EQUILIBRIUM STATES FOR FACTOR MAPS BETWEEN SUBSHIFTS

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Abstract. Let $\pi : X \to Y$ be a factor map, where (X, σ_X) and (Y, σ_Y) are subshifts over finite alphabets. Assume that X satisfies weak specification. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ with $a_1 > 0$ and $a_2 \geq 0$. Let f be a continuous function on X with sufficient regularity (Hölder continuity, for instance). We show that there is a unique shift invariant measure μ on X that maximizes $\int f \, d\mu + a_1 h_\mu(\sigma_X) + a_2 h_{\mu\circ\pi^{-1}}(\sigma_Y)$. In particular, taking $f \equiv 0$ we see that there is a unique invariant measure μ on X that maximizes the weighted entropy $a_1 h_\mu(\sigma_X) + a_2 h_{\mu\circ\pi^{-1}}(\sigma_Y)$, which answers an open question raised by Gatzouras and Peres in [16]. An extension is given to high dimensional cases. As an application, we show that for each compact invariant set K on the k-torus under a diagonal endomorphism, if the symbolic coding of K satisfies weak specification, then there is a unique invariant measure μ supported on K so that $\dim_H \mu = \dim_H K$.

1. Introduction

In this paper, we study the thermodynamic formalism on subshifts and give an application in non-conformal dynamical systems.

Let $k \geq 2$ be an integer. Assume that (X_i, σ_{X_i}), $i = 1, \ldots, k$, are one-sided (or two-sided) subshifts over finite alphabets, and X_{i+1} is a factor of X_i with a factor map $\pi_i : X_i \to X_{i+1}$ for $i = 1, \ldots, k-1$. (See Sect. 2 for the definitions). For convenience, we use π_0 to denote the identity map on X_1. Define $\tau_i : X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_0$ for $i = 0, 1, \ldots, k-1$. Let $\mathcal{M}(X_i, \sigma_{X_i})$ denote the set of all σ_{X_i}-invariant Borel probability measures on X_i, endowed with the weak-star topology. Fix $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$. For $\mu \in \mathcal{M}(X_1, \sigma_{X_1})$, we call

$$h_\mu^{\mathbf{a}}(\sigma_{X_1}) := \sum_{i=1}^{k} a_i h_{\mu\circ\tau_{i-1}^{-1}}(\sigma_{X_i})$$

the $\mathbf{a}$-*weighted measure-theoretic entropy of* μ *with respect to* σ_{X_1}, or simply, the $\mathbf{a}$-*weighted entropy of* μ, where $h_{\mu\circ\tau_{i-1}^{-1}}(\sigma_{X_i})$ denotes the measure-theoretic entropy of $\mu \circ \tau_{i-1}^{-1}$ with respect to σ_{X_i}. For a real continuous function f on X_1, we say that

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$\mu \in \mathcal{M}(X_1, \sigma_{X_1})$ is an $\mathbf{a}$-*weighted equilibrium state of f for the factor maps π_i's*, or simply, $\mathbf{a}$-*weighted equilibrium state of f* if

$$\int f \, d\mu + h^{\mathbf{a}}_{\mu}(\sigma_{X_1}) = \sup \left\{ \int f \, d\eta + h^{\mathbf{a}}_{\eta}(\sigma_{X_1}) : \ \eta \in \mathcal{M}(X_1, \sigma_{X_1}) \right\}. \tag{1.1}$$

The value in the right hand side of (1.1) is called the $\mathbf{a}$-*weighted topological pressure of f* and is denoted by $P^{\mathbf{a}}(\sigma_{X_1}, f)$. The existence of at least one $\mathbf{a}$-weighted equilibrium state of f follows from the upper semi-continuity of the entropy functions $h_{(\cdot)}(\sigma_{X_i})$.

The notions of weighted topological pressure and weighted equilibrium state were recently introduced by Barral and the author in [1], mainly motivated from the study of the multifractal analysis on self-affine sponges. When $\mathbf{a} = (1, 0, \ldots, 0)$, the $\mathbf{a}$-weighted topological pressure and $\mathbf{a}$-weighted equilibrium states are reduced back to the classical topological pressure and equilibrium states (cf. [32, 35, 29]).

The main objective of this paper is to study the dynamical property of general weighted equilibrium states. We want to give conditions on f and X_i's to guarantee a unique $\mathbf{a}$-weighted equilibrium state. This study is mainly motivated from the following question raised by Gatzouras and Peres in [16, Problem 3]:

Question 1.1. *Let $\pi : X \to Y$ be a factor map between subshifts X and Y, where X is an irreducible subshift of finite type. Let $\alpha > 0$. Is there a unique invariant measure μ on X maximizing the weighted entropy $h_\mu(\sigma_X) + \alpha h_{\mu \circ \pi^{-1}}(\sigma_Y)$?*

Question 1.1 is closely related to dimension theory of certain non-conformal dynamical systems [16] (we will address it a little later). It still remains open except some partial results (see the remarks after Theorem 1.3). One of the difficulties is due to the complex structure of the fibres

$$\pi^{-1}(\nu) := \{\eta \in \mathcal{M}(X, \sigma_X) : \ \eta \circ \pi^{-1} = \nu\}$$

for invariant measures ν on Y. For instance, there may exist different elements $\mu \in \pi^{-1}(\nu)$ such that $h_\mu(\sigma_X) = \sup\{h_\eta(\sigma_X) : \ \eta \in \pi^{-1}(\nu)\}$ [30]. The reader is referred to [7] for some related open questions about the structure of $\pi^{-1}(\nu)$.

Let us return back to our general issue. We say that the subshift X_1 satisfies *weak specification* if there exists $p \in \mathbb{N}$ such that, for any two words I and J that are legal in X_1 (i.e., may be extended to sequences in X_1), there is a word K of length not exceeding p such that the word IKJ is legal in X_1. Similarly, say that X_1 satisfies *specification* if there exists $p \in \mathbb{N}$ such that, for any two words I and J that are legal in X_1, there is a word K of length p such that the word IKJ is legal in X_1. For more details about the definitions, see Sect. 2.

Let $C(X_1)$ denote the collection of real continuous functions on X_1. For $f \in C(X_1)$ and $n \geq 1$ let

$$S_n f(x) = \sum_{i=0}^{n-1} f(\sigma_{X_1}^i x), \quad x \in X_1. \tag{1.2}$$

Let $V(\sigma_{X_1})$ denote the set of $f \in C(X_1)$ such that there exists $c > 0$ such that

$$|S_n f(x) - S_n f(y)| \leq c \quad \text{whenever } x_i = y_i \text{ for all } 0 < i \leq n, \tag{1.3}$$

where $x = (x_i)_{i=1}^\infty$ and $y = (y_i)_{i=1}^\infty$. Endow X_1 with the usual metric (see Sect. 2). Clearly $V(\sigma_{X_1})$ contains all Hölder continuous functions on X_1. The main result of the paper is the following.

Theorem 1.2. *Assume that X_1 satisfies weak specification. Then for any $f \in V(\sigma_{X_1})$ and $\mathbf{a} = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$, f has a unique $\mathbf{a}$-weighted equilibrium state μ. The measure μ is ergodic and, there exist $p \in \mathbb{N}$ and $c > 0$ such that*

$$\liminf_{n \to \infty} \sum_{i=0}^{p} \mu\left(A \cap \sigma_{X_1}^{-n-i}(B)\right) \geq c\mu(A)\mu(B), \quad \forall \text{ Borel sets } A, B \subseteq X_1.$$

Furthermore, if X_1 satisfies specification, then there exists $c > 0$ such that

$$\liminf_{n \to \infty} \mu\left(A \cap \sigma_{X_1}^{-n}(B)\right) \geq c\mu(A)\mu(B), \quad \forall \text{ Borel sets } A, B \subseteq X_1. \tag{1.4}$$

The measure μ in Theorem 1.2 can be constructed as the limit of a sequence of discrete measures in the weak-star topology (see Remark 7.4). Taking $f = 0$ in Theorem 1.2 yields, whenever X_1 satisfies weak specification, there is a unique invariant measure μ on X_1 maximizing the $\mathbf{a}$-weighted entropy. This yields a confirmative answer to Question 1.1, because each irreducible subshift of finite type satisfies weak specification (cf. Sect. 2).

We remark that Theorem 1.2 is a natural extension of Bowen's result [5] about the classical equilibrium states. Restricted to the subshift case, Bowen [5] proved whenever X_1 satisfies specification and $f \in V(\sigma_{X_1})$, there is a unique invariant measure μ on X_1 such that

$$\int f \, d\mu + h_\mu(\sigma_{X_1}) = \sup\left\{\int f \, d\eta + h_\eta(\sigma_{X_1}) : \ \eta \in \mathcal{M}(X_1, \sigma_{X_1})\right\},$$

and furthermore, μ satisfies (1.4). This result corresponds to the special case $\mathbf{a} = (1, 0, \ldots, 0)$ in Theorem 1.2.

In the literature, an invariant measure μ satisfying (1.4) is called *partially mixing* with respect to σ_{X_1}. Recall that an invariant measure μ is called *weakly mixing* if

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\left|\mu\left(A\cap\sigma_{X_1}^{-i}(B)\right)-\mu(A)\mu(B)\right|=0,\quad \forall \text{ Borel sets } A,B\subseteq X_1,$$

and μ is called *mixing* if

$$\lim_{n\to\infty}\mu\left(A\cap\sigma_{X_1}^{-n}(B)\right)=\mu(A)\mu(B),\quad \forall \text{ Borel sets } A,B\subseteq X_1.$$

It is known that mixing implies partial mixing, and partial mixing implies weak mixing; these three properties are essentially different (cf. [15] and references therein).

