} + V_{24}; \\
V_{29} \to V_{25}; &
\end{array}
$$

The reader may check that for any $1 \leq i \leq 29$, the I-color V_i can generate out ultimately V_{26}.

Acknowledgment: The author would like to express his deep gratitude to Prof. Z.Y. Wen and Dr. J. Wu for many valuable discussions and suggestions.

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