Theorem 1.2 has an interesting application in characterizing invariant measures of maximal Hausdorff dimension for certain non-conformal dynamical systems. Let T be the endmorphism on the k-dimensional torus $\mathbb{T}^k=\mathbb{R}^k/\mathbb{Z}^k$ represented by an integral diagonal matrix

$$\Lambda=\mathrm{diag}(m_1,m_2,\ldots,m_k),$$

where $2\leq m_1\leq\ldots\leq m_k$. That is, $Tu=\Lambda u \pmod 1$ for $u\in\mathbb{T}^k$. Let $\mathcal{A}$ denote the Cartesian product

$$\prod_{i=1}^{k}\{0,1,\ldots,m_i-1\}$$

let $R:\ \mathcal{A}^{\mathbb{N}}\to\mathbb{T}^k$ be the canonical coding map given by

$$R(x)=\sum_{n=1}^{\infty}\Lambda^{-n}x_i,\quad x=(x_i)_{i=1}^{\infty}\in\mathcal{A}^{\mathbb{N}},$$

where each element in $\mathcal{A}$ is viewed as a k-dimensional column vector. For any non-empty $\mathcal{D}\subseteq\mathcal{A}$, the set $R(\mathcal{D}^{\mathbb{N}})$ is called a *self-affine Sierpinski sponge*. Whenever $k=2$, McMullen [26] and Bedford [4] determined the explicit value of the Hausdorff dimension of $R(\mathcal{D}^{\mathbb{N}})$, and showed that there exists a Bernoulli product measure μ on $\mathcal{D}^{\mathbb{N}}$ such that $\dim_H\mu\circ R^{-1}=\dim_H R(\mathcal{D}^{\mathbb{N}})$. Kenyon and Peres [19] extended this result to the general case $k\geq 2$, and moreover, they proved for each compact T-invariant set $K\subseteq\mathbb{T}^k$, there is an ergodic σ-invariant μ on $\mathcal{A}^{\mathbb{N}}$ so that $\mu(R^{-1}(K))=1$ and $\dim_H\mu\circ R^{-1}=\dim_H K$. Furthermore, Kenyon and Peres [19] proved the uniqueness of $\mu\in\mathcal{M}(\mathcal{D}^{\mathbb{N}},\sigma)$ satisfying $\dim_H\mu\circ R^{-1}=\dim_H R(\mathcal{D}^{\mathbb{N}})$, by setting up the following formula for any ergodic $\eta\in\mathcal{M}(\mathcal{A}^{\mathbb{N}},\sigma)$:

$$\dim_H\eta\circ R^{-1}=\frac{1}{\log m_k}h_\eta(\sigma)+\sum_{i=1}^{k-1}\left(\frac{1}{\log m_{k-i}}-\frac{1}{\log m_{k-i+1}}\right)h_{\eta\circ\tau_i^{-1}}(\sigma_i),\tag{1.5}$$

where τ_i denotes the one-block map from $\mathcal{A}^{\mathbb{N}}$ to $\mathcal{A}_i^{\mathbb{N}}$, with $\mathcal{A}_i=\prod_{j=1}^{k-i}\{0,1,\ldots,m_j-1\}$, so that each element in $\mathcal{A}$ (viewed as a k-dimensional vector) is projected into its first $(k-i)$ coordinates; and σ_i denotes the left shift on $\mathcal{A}_i^{\mathbb{N}}$. Formula (1.5) is an analogue of that for the Hausdorff dimension of $C^{1+\alpha}$ hyperbolic measures along an unstable (respectively, a stable) manifold established by Ledrappier and Young [24]. As Gatzouras and Peres pointed out in [16], the uniqueness has not been known for more general invariant subsets K, even if $K = R(X)$, where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a general irreducible subshift of finite type. However, as a direct application of (1.5) and Theorem 1.2, we have the following answer.

Theorem 1.3. *Let $K = R(X)$, where $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a subshift satisfying weak specification. Then there is a unique T-invariant measure μ on K such that $\dim_H \mu = \dim_H K$.*

Now we give some historic remarks about the study of Question 1.1. Assume that π is a factor map between subshifts X and Y, where X is an irreducible subshift of finite type. Recall that a *compensation function* for π is a continuous function $F: X \to \mathbb{R}$ such that

$$\sup_{\nu\in\mathcal{M}(Y,\sigma_Y)} \left(\int \phi \, d\nu + h_\nu(\sigma_Y)\right) = \sup_{\mu\in\mathcal{M}(X,\sigma_X)} \left(\int (\phi\circ\pi + F)\, d\mu + h_\mu(\sigma_X)\right)$$

for all $\phi \in C(Y)$. Compensation functions were introduced in [8] and studied systematically in [36]. Shin [33] showed that if there exists a compensation function of the form $f\circ\pi$, with $f \in C(Y)$, and if $\frac{\alpha}{1+\alpha} f\circ\pi$ has a unique equilibrium state, then there is a unique measure μ maximizing the weighted entropy $h_\mu(\sigma_X) + \alpha h_{\mu\circ\pi^{-1}}(\sigma_Y)$. However, there exist factor maps between irreducible subshifts of finite type for which there are no such compensation functions [34]. Later, Petersen, Quas and Shin [30] proved that for each ergodic measure ν on Y, the number of ergodic measures μ of maximal entropy in the fibre $\pi^{-1}(\nu)$ is uniformly bounded; in particular, if π is a one-block map and there is a symbol b in the alphabet of Y such that the pre-image of b is a singleton (in this case, $\pi : X \to Y$ is said to *have a singleton clump*), then there is a unique measures μ of maximal entropy in the fibre $\pi^{-1}(\nu)$ for each ergodic measure ν on Y. Recently, Yayama [37, 38] showed the uniqueness of invariant measures of maximal weighted entropy if $\pi : X \to Y$ has a singleton clump. The uniqueness is further proved by Olivier [27] and Yayama [38] under an assumption that the projection of the "Parry measure" on X has certain Gibbs property (however the assumption only fulfils in some special cases).

We remark that a special case of Theorem 1.2 was studied in [1]. It was proved in [1] that, whenever $\pi_i : X_i \to X_{i+1}$ $(i = 1, \ldots, k-1)$ are one-block factor maps between one-sided full shifts (X_i, σ_{X_i}), each $f \in V(\sigma_{X_1})$ has a unique **a**-weighted equilibrium state, which is Gibbs and mixing. This result has an interesting application in the multifractal analysis [1]. See [2, 3, 20, 28] for related results. The approach given in [1] is based on the (relativized) thermodynamic formalism of almost additive potentials, which depends strongly upon the simple fibre structure in this special setting.

For the proof of Theorem 1.2 in the general case, due to the complexity of fibre structures, it seems rather intractable to use classical thermodynamic formalism as in [5, 10] or take an approach as in [1]. In this paper, we manage to prove Theorem 1.2 by showing the uniqueness of equilibrium states and conditional equilibrium states for certain sub-additive potentials. A crucial step in our approach is to prove, for certain functions f defined on $\mathcal{A}^*$ (the set of finite words over $\mathcal{A}$), there exists an ergodic invariant measures μ on the full shift space $\mathcal{A}^{\mathbb{N}}$ and $c > 0$, so that $\mu(I) \geq cf(I)$ for $I \in \mathcal{A}^*$ (see Proposition 4.3).

After we completed the first version of this paper, François Ledrappier informed us the following result which improves Theorem 1.2.

Theorem 1.4. *When X_1 satisfies specification, the unique* **a***-weighted equilibrium state μ in Theorem 1.2 is in fact a K-system; in particular, it is mixing.*

This result can be proved by using a similar argument as in [22, Proposition 4]. To see it, consider on $X_1 \times X_1$ the function $(x, x') \mapsto F(x, x') = f(x) + f(x')$. By Theorem 1.2, F has a unique **a**-weighted equilibrium state. Let $\mathcal{P}_\mu$ be the Pinsker σ-algebra of (X_1, μ) (see [35]) and let η be relatively independent product of μ by μ over $\mathcal{P}_\mu$, i.e.,

$$\eta(A \times B) = \int E(\chi_A|\mathcal{P}_\mu)E(\chi_B|\mathcal{P}_\mu)\, d\mu, \qquad \forall \text{ Borel sets } A, B \subseteq X_1,$$

where $E(\cdot|\cdot)$ denotes the conditional expectation, and χ_A, χ_B the indicator functions of A, B respectively. Then both $\mu \times \mu$ and η are **a**-weighted equilibrium states of F. So they coincide and thus the σ-algebra $\mathcal{P}_\mu$ has to be trivial. Therefore μ is a K-system.

The paper is organized as follows: In Sect. 2, we give some basic notation and definitions about subshifts. In Sect. 3, we present and prove some variational principles about certain sub-additive potentials. In Sect. 4, we prove Proposition 4.3. In Sect. 5, we prove the uniqueness of equilibrium states for certain sub-additive potentials. In Sect. 6, we prove the uniqueness of weighted equilibrium states for certain sub-additive potentials in the case $k = 2$. The extension to the general case $k \geq 2$ is given in Sect. 7, together with the proof of Theorem 1.2.

2. Preliminaries about subshifts

In this section, we introduce some basic notation and definitions about subshifts. The reader is referred to [25] for the background and more details.

2.1. **One-sided subshifts over finite alphabets.** Let $\mathcal{A}$ be a finite set of symbols which we will call the *alphabet*. Let

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$$

denote the set of all finite words with letters from $\mathcal{A}$, including the empty word ε. Let

$$\mathcal{A}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty} :\ x_i \in \mathcal{A} \text{ for } i \geq 1\}$$

denote the collection of infinite sequences with entries from $\mathcal{A}$. Then $\mathcal{A}^{\mathbb{N}}$ is a compact metric space endowed with the metric

$$d(x, y) = 2^{-\inf\{k:\ x_k \neq y_k\}}, \quad x = (x_i)_{i=1}^{\infty},\ y = (y_i)_{i=1}^{\infty}.$$

For any $n \in \mathbb{N}$ and $I \in \mathcal{A}^n$, we write

$$[I] = \{(x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} :\ x_1 \dots x_n = I\} \tag{2.1}$$

and call it an *n-th cylinder set* in $\mathcal{A}^{\mathbb{N}}$.

In this paper, a *topological dynamical system* is a continuous self map of a compact metrizable space. The shift transformation $\sigma :\ \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ is defined by $(\sigma x)_i = x_{i+1}$ for all $i \in \mathbb{N}$. The pair $(\mathcal{A}^{\mathbb{N}}, \sigma)$ forms a topological dynamical system which is called the *one-sided full shift over $\mathcal{A}$*.

If X is a compact σ-invariant subset of $\mathcal{A}^{\mathbb{N}}$, that is, $\sigma(X) \subseteq X$, then the topological dynamical system (X, σ) is called a *one-sided subshift over $\mathcal{A}$*, or simply, a *subshift*. Sometimes, we denote a subshift (X, σ) by X, or (X, σ_X).

A subshift X over $\mathcal{A}$ is called a *subshift of finite type* if, there exists a matrix $A = (A(\alpha, \beta))_{\alpha,\beta\in\mathcal{A}}$ with entries 0 or 1 such that

$$X = \left\{(x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} :\ A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\right\}.$$

If A is irreducible (in the sense that, for any $\alpha, \beta \in \mathcal{A}$, there exists $n > 0$ such that $A^n(\alpha, \beta) > 0$), X is called an *irreducible subshift of finite type*. Moreover if A is primitive (in the sense that, there exists $n > 0$ such that $A^n(\alpha, \beta) > 0$ for all $\alpha, \beta \in \mathcal{A}$), X is called a *mixing subshift of finite type*.

The *language* $\mathcal{L}(X)$ of a subshift X is the set of all finite words (including the empty word ε) that occur as consecutive strings $x_1 \dots x_n$ in the sequences $x = (x_i)_{i=1}^{\infty}$ which comprise X. That is,

$$\mathcal{L}(X) = \{I \in \mathcal{A}^* :\ I = x_1 \dots x_n \text{ for some } x = (x_i)_{i=1}^{\infty} \in X \text{ and } n \geq 1\} \cup \{\varepsilon\}.$$

Denote $|I|$ the length of a word I. For $n \geq 0$, denote

$$\mathcal{L}_n(X) = \{I \in \mathcal{L}(X) :\ |I| = n\}.$$

Let $p \in \mathbb{N}$. A subshift X is said to satisfy *p-specification* if for any $I, J \in \mathcal{L}(X)$, there exists $K \in \mathcal{L}_p(X)$ such that $IKJ \in \mathcal{L}(X)$. We say that X satisfies *specification* if it satisfies p-specification for some $p \in \mathbb{N}$. Similarly, X is said to satisfy *weak p-specification* if for any $I, J \in \mathcal{L}(X)$, there exists $K \in \bigcup_{i=0}^{p} \mathcal{L}_i(X)$ such that $IKJ \in \mathcal{L}(X)$; and X is said to satisfy *weak specification* if it satisfies weak p-specification for some $p \in \mathbb{N}$. It is easy to see that an irreducible subshift of finite type satisfies weak specification, whilst a mixing subshift of finite type satisfies specification.

Let (X, σ_X) and (Y, σ_Y) be two subshifts over finite alphabets $\mathcal{A}$ and $\mathcal{A}'$, respectively. We say that Y is a *factor* of X if, there is a continuous surjective map $\pi: X \to Y$ such that $\pi T = S\pi$. Here π is called a *factor map*. Furthermore π is called a 1-*block map* if there exists a map $\pi: \mathcal{A} \to \mathcal{A}'$ such that

$$\pi(x) = (\pi(x_i))_{i=1}^{\infty}, \qquad x = (x_i)_{i=1}^{\infty} \in X.$$

It is well known (see, e.g. [25, Proposition 1.5.12]) that each factor map $\pi: X \to Y$ between two subshifts X and Y, will become a 1-block factor map if we enlarge the alphabet $\mathcal{A}$ and recode X through a so-called *higher block representation of X*. Whenever $\pi: X \to Y$ is 1-block, we write $\pi I = \pi(x_1) \dots \pi(x_n)$ for $I = x_1 \dots x_n \in \mathcal{L}_n(X)$; clearly $\pi I \in \mathcal{L}_n(Y)$.

2.2. **Two-sided subshifts over finite alphabets.**

For a finite alphabet $\mathcal{A}$, let

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}}: \ x_i \in \mathcal{A} \text{ for all } i \in \mathbb{Z}\}$$

denote the collection of all bi-infinite sequences of symbols from $\mathcal{A}$. Similarly, $\mathcal{A}^{\mathbb{Z}}$ is a compact metric space endowed with the metric

$$d(x, y) = 2^{-\inf\{|k|: \, x_k \neq y_k\}}, \quad x = (x_i)_{i \in \mathbb{Z}}, \ y = (y_i)_{i \in \mathbb{Z}}.$$

The shift map $\sigma: \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $(\sigma x)_i = x_{i+1}$ for $x = (x_i)_{i \in \mathbb{Z}}$. The topological dynamical system $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ is called the *two-sided full shift over $\mathcal{A}$*.

If $X \subseteq \mathcal{A}^{\mathbb{Z}}$ is compact and $\sigma(X) = X$, the topological dynamical system (X, σ) is called a *two-sided subshift over $\mathcal{A}$*.

The definitions of $\mathcal{L}(X)$, (weak) specification and factor maps for two-sided subshifts can be given in a way similar to the one-sided case.

2.3. **Some notation.**

For two families of real numbers $\{a_i\}_{i \in \mathcal{I}}$ and $\{b_i\}_{i \in \mathcal{I}}$, we write

$a_i \approx b_i$	if there is $c > 0$ such that $\frac{1}{c} b_i \leq a_i \leq c b_i$ for $i \in \mathcal{I}$;
$a_i \succcurlyeq b_i$	if there is $c > 0$ such that $a_i \geq c b_i$ for $i \in \mathcal{I}$;
$a_i \preccurlyeq b_i$	if there is $c > 0$ such that $a_i \leq c b_i$ for $i \in \mathcal{I}$;
$a_i = b_i + O(1)$	if there is $c > 0$ such that $\lvert a_i - b_i \rvert \leq c$ for $i \in \mathcal{I}$;
$a_i \geq b_i + O(1)$	if there is $c > 0$ such that $a_i - b_i \geq -c$ for $i \in \mathcal{I}$;
$a_i \leq b_i + O(1)$	if there is $c > 0$ such that $a_i - b_i \leq c$ for $i \in \mathcal{I}$.

3. Variational principles for sub-additive potentials

In this section we present and prove some variational principles for certain sub-additive potentials. This is the starting point in our work.

First we give some notation and definitions. Let (X, σ_X) be a one-sided subshift over a finite alphabet $\mathcal{A}$. We use $\mathcal{M}(X)$ to denote the set of all Borel probability measures on X. Endow $\mathcal{M}(X)$ with the weak-star topology. Let $\mathcal{M}(X, \sigma_X)$ denote the set of all σ_X-invariant Borel probability measures on X. The sets $\mathcal{M}(X)$ and $\mathcal{M}(X, \sigma_X)$ are non-empty, convex and compact (cf. [35]). Let $\mathcal{L}(X)$ denote the language of X (cf. Sect 2). For convenience, for $\mu \in \mathcal{M}(X)$ and $I \in \mathcal{L}(X)$, we would like to write

$$\mu(I) := \mu([I] \cap X),$$

where $[I]$ denotes the n-th cylinder in $\mathcal{A}^{\mathbb{N}}$ defined as in (2.1).

For $\mu \in \mathcal{M}(X, \sigma_X)$, the *measure theoretic entropy* of μ with respect to σ_X is defined as

$$h_\mu(\sigma_X) := -\lim_{n\to\infty} \frac{1}{n} \sum_{I \in \mathcal{L}_n(X)} \mu(I) \log \mu(I). \tag{3.1}$$

The above limit exists since the sequence $(a_n)_{n=1}^{\infty}$, where

$$a_n = -\sum_{I \in \mathcal{L}_n(X)} \mu(I) \log \mu(I),$$

satisfies $a_{n+m} \le a_n + a_m$ for $n, m \in \mathbb{N}$. It follows that

$$h_\mu(\sigma_X) = \inf_{n\in\mathbb{N}} \left(-\frac{1}{n} \sum_{I \in \mathcal{L}_n(X)} \mu(I) \log \mu(I) \right). \tag{3.2}$$

The function $\mu \mapsto h_\mu(\sigma_X)$ is affine and upper semi-continuous on $\mathcal{M}(X, \sigma_X)$ (cf. [35]).

A sequence $\Phi = (\log \phi_n)_{n=1}^{\infty}$ of functions on a subshift X is called a *sub-additive potential on* X, if each ϕ_n is a non-negative continuous function on X and there exists $c > 0$ such that

$$\phi_{n+m}(x) \le c\phi_n(x)\phi_m(\sigma_X^n x), \quad \forall\, x \in X,\ n, m \in \mathbb{N}. \tag{3.3}$$

For convenience, we denote by $\mathcal{C}_{sa}(X, \sigma_X)$ the collection of sub-additive potentials on X. For $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{sa}(X, \sigma_X)$, define $\Phi_* :\ \mathcal{M}(X, \sigma_X) \to \mathbb{R} \cup \{-\infty\}$ by

$$\Phi_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x) d\mu(x). \tag{3.4}$$

The limit in (3.4) exists by the sub-additivity of $\int \log(c\phi_n) d\mu$.

Remark 3.1. One observes that for $f \in C(X)$, if $\Phi = (\log \phi_n)_{n=1}^\infty$ is given by $\phi_n(x) = \exp(S_n f(x))$, then $\Phi \in \mathcal{C}_{sa}(X, \sigma_X)$ and $\Phi_*(\mu) = \int f \, d\mu$ for each $\mu \in \mathcal{M}(X, \sigma_X)$.

By the sub-additivity (3.3), we have the following simple lemma (cf. Proposition 3.1 in [13]).

Lemma 3.2. (i) *Φ_* is affine and upper semi-continuous on $\mathcal{M}(X, \sigma_X)$.*
(ii) *There is a constant $C \in \mathbb{R}$ such that $\int \log \phi_n(x) d\mu(x) \geq n\Phi_*(\mu) - C$ for $n \in \mathbb{N}$ and $\mu \in \mathcal{M}(X, \sigma_X)$.*

Definition 3.3. For $\Phi \in \mathcal{C}_{sa}(X, \sigma_X)$, $\mu \in \mathcal{M}(X, \sigma_X)$ is called an *equilibrium state of* Φ if

$$\Phi_*(\mu) + h_\mu(\sigma_X) = \sup\{\Phi_*(\eta) + h_\eta(\sigma_X) : \ \eta \in \mathcal{M}(X, \sigma_X)\}.$$

Let $\mathcal{I}(\Phi)$ denote the collection of all equilibrium states of Φ.

Definition 3.4. A function $\phi : \ \mathcal{L}(X) \to [0, \infty)$ is said to be *sub-multiplicative* if, $\phi(\varepsilon) = 1$ and there exists a constant $c > 0$ such that $\phi(IJ) \leq c\phi(I)\phi(J)$ for any $IJ \in \mathcal{L}(X)$. Furthermore, say $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X, \sigma_X)$ is *generated* by ϕ if

$$\phi_n(x) = \phi(x_1 \dots x_n), \qquad x = (x_i)_{i=1}^\infty \in X.$$

Proposition 3.5. *Assume that $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X, \sigma_X)$ is generated by a sub-multiplicative function $\phi : \ \mathcal{L}(X) \to [0, \infty)$. Then*

(i) $\sup\{\Phi_*(\mu) + h_\mu(\sigma_X) : \ \mu \in \mathcal{M}(X, \sigma_X)\} = \lim_{n\to\infty} \frac{1}{n} \log u_n$, *where u_n is given by*

$$u_n = \sum_{I \in \mathcal{L}_n(X)} \phi(I).$$

(ii) *$\mathcal{I}(\Phi)$ is a non-empty compact convex subset of $\mathcal{M}(X, \sigma_X)$. Furthermore each extreme point of $\mathcal{I}(\Phi)$ is an ergodic measure.*

We remark that Proposition 3.5(i) is a special case of Theorem 1.1 in [9] on the variational principle for general sub-additive potentials. It was also obtained in [18] for the case that $\phi > 0$. Proposition 3.5(ii) actually holds for any $\Phi \in \mathcal{C}_{sa}(X, \sigma_X)$, by the affinity and upper semi-continuity of $\Phi_*(\cdot)$ and $h_{(\cdot)}(\sigma_X)$ on $\mathcal{M}(X, \sigma_X)$ (see the proof of Proposition 3.7(ii) for details).

Now let (X, σ_X) and (Y, σ_Y) be one-sided subshifts over $\mathcal{A}, \mathcal{A}'$, respectively. Assume that Y is a factor of X with a 1-block factor map $\pi : \ X \to Y$.

Definition 3.6. For $\nu \in \mathcal{M}(Y, \sigma_Y)$, $\mu \in \mathcal{M}(X, \sigma_X)$ is called a *conditional equilibrium state of Φ with respect to ν* if, $\mu \circ \pi^{-1} = \nu$ and

$$\Phi_*(\mu) + h_\mu(\sigma_X) = \sup\{\Phi_*(\eta) + h_\eta(\sigma_X) : \ \eta \in \mathcal{M}(X, \sigma_X), \ \eta \circ \pi^{-1} = \nu\}.$$

Let $\mathcal{I}_\nu(\Phi)$ denote the collection of all conditional equilibrium states of Φ with respect to ν.

The following result is a relativized version of Proposition 3.5.

Proposition 3.7. *Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X, \sigma_X)$ be generated by a sub-multiplicative function $\phi: \mathcal{L}(X) \to [0, \infty)$. Let $\nu \in \mathcal{M}(Y, \sigma_Y)$. Then*

(i) $\sup\{\Phi_*(\mu) + h_\mu(\sigma_X) - h_\nu(\sigma_Y) :\ \mu \in \mathcal{M}(X, \sigma_X),\ \mu\circ\pi^{-1} = \nu\} = \Psi_*(\nu)$, *where* $\Psi = (\log \psi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(Y, \sigma_Y)$ *is generated by a sub-multiplicative function* $\psi: \mathcal{L}(Y) \to [0, \infty)$, *which satisfies*

$$\psi(J) = \sum_{I \in \mathcal{L}(X):\, \pi I = J} \phi(I), \qquad \forall\ J \in \mathcal{L}(Y). \tag{3.5}$$

(ii) $\mathcal{I}_\nu(\Phi)$ *is a non-empty compact convex subset of* $\mathcal{M}(X, \sigma_X)$. *Furthermore, if* ν *is ergodic, then each extreme point of* $\mathcal{I}_\nu(\Phi)$ *is an ergodic measure on* X.

We remark that Proposition 3.5 can be obtained from Proposition 3.7 by considering the special case that Y is a singleton (correspondingly, $\mathcal{A}'$ consists of one symbol).

To prove Proposition 3.7, we need the following lemmas.

Lemma 3.8 ([6], p. 34)**.** *Suppose* $0 \le p_1, \ldots, p_m \le 1$, $s = p_1 + \cdots + p_m \le 1$ *and* $a_1, \ldots, a_m \ge 0$. *Then*

$$\sum_{i=1}^m p_i(\log a_i - \log p_i) \le s\log(a_1 + \cdots + a_m) - s\log s.$$

Lemma 3.9 ([9], Lemma 2.3)**.** *Let* $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X, \sigma_X)$. *Suppose* $(\eta_n)_{n=1}^\infty$ *is a sequence in* $\mathcal{M}(X)$. *We form the new sequence* $(\mu_n)_{n=1}^\infty$ *by* $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1} \eta_n \circ \sigma_X^{-i}$. *Assume that* μ_{n_i} *converges to* μ *in* $\mathcal{M}(X)$ *for some subsequence* (n_i) *of natural numbers. Then* $\mu \in \mathcal{M}(X, \sigma_X)$, *and moreover*

$$\limsup_{i\to\infty} \frac{1}{n_i} \int \log \phi_{n_i}(x)\, d\eta_{n_i}(x) \le \Phi_*(\mu).$$

Lemma 3.10 ([9], Lemma 2.4)**.** *Denote* $k = \#\mathcal{A}$. *Then for any* $\xi \in \mathcal{M}(X)$, *and positive integers* n, ℓ *with* $n \ge 2\ell$, *we have*

$$\frac{1}{n}\sum_{I \in \mathcal{L}_n(X)} \xi(I)\log\xi(I) \ge \frac{1}{\ell}\sum_{I \in \mathcal{L}_\ell(X)} \xi_n(I)\log\xi_n(I) - \frac{2\ell}{n}\log k,$$

where $\xi_n = \frac{1}{n}\sum_{i=0}^{n-1} \xi \circ \sigma_X^{-i}$.

Proof of Proposition 3.7. Fix $\nu \in \mathcal{M}(Y,\sigma_Y)$. For any $\mu \in \mathcal{M}(X,\sigma_X)$ with $\mu\circ\pi^{-1} = \nu$, and $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{I\in\mathcal{L}_n(X)} \mu(I)\log\phi(I) - \mu(I)\log\mu(I)\\
&\qquad = \sum_{J\in\mathcal{L}_n(Y)}\ \sum_{I\in\mathcal{L}_n(X):\,\pi I=J} \mu(I)\log\phi(I) - \mu(I)\log\mu(I)\\
&\qquad \le \sum_{J\in\mathcal{L}_n(Y)} \nu(J)\log\psi(J) - \nu(J)\log\nu(J) \qquad \text{(by Lemma 3.8)}.
\end{aligned}$$

Dividing both sides by n and letting $n\to\infty$, we obtain

$$\Phi_*(\mu) + h_\mu(\sigma_X) - h_\nu(\sigma_Y) \le \Psi_*(\nu).$$

Thus to complete the proof of (i), it suffices to show that there exists μ with $\mu\circ\pi^{-1} = \nu$, such that $\Phi_*(\mu)+h_\mu(\sigma_X)-h_\nu(\sigma_Y) \ge \Psi_*(\nu)$. For this purpose, construct a sequence $(\eta_n)_{n=1}^\infty$ in $\mathcal{M}(X)$ such that

$$\eta_n(I) = \frac{\nu(\pi I)\phi(I)}{\psi(\pi I)}, \quad \forall\, I \in \mathcal{L}_n(X),$$

where we take the convention $\frac{0}{0} = 0$. Clearly, $\eta_n\circ\pi^{-1}(J) = \nu(J)$ for all $J\in\mathcal{L}_n(Y)$. Set $\mu_n = \frac{1}{n}\sum_{i=0}^{n-1}\eta_n\circ\sigma_X^{-i}$. Assume that μ_{n_i} converges to μ in $\mathcal{M}(X)$ for some subsequence (n_i) of natural numbers. By Lemma 3.9, $\mu\in\mathcal{M}(X,\sigma_X)$ and

$$\Phi_*(\mu) \ge \limsup_{i\to\infty}\frac{1}{n_i}\int \log\phi_{n_i}(x)\,d\eta_{n_i}(x) = \limsup_{i\to\infty}\frac{1}{n_i}\sum_{I\in\mathcal{L}_{n_i}(X)}\eta_{n_i}(I)\log\phi(I). \tag{3.6}$$

We first show that $\mu\circ\pi^{-1} = \nu$. Let $J\in\mathcal{L}(Y)$. Denote $\ell = |J|$. For $n>\ell$ and $0\le i\le n-\ell$, we have

$$\begin{aligned}
\eta_n\circ\sigma_X^{-i}\circ\pi^{-1}(J) &= \eta_n\circ\pi^{-1}\circ\sigma_Y^{-i}(J)\\
&= \sum_{J_1\in\mathcal{L}_i(Y),\ J_2\in\mathcal{L}_{n-i-\ell}(Y):\ J_1JJ_2\in\mathcal{L}_n(Y)} \eta_n\circ\pi^{-1}(J_1JJ_2)\\
&= \sum_{J_1\in\mathcal{L}_i(Y),\ J_2\in\mathcal{L}_{n-i-\ell}(Y):\ J_1JJ_2\in\mathcal{L}_n(Y)} \nu(J_1JJ_2) = \nu(J).
\end{aligned}$$

It follows that $\mu_n\circ\pi^{-1}(J) = \frac{1}{n}\sum_{i=0}^{n-1}\eta_n\circ\sigma_X^{-i}\circ\pi^{-1}(J) \to \nu(J)$, as $n\to\infty$. Therefore $\mu\circ\pi^{-1}(J) = \nu(J)$. Since $J\in\mathcal{L}(Y)$ is arbitrary, we have $\mu\circ\pi^{-1} = \nu$.

We next show that

$$\Phi_*(\mu) + h_\mu(\sigma_X) - h_\nu(\sigma_Y) \ge \Psi_*(\nu). \tag{3.7}$$

Fix $\ell\in\mathbb{N}$. By Lemma 3.10, we have for $n\ge 2\ell$,

$$\frac{1}{n}\sum_{I\in\mathcal{L}_n(X)}\eta_n(I)\log\eta_n(I) \ge \frac{1}{\ell}\sum_{I\in\mathcal{L}_\ell(X)}\mu_n(I)\log\mu_n(I) - \frac{2\ell}{n}\log k,$$

where $k := \#\mathcal{A}$. Since $\mu_{n_i} \to \mu$ as $i \to \infty$, we obtain

$$\liminf_{i\to\infty} \frac{1}{n_i} \sum_{I\in\mathcal{L}_{n_i}(X)} \eta_{n_i}(I)\log\eta_{n_i}(I) \geq \frac{1}{\ell}\sum_{I\in\mathcal{L}_\ell(X)} \mu(I)\log\mu(I).$$

Taking $\ell \to \infty$ yields

$$\liminf_{i\to\infty} \frac{1}{n_i} \sum_{I\in\mathcal{L}_{n_i}(X)} \eta_{n_i}(I)\log\eta_{n_i}(I) \geq -h_\mu(\sigma_X). \tag{3.8}$$

Observe that

$$\begin{aligned}\sum_{I\in\mathcal{L}_n(X)} \eta_n(I)\log\phi(I) &= \sum_{I\in\mathcal{L}_n(X)} \eta_n(I)\log\frac{\eta_n(I)\psi(\pi I)}{\nu(\pi I)} \\ &= \sum_{I\in\mathcal{L}_n(X)} \eta_n(I)\log\eta_n(I) \\ &\qquad + \sum_{J\in\mathcal{L}_n(Y)} \nu(J)(\log\psi(J)-\log\nu(J)).\end{aligned}$$

This together with (3.8) yields

$$\liminf_{i\to\infty} \frac{1}{n_i} \sum_{I\in\mathcal{L}_{n_i}(X)} \eta_{n_i}(I)\log\phi(I) \geq -h_\mu(\sigma_X) + \Psi_*(\nu) + h_\nu(\sigma_Y).$$

Applying (3.6), we have $\Phi_*(\mu) \geq -h_\mu(\sigma_X) + \Psi_*(\nu) + h_\nu(\sigma_Y)$. This proves (3.7). Hence the proof of (i) is complete.

Now we show (ii). By the above proof, we see that $\mathcal{I}_\nu(\Phi) \neq \emptyset$. The convexity of $\mathcal{I}_\nu(\Phi)$ follows directly from the affinity of $\Phi_*(\cdot)$ and $h_{(\cdot)}(\sigma_X)$ on $\mathcal{M}(X,\sigma_X)$. Furthermore, the compactness of $\mathcal{I}_\nu(\Phi)$ follows from the upper semi-continuity of $\Phi_*(\cdot)$ and $h_{(\cdot)}(\sigma_X)$ on $\mathcal{M}(X,\sigma_X)$. Next, assume that ν is ergodic and let μ be an extreme point of $\mathcal{I}_\nu(\Phi)$. We are going to show that μ is ergodic. Assume it is not true, that is, there exist $\mu_1, \mu_2 \in \mathcal{M}(X,\sigma_X)$ with $\mu_1 \neq \mu_2$, and $\alpha_1, \alpha_2 \in (0,1)$ with $\alpha_1+\alpha_2 = 1$, such that $\mu = \sum_{i=1}^2 \alpha_i\mu_i$. Then $\nu = \mu\circ\pi^{-1} = \sum_{i=1}^2 \alpha_i\mu_i\circ\pi^{-1}$. Since $\mu_i\circ\pi^{-1} \in \mathcal{M}(Y,\sigma_Y)$ for $i=1,2$ and ν is ergodic, we have $\mu_1\circ\pi^{-1} = \mu_2\circ\pi^{-1} = \nu$. Note that

$$\Psi_*(\nu) = \Phi_*(\mu) + h_\mu(\sigma_X) - h_\nu(\sigma_Y) = \sum_{i=1}^2 \alpha_i(\Phi_*(\mu_i) + h_{\mu_i}(\sigma_X) - h_\nu(\sigma_Y))$$

and $\Phi_*(\mu_i) + h_{\mu_i}(\sigma_X) - h_\nu(\sigma_Y) \leq \Psi_*(\nu)$ by (i). Hence we have

$$\Phi_*(\mu_i) + h_{\mu_i}(\sigma_X) - h_\nu(\sigma_Y) = \Psi_*(\nu), \quad i = 1,2.$$

That is, $\mu_i \in \mathcal{I}_\nu(\Phi)$ for $i=1,2$. However $\mu = \sum_{i=1}^2 \alpha_i\mu_i$. It contradicts the assumption that μ is an extreme point of $\mathcal{I}_\nu(\Phi)$. This finishes the proof of the proposition. $\square$

Definition 3.11. Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ with $a_1 > 0$ and $a_2 \geq 0$. For $\Phi \in \mathcal{C}_{sa}(X, \sigma_X)$, $\mu \in \mathcal{M}(X, \sigma_X)$ is called an $\mathbf{a}$-*weighted equilibrium state of* Φ *for the factor map* π, or simply, $\mathbf{a}$-*weighted equilibrium state of* Φ, if

$$\begin{aligned}\Phi_*(\mu) &+ a_1 h_\mu(\sigma_X) + a_2 h_{\mu\circ\pi^{-1}}(\sigma_Y) \\ &= \sup\{\Phi_*(\eta) + a_1 h_\eta(\sigma_X) + a_2 h_{\eta\circ\pi^{-1}}(\sigma_Y) :\ \eta \in \mathcal{M}(X, \sigma_X)\}.\end{aligned} \tag{3.9}$$

We use $\mathcal{I}(\Phi, \mathbf{a})$ to denote the collection of all $\mathbf{a}$-weighted equilibrium states of Φ. The value in the right hand side of (3.9) is called the $\mathbf{a}$-*weighted topological pressure of* Φ and is denoted by $P^{\mathbf{a}}(\sigma_X, \Phi)$.

As a corollary of Propositions 3.5 and 3.7, we have

Corollary 3.12. *Let* $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ *with* $a_1 > 0$ *and* $a_2 \geq 0$. *Let* $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X, \sigma_X)$ *be generated by a sub-multiplicative function* $\phi :\ \mathcal{L}(X) \to [0, \infty)$. *Define* $\phi^{(2)} :\ \mathcal{L}(Y) \to [0, \infty)$ *by*

$$\phi^{(2)}(J) = \Big(\sum_{I \in \mathcal{L}_n(X):\ \pi I = J} \phi(I)^{\frac{1}{a_1}} \Big)^{a_1} \quad \textit{for } J \in \mathcal{L}_n(Y),\ n \in \mathbb{N}.$$

Let $\Phi^{(2)} = (\log \psi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(Y, \sigma_Y)$ *be generated by* $\phi^{(2)}$. *Then*

- (i) $\mu \in \mathcal{I}(\Phi, \mathbf{a})$ *if and only if* $\mu \circ \pi^{-1} \in \mathcal{I}(\frac{1}{a_1+a_2}\Phi^{(2)})$ *and* $\mu \in \mathcal{I}_{\mu\circ\pi^{-1}}(\frac{1}{a_1}\Phi)$, *where* $\frac{1}{a_1+a_2}\Phi^{(2)} := (\log(\psi_n^{1/(a_1+a_2)}))_{n=1}^\infty$ *and* $\frac{1}{a_1}\Phi := (\log(\phi_n^{1/a_1}))_{n=1}^\infty$.
- (ii) *Furthermore,* $\mathcal{I}(\Phi, \mathbf{a})$ *is a non-empty compact convex set, and each extreme point of* $\mathcal{I}(\Phi, \mathbf{a})$ *is ergodic.*
- (iii) $\mathcal{I}(\Phi, \mathbf{a})$ *is a singleton if and only if* $\mathcal{I}(\frac{1}{a_1+a_2}\Phi^{(2)})$ *is a singleton* $\{\nu\}$ *and,* $\mathcal{I}_\nu(\frac{1}{a_1}\Phi)$ *contains a unique ergodic measure.*

Proof. Note that for $\mu \in \mathcal{M}(X, \sigma_X)$,

$$\begin{aligned}\Phi_*(\mu) &+ a_1 h_\mu(\sigma_X) + a_2 h_{\mu\circ\pi^{-1}}(\sigma_Y) \\ &= \Phi_*(\mu) + a_1(h_\mu(\sigma_X) - h_{\mu\circ\pi^{-1}}(\sigma_Y)) + (a_1 + a_2) h_{\mu\circ\pi^{-1}}(\sigma_Y).\end{aligned}$$

By Proposition 3.7,

$$\begin{aligned}\sup &\left\{\Phi_*(\eta) + a_1(h_\eta(\sigma_X) - h_{\eta\circ\pi^{-1}}(\sigma_Y)) :\ \eta \in \mathcal{M}(X, \sigma_X),\ \eta \circ \pi^{-1} = \mu \circ \pi^{-1}\right\} \\ &= \Phi_*^{(2)}(\mu \circ \pi^{-1}).\end{aligned}$$

Hence $\mu \in \mathcal{I}(\Phi, \mathbf{a})$ if and only if that

$$\Phi_*(\mu) + a_1(h_\mu(\sigma_X) - h_{\mu\circ\pi^{-1}}(\sigma_Y)) = \Phi_*^{(2)}(\mu \circ \pi^{-1}) \text{ and}$$

$$\Phi_*^{(2)}(\mu \circ \pi^{-1}) + (a_1 + a_2) h_{\mu\circ\pi^{-1}}(\sigma_Y) = \sup_{\nu \in \mathcal{M}(Y, \sigma_Y)} \left\{\Phi^{(2)}(\nu) + (a_1 + a_2) h_\nu(\sigma_Y)\right\}$$

hold simultaneously. That is, $\mu \in \mathcal{I}(\Phi, \mathbf{a})$ if and only if $\mu \in \mathcal{I}_{\mu\circ\pi^{-1}}(\frac{1}{a_1}\Phi)$ and $\mu \circ \pi^{-1} \in \mathcal{I}(\frac{1}{a_1+a_2}\Phi^{(2)})$. This proves (i). The proof of (ii) is essentially identical to that of Proposition 3.7(ii). Part (iii) follows from (i) and (ii). $\square$

Remark 3.13. Proposition 3.7 was proved in [1] in the special case that $\pi: X \to Y$ is a one-block factor map between full shifts. Independently, Proposition 3.7 and Corollary 3.12 were set up in [38] for the special case that $\phi \equiv 1$ and X is an irreducible subshift of finite type, by a direct combination of [23, Theorem 2.1] and [31, Corollary].

4. Ergodic invariant measures associated with certain functions on $\mathcal{A}^*$

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$. We define two collections of functions over $\mathcal{A}^*$.

Definition 4.1. *Let $p \in \mathbb{N}$. Define $\Omega_w(\mathcal{A}^*, p)$ to be the collection of functions $f : \mathcal{A}^* \to [0,1]$ such that there exists $0 < c \leq 1$ so that*

(H1) $\sum_{I \in \mathcal{A}^n} f(I) = 1$ *for any* $n \geq 0$.
(H2) *For any $I, J \in \mathcal{A}^*$, there exists $K \in \bigcup_{i=0}^{p} \mathcal{A}^i$ such that $f(IKJ) \geq cf(I)f(J)$.*
(H3) *For each $I \in \mathcal{A}^*$, there exist $i, j \in \mathcal{A}$ such that*

$$f(iI) \geq cf(I), \quad f(Ij) \geq cf(I).$$

Definition 4.2. *Let $p \in \mathbb{N}$. Let $\Omega(\mathcal{A}^*, p)$ denote the collection of functions $g : \mathcal{A}^* \to [0,1]$ such that there exists $0 < c \leq 1$ so that*

(A1) $\sum_{I \in \mathcal{A}^n} g(I) = 1$ *for any* $n \geq 0$.
(A2) *For any $I, J \in \mathcal{A}^*$, there exists $K \in \mathcal{A}^p$ such that $g(IKJ) \geq cg(I)g(J)$.*

For $f \in \Omega_w(\mathcal{A}^*, p) \cup \Omega(\mathcal{A}^*, p)$, define a map $f^* : \mathcal{A}^* \to [0, \infty)$ by

$$f^*(I) = \sup_{m,n \geq 0} f_{m,n}(I), \qquad I \in \mathcal{A}^*, \tag{4.1}$$

where $f_{m,n}(I) := \sum_{I_1 \in \mathcal{A}^m} \sum_{I_2 \in \mathcal{A}^n} f(I_1 I I_2)$. Clearly, $f(I) = f_{0,0}(I) \leq f^*(I) \leq 1$ for any $I \in \mathcal{A}^*$.

The main result in this section is the following proposition, which plays a key role in our proof of Theorem 1.2.

Proposition 4.3. *Let $f \in \Omega_w(\mathcal{A}^*, p) \cup \Omega(\mathcal{A}^*, p)$ and f^* be defined as in (4.1). Let $(\eta_n)_{n=1}^{\infty}$ be a sequence of Borel probability measures on $\mathcal{A}^{\mathbb{N}}$ satisfying*

$$\eta_n(I) = f(I), \qquad \forall\, I \in \mathcal{A}^n.$$

We form the new sequence $(\mu_n)_{n=1}^{\infty}$ by $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \eta_n \circ \sigma^{-n}$. Assume that μ_{n_i} converges to μ for some subsequence (n_i) of natural numbers. Then $\mu \in \mathcal{M}(\mathcal{A}^{\mathbb{N}}, \sigma)$ and it satisfies the following properties:

(i) *There is a constant $C_1 > 0$ such that $C_1 f^*(I) \leq \mu(I) \leq f^*(I)$ for all $I \in \mathcal{A}^*$.*

(ii) *There is a constant* $C_2 > 0$ *such that*

$$\liminf_{n\to\infty} \sum_{i=0}^{p} \mu(A \cap \sigma^{-n-i}(B)) \geq C_2 \mu(A)\mu(B)$$

for any Borel sets $A, B \subseteq \mathcal{A}^{\mathbb{N}}$.

(iii) μ *is ergodic.*

(iv) μ *is the unique ergodic measure on* $\mathcal{A}^{\mathbb{N}}$ *such that* $\mu(I) \geq C_3 f(I)$ *for all* $I \in \mathcal{A}^*$ *and some constant* $C_3 > 0$.

(v) $\frac{1}{n}\sum_{i=0}^{n-1} \eta_n \circ \sigma^{-n}$ *converges to* μ *in the weak-star topology, as* $n \to \infty$.

Furthermore if $f \in \Omega(\mathcal{A}^*, p)$, *we have*

(vi) *There is a constant* $C_4 > 0$ *such that*

$$\liminf_{n\to\infty} \mu(A \cap \sigma^{-n}(B)) \geq C_4 \mu(A)\mu(B)$$

for any Borel sets $A, B \subseteq \mathcal{A}^{\mathbb{N}}$.

To prove the above proposition, we need several lemmas.

Lemma 4.4. *Let* $f \in \Omega_w(\mathcal{A}^*, p) \cup \Omega(\mathcal{A}^*, p)$. *Then there is a constant* $C > 0$, *which depends on* f, *such that*

(i) $f_{m',n'}(I) \geq C f_{m,n}(I)$ *for any* $I \in \mathcal{A}^*$, $m' \geq m + p$ *and* $n' \geq n + p$.

(ii) *For each* $I \in \mathcal{A}^*$, *there exists an integer* $N = N(I)$ *such that*

$$f_{m,n}(I) \geq (C/2) f^*(I), \qquad \forall\, m, n \geq N.$$

Proof. To show (i), we first assume $f \in \Omega_w(\mathcal{A}^*, p)$. Let c be the constant associated with f in Definition 4.1. Fix $I \in \mathcal{A}^*$ and $m, n, m', n' \in \mathbb{N} \cup \{0\}$ such that $m' \geq m + p$ and $n' \geq n + p$. By (H2), for given $I_1 \in \mathcal{A}^m$, $I_2 \in \mathcal{A}^n$, $I_3 \in \mathcal{A}^{m'-m-p}$ and $I_4 \in \mathcal{A}^{n'-n-p}$, there exist $K_1, K_2 \in \bigcup_{i=0}^{p} \mathcal{A}^i$ so that

$$f(I_3K_1I_1II_2K_2I_4) \geq c^2 f(I_3) f(I_1II_2) f(I_4).$$

Furthermore by (H3), there exist $K_3, K_4 \in \bigcup_{i=0}^{p} \mathcal{A}^i$ so that $|K_1| + |K_3| = p$, $|K_2| + |K_4| = p$ and

$$f(K_3I_3K_1I_1II_2K_2I_4K_4) \geq c^{2p} f(I_3K_1I_1II_2K_2I_4) \geq c^{2p+2} f(I_3) f(I_1II_2) f(I_4). \tag{4.2}$$

Summing over $I_1 \in \mathcal{A}^m$, $I_2 \in \mathcal{A}^n$, $I_3 \in \mathcal{A}^{m'-m-p}$ and $I_4 \in \mathcal{A}^{n'-n-p}$, and using (H1), we obtain

$$f_{m',n'}(I) \geq \frac{1}{M} c^{2p+2} f_{m,n}(I),$$

where M denotes the number of different tuples $(J_1, J_2, J_3, J_4) \in (\mathcal{A}^*)^4$ with $|J_1| + |J_3| = p$ and $|J_2| + |J_4| = p$.

Now assume $f \in \Omega(\mathcal{A}^*, p)$. Instead of (4.2), by (A2), we can find $K_1, K_2 \in \mathcal{A}^p$ such that

$$f(I_3K_1I_1II_2K_2I_4) \geq c^2 f(I_3)f(I_1II_2)f(I_4).$$

Summing over I_1, I_2, I_3, I_4 yields

$$f_{m',n'}(I) \geq c^2 f_{m,n}(I).$$

This proves (i) by taking $C = \min\{c^2, \frac{1}{M}c^{2p+2}\} = \frac{1}{M}c^{2p+2}$.

To show (ii), note that $f^*(I) = \sup_{m,n\geq 0} f_{m,n}(I)$. Hence we can pick m_0, n_0 such that $f_{m_0,n_0}(I) \geq f^*(I)/2$. Let $N = m_0 + n_0 + p$. Then by (i), for any $m, n \geq N$, we have

$$f_{m,n}(I) \geq Cf_{m_0,n_0}(I) \geq \frac{C}{2}f^*(I).$$

This finishes the proof of the lemma. $\square$

Lemma 4.5. *Let $f \in \Omega_w(\mathcal{A}^*, p) \cup \Omega(\mathcal{A}^*, p)$. Then there exists a constant $C' > 0$ such that for any $I, J \in \mathcal{A}^*$, there exists an integer $N = N(I, J)$ such that*

$$\sum_{i=0}^{p} \sum_{K\in\mathcal{A}^{n+i}} f^*(IKJ) \geq C' f^*(I)f^*(J), \qquad \forall\, n \geq N.$$

In particular, if $f \in \Omega(\mathcal{A}^*, p)$, then the above inequality can be strengthened as

$$\sum_{K\in\mathcal{A}^n} f^*(IKJ) \geq C' f^*(I)f^*(J), \qquad \forall\, n \geq N.$$

Proof. First assume $f \in \Omega_w(\mathcal{A}^*, p)$. Let C be the constant associated with f in Lemma 4.4. Fix $I, J \in \mathcal{A}^*$. By Lemma 4.4(ii), there exists $k \in \mathbb{N}$ such that for $m_1, m_2, m_3, m_4 \geq k$,

$$f_{m_1,m_2}(I) \geq \frac{C}{2}f^*(I), \quad f_{m_3,m_4}(J) \geq \frac{C}{2}f^*(J).$$

Take $N = 2k$. Let $n \geq N$. Then we have

$$f_{k,n-k}(I) \geq \frac{C}{2}f^*(I), \quad f_{k,k}(J) \geq \frac{C}{2}f^*(J).$$

By (H2), for any $I_1 \in \mathcal{A}^k$, $I_2 \in \mathcal{A}^{n-k}$, $J_1, J_2 \in \mathcal{A}^k$, we have

$$\sum_{i=0}^{p} \sum_{U\in\mathcal{A}^i} f(I_1II_2UJ_1JJ_2) \geq cf(I_1II_2)f(J_1JJ_2). \tag{4.3}$$

Summing over I_1, I_2, J_1, J_2 yields

$$\sum_{i=0}^{p} \sum_{K\in\mathcal{A}^{n+i}} f_{k,k}(IKJ) \geq cf_{k,n-k}(I)f_{k,k}(J).$$

Hence, we have

$$\sum_{i=0}^{p}\sum_{K\in\mathcal{A}^{n+i}} f^*(IKJ) \geq cf_{k,n-k}(I)f_{k,k}(J) \geq c(C/2)^2 f^*(I)f^*(J).$$

Next assume $f \in \Omega(\mathcal{A}^*, p)$. By (A2), instead of (4.3), we have

$$\sum_{U\in\mathcal{A}^p} f(I_1 I I_2 U J_1 J J_2) \geq cf(I_1 I I_2)f(J_1 J J_2)$$

for any $I_1 \in \mathcal{A}^k$, $I_2 \in \mathcal{A}^{n-k}$, $J_1, J_2 \in \mathcal{A}^k$. Summing over I_1, I_2, J_1, J_2 we obtain

$$\sum_{K\in\mathcal{A}^{n+p}} f_{k,k}(IKJ) \geq cf_{k,n-k}(I)f_{k,k}(J) \geq c(C/2)^2 f^*(I)f^*(J).$$

Hence $\sum_{K\in\mathcal{A}^{n+p}} f^*(IKJ) \geq c(C/2)^2 f^*(I)f^*(J)$. This finishes the proof of the lemma. □

Proof of Proposition 4.3. By [35, Theorem 6.9], μ is σ-invariant. Fix $I \in \mathcal{A}^*$. Let $m = |I|$. For $n > m$, we have

$$\begin{aligned}\mu_n(I) &= \frac{1}{n}\left(\sum_{i=0}^{n-m} \eta_n \circ \sigma^{-i}(I) + \sum_{j=n-m+1}^{n-1} \eta_n \circ \sigma^{-j}(I)\right)\\ &= \frac{1}{n}\left(\sum_{i=0}^{n-m} f_{i,n-m-i}(I) + \sum_{j=n-m+1}^{n-1} \eta_n \circ \sigma^{-j}(I)\right).\end{aligned}$$

Applying Lemma 4.4(ii) to the above equality yields

$$\frac{C}{2}f^*(I) \leq \liminf_{n\to\infty} \mu_n(I) \leq \limsup_{n\to\infty} \mu_n(I) \leq f^*(I),$$

where $C > 0$ is a constant independent of I. Hence

$$(C/2)f^*(I) \leq \mu(I) \leq f^*(I).$$

This proves (i) by taking $C_1 = C/2$.

By (i) and Lemma 4.5, we have

$$\begin{aligned}\liminf_{n\to\infty}\sum_{i=0}^{p} \mu([I]\cap\sigma^{-n-i}([J])) &\geq C_1 \liminf_{n\to\infty}\sum_{i=0}^{p}\sum_{K\in\mathcal{A}^{n+i}} f^*(IKJ)\\ &\geq C_1C' f^*(I)f^*(J) \geq C_1C'\mu(I)\mu(J)\end{aligned} \tag{4.4}$$

for some constant $C' > 0$ and all $I, J \in \mathcal{A}^*$. Take $C_2 = C_1C'$. Since $\{[I] : I \in \mathcal{A}^*\}$ generates the Borel σ-algebra of $\mathcal{A}^{\mathbb{N}}$, (ii) follows from (4.4) by a standard argument.

As a consequence of (ii), for any Borel sets $A, B \subseteq \mathcal{A}^{\mathbb{N}}$ with $\mu(A) > 0$ and $\mu(B) > 0$, there exists n such that $\mu(A \cap \sigma^{-n}(B)) > 0$. This implies that μ is ergodic (cf. [35, Theorem 1.5]). This proves (iii).

To prove (iv), assume that η is an ergodic measure on $\mathcal{A}^{\mathbb{N}}$ so that there exists $C_3 > 0$ such that
$$\eta(I) \geq C_3 f(I), \quad \forall\, I \in \mathcal{A}^*.$$
Then for any $I \in \mathcal{A}^*$ and $m, n \in \mathbb{N}$,
$$\eta(I) = \sum_{I_1 \in \mathcal{A}^m} \sum_{I_2 \in \mathcal{A}^n} \eta(I_1 I I_2) \geq C_3 \sum_{I_1 \in \mathcal{A}^m} \sum_{I_2 \in \mathcal{A}^n} f(I_1 I I_2) = C_3 f_{m,n}(I).$$
Hence $\eta(I) \geq C_3 f^*(I) \geq C_3 \mu(I)$. It implies that μ is absolutely continuous with respect to η. Since any two different ergodic measures on $\mathcal{A}^{\mathbb{N}}$ are singular to each other (cf. [35, Theorem 6.10(iv)]), we have $\eta = \mu$. This proves (iv). Notice that (v) follows directly from (i), (iii) and (iv).

Now assume that $f \in \Omega(\mathcal{A}^*, p)$. Instead of (4.4), by (i) and Lemma 4.5 we have
$$\begin{aligned}\liminf_{n\to\infty} \mu([I] \cap \sigma^{-n}([J])) &\geq C_1 \liminf_{n\to\infty} \sum_{K \in \mathcal{A}^n} f^*(IKJ) \\ &\geq C_1 C' f^*(I) f^*(J) \geq C_1 C' \mu(I)\mu(J) = C_2 \mu(I)\mu(J),\end{aligned}$$
from which (vi) follows. This finishes the proof of Proposition 4.3. □

5. Equilibrium states for certain sub-additive potentials

In this section, we show the uniqueness of equilibrium states for certain sub-additive potentials on one-sided subshifts.

Let (X, σ_X) be a subshift over a finite alphabet $\mathcal{A}$. Let $p \in \mathbb{N}$. We use $\mathcal{D}_w(X, p)$ to denote the collection of functions $\phi: \mathcal{L}(X) \to [0, \infty)$ such that $\phi(W) > 0$ for at least one $W \in \mathcal{L}(X) \backslash \{\varepsilon\}$, and there exists $0 < c \leq 1$ so that

(1) $\phi(IJ) \leq c^{-1} \phi(I)\phi(J)$ for any $IJ \in \mathcal{L}(X)$.
(2) For any $I, J \in \mathcal{L}(X)$, there exists $K \in \bigcup_{i=0}^{p} \mathcal{L}_i(X)$ such that $IKJ \in \mathcal{L}(X)$ and $\phi(IKJ) \geq c\phi(I)\phi(J)$.

Furthermore, we use $\mathcal{D}(X, p)$ to denote the collection of functions $\phi: \mathcal{L}(X) \to [0, \infty)$ such that $\phi(W) > 0$ for at least one $W \in \mathcal{L}(X) \backslash \{\varepsilon\}$, and there exists $0 < c \leq 1$ so that ϕ satisfies the above condition (1), and

(2') For any $I, J \in \mathcal{L}(X)$, there exists $K \in \mathcal{L}_p(X)$ such that $IKJ \in \mathcal{L}(X)$ and $\phi(IKJ) \geq c\phi(I)\phi(J)$.

Remark 5.1. (i) $\mathcal{D}(X, p) \subseteq \mathcal{D}_w(X, p)$.
(ii) $\mathcal{D}_w(X, p) \neq \emptyset$ if and only if X satisfies weak p-specification. The necessity is obvious. For the sufficiency, if X satisfies weak p-specification, then the constant function $\phi \equiv 1$ on $\mathcal{L}(X)$ is an element in $\mathcal{D}_w(X, p)$. Similarly, $\mathcal{D}(X, p) \neq \emptyset$ if and only if X satisfies p-specification.

Lemma 5.2. *Suppose $\phi \in \mathcal{D}_w(X,p)$. Then the following two properties hold:*

(i) *There exists a constant $\gamma > 0$ such that for each $I \in \mathcal{L}(X)$, there exist $i, j \in \mathcal{A}$ such that $\phi(iI) \geq \gamma\phi(I)$ and $\phi(Ij) \geq \gamma\phi(I)$.*

(ii) *Let $u_n = \sum_{J\in\mathcal{L}_n(X)} \phi(J)$. Then the limit $u = \lim_{n\to\infty}(1/n)\log u_n$ exists and $u_n \approx \exp(nu)$.*

Proof. Let $\phi \in \mathcal{D}_w(X,p)$ with the corresponding constant $c \in (0,1]$. For (i), we only prove there exists a constant $\gamma > 0$ such that for each $I \in \mathcal{L}(X)$, there exists $j \in \mathcal{A}$ such that $\phi(Ij) \geq \gamma\phi(I)$. The other statement (there exists $i \in \mathcal{A}$ so that $\phi(iI) \geq \gamma\phi(I)$) follows by an identical argument. Fix a word $W \in \mathcal{L}(X)\backslash\{\varepsilon\}$ such that $\phi(W) > 0$. Let $I \in \mathcal{L}(X)$ so that $\phi(I) > 0$. Then there exists $K \in \bigcup_{i=0}^{p} \mathcal{L}_i(X)$ such that $\phi(IKW) \geq c\phi(I)\phi(W)$. Write $KW = jU$, where j is the first letter in the word KW. Then

$$\phi(Ij)\phi(U) \geq c\phi(IjU) = c\phi(IKW) \geq c^2\phi(I)\phi(W).$$

Hence $\phi(U) > 0$ and $\phi(Ij) \geq c^2\phi(I)\phi(W)/\phi(U)$. Since there are only finite possible U (for $|U| \leq |W| + p$), $\phi(Ij)/\phi(I) \geq \gamma$ for some constat $\gamma > 0$.

To see (ii), we have

$$\begin{aligned}
u_{n+m} &= \sum_{I\in\mathcal{L}_n(X),\ J\in\mathcal{L}_m(X):\ IJ\in\mathcal{L}_{n+m}(X)} \phi(IJ) \\
&\leq \sum_{I\in\mathcal{L}_n(X),\ J\in\mathcal{L}_m(X)} c^{-1}\phi(I)\phi(J) = c^{-1}u_n u_m
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
\sum_{k=0}^{p} u_{n+m+k} &= \sum_{I\in\mathcal{L}_n(X),\ J\in\mathcal{L}_m(X)} \ \sum_{K\in\bigcup_{i=0}^{p}\mathcal{L}_i(X):\ IKJ\in\mathcal{L}(X)} \phi(IKJ) \\
&\geq \sum_{I\in\mathcal{L}_n(X),\ J\in\mathcal{L}_m(X)} c\phi(I)\phi(J) = cu_n u_m.
\end{aligned} \tag{5.2}$$

On the other hand,

$$u_{n+1} = \sum_{I\in\mathcal{L}_n(X)} \ \sum_{j\in\mathcal{A}:\ Ij\in\mathcal{L}_{n+1}(X)} \phi(Ij) \geq \gamma \sum_{I\in\mathcal{L}_n(X)} \phi(I) = \gamma u_n,$$

and $u_{n+1} \leq c^{-1}u_1 u_n$ by (5.1). Hence $u_{n+1} \approx u_n$. This together with (5.1) and (5.2) yields $u_{n+m} \approx u_n u_m$, from which (ii) follows. $\square$

Note that we have introduced $\Omega_w(\mathcal{A}^*, p)$ and $\Omega(\mathcal{A}^*, p)$ in Sect. 4. As a direct consequence of Lemma 5.2, we have

Lemma 5.3. *Let $\phi \in \mathcal{D}_w(X,p)$. Define $f : \mathcal{A}^* \to [0,1]$ by*

$$f(I) = \begin{cases} \dfrac{\phi(I)}{\sum_{J\in\mathcal{L}_n(X)}\phi(J)} & \text{if } I \in \mathcal{L}_n(X),\ n \geq 0, \\ \\ 0 & \text{if } I \in \mathcal{A}^*\backslash\mathcal{L}(X), \end{cases} \tag{5.3}$$

Then $f \in \Omega_w(\mathcal{A}^*,p)$, and $f(IJ) \preccurlyeq f(I)f(J)$ for $I, J \in \mathcal{A}^*$. Moreover if $\phi \in \mathcal{D}(X,p)$, then $f \in \Omega(\mathcal{A}^*,p)$.

Lemma 5.4. *Let $\eta, \mu \in \mathcal{M}(X,\sigma_X)$. Assume that η is not absolutely continuous with respect to μ. Then*

$$\lim_{n\to\infty} \sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\mu(I) - \eta(I)\log\eta(I) = -\infty.$$

Proof. We take a slight modification of the proof of Theorem 1.22 in [6]. Since η is not absolutely continuous with respect to μ, there exists $c \in (0,1)$ such that for any $0 < \epsilon < c/2$, there exists a Borel set $A \subset X$ so that

$$\eta(A) > c \quad \text{and} \quad \mu(A) < \epsilon.$$

Applying [6, Lemma 1.23], we see that for each sufficiently large n, there exists $F_n \subset \mathcal{L}_n(X)$ so that

$$\mu(A \triangle A_n) + \eta(A \triangle A_n) < \epsilon \quad \text{with } A_n := \bigcup_{I\in F_n} [I] \cap X,$$

which implies $\eta(A_n) > c - \epsilon > c/2$ and $\mu(A_n) < 2\epsilon$. Using Lemma 3.8, we obtain

$$\begin{aligned} \sum_{I\in F_n} \eta(I)\log\mu(I) - \eta(I)\log\eta(I) &\leq \eta(A_n)\log\mu(A_n) + \sup_{0\leq s\leq 1} s\log(1/s) \\ &\leq (c/2)\log(2\epsilon) + \log 2 \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &\sum_{I\in\mathcal{L}_n(X)\backslash F_n} \eta(I)\log\mu(I) - \eta(I)\log\eta(I) \\ &\leq \eta(X\backslash A_n)\log\mu(X\backslash A_n) + \sup_{0\leq s\leq 1} s\log(1/s) \leq \log 2. \end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5) yields

$$\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\mu(I) - \eta(I)\log\eta(I) \leq (c/2)\log(2\epsilon) + 2\log 2,$$

from which the lemma follows. ∎

The main result in this section is the following

Theorem 5.5. *Let $\phi \in \mathcal{D}_w(X,p)$. Let $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{sa}(X,\sigma_X)$ be generated by ϕ, i.e. $\phi_n(x) = \phi(x_1 \dots x_n)$ for $x = (x_i)_{i=1}^\infty \in X$. Then Φ has a unique equilibrium state μ. The measure μ is ergodic and has the following Gibbs property*

$$\mu(I) \approx \frac{\phi(I)}{\sum_{J\in\mathcal{L}_n(X)} \phi(J)} \approx \exp(-nP)\phi(I), \qquad I \in \mathcal{L}_n(X), \quad n \in \mathbb{N}, \tag{5.6}$$

where $P = \lim_{n\to\infty} \frac{1}{n} \log \sum_{J\in\mathcal{L}_n(X)} \phi(J)$. Furthermore, we have the following estimates:

$$\sum_{I\in\mathcal{L}_n(X)} \mu(I)\log\phi(I) = n\Phi_*(\mu) + O(1), \sum_{I\in\mathcal{L}_n(X)} \mu(I)\log\mu(I) = -nh_\mu(\sigma_X) + O(1).$$

Proof. Define $f : \mathcal{A}^* \to [0,1]$ as in (5.3). By Lemma 5.3, $f \in \Omega_w(\mathcal{A}^*, p)$ and f satisfies $f(IJ) \preccurlyeq f(I)f(J)$ for $I, J \in \mathcal{A}^*$. Let $f^* : \mathcal{A}^* \to [0,\infty)$ be defined as

$$f^*(I) = \sup_{n,m\geq 0} \sum_{I_1\in\mathcal{A}^n} \sum_{I_2\in\mathcal{A}^m} f(I_1 I I_2), \qquad I \in \mathcal{A}^*.$$

Since $f(IJ) \preccurlyeq f(I)f(J)$ for $I, J \in \mathcal{A}^*$, we have $f^*(I) \approx f(I)$. Hence by Proposition 4.3, there exists an ergodic measure μ on $\mathcal{A}^{\mathbb{N}}$ such that $\mu(I) \approx f(I)$, $I \in \mathcal{A}^*$. Since $f(I) = 0$ for $I \in \mathcal{A}^* \backslash \mathcal{L}(X)$, μ is supported on X. By Lemma 5.2(ii), $\sum_{I\in\mathcal{L}_n(X)} \phi(I) \approx \exp(nP)$, hence we have

$$\mu(I) \approx f(I) \approx \exp(-nP)\phi(I), \quad I \in \mathcal{L}_n(X),\ n \in \mathbb{N}.$$

Let η be an ergodic equilibrium state of Φ. By Proposition 3.5(i), $\Phi_*(\eta)+h_\eta(\sigma_X) = P$. By Lemma 3.2 and (3.2), we have

$$\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\phi(I) \geq n\Phi_*(\eta) + O(1),\ -\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\eta(I) \geq nh_\eta(\sigma_X) + O(1). \tag{5.7}$$

Thus we have

$$\begin{aligned} O(1) &\leq \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\phi(I) - \eta(I)\log\eta(I)\Big) - nP \\ &\leq \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\mu(I) - \eta(I)\log\eta(I)\Big) + O(1). \end{aligned} \tag{5.8}$$

That is, $\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\mu(I) - \eta(I)\log\eta(I) \geq O(1)$. By Lemma 5.4, η is absolutely continuous with respect to μ. Since both μ and η are ergodic, we have $\eta = \mu$ (cf. [35, Theorem 6.10(iv)]). This implies that μ is the unique ergodic equilibrium state of Φ. By Proposition 3.5(ii), μ is the unique equilibrium state of Φ.

Since $\eta = \mu$, by (5.8), we have

$$\sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\phi(I) - \eta(I)\log\eta(I)\Big) - n\Phi_*(\eta) - nh_\eta(\sigma_X)$$
$$= \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\phi(I) - \eta(I)\log\eta(I)\Big) - nP \le O(1).$$

This together with (5.7) yields the estimates:

$$\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\phi(I) = n\Phi_*(\eta) + O(1), \; -\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\eta(I) = nh_\eta(\sigma_X) + O(1).$$

This completes the proof of Theorem 5.5. □

Remark 5.6. The introduction of $\mathcal{D}_w(X,p)$ and $\mathcal{D}(X,p)$ was inspired by the work [14]. Indeed, Theorem 5.5 was first setup in [14] for a class of $\phi \in \mathcal{D}_w(X,p)$, where X is an irreducible subshift of finite type and, ϕ is given by the norm of products of non-negative matrices satisfying an irreducibility condition (see [14, Theorem 3.2], [11, Theorem 3.1]). Although the approach in [14] can be adapted to prove (5.6) under our general settings, we like to provide the above short proof using Proposition 4.3. Independently, Theorem 5.5 was set up in [38] in the special case that X is a mixing subshift of finite type, and ϕ a certain element in $\mathcal{D}(X,p)$, through an approach similar to [14].

In the end of this section, we give the following easy-checked, but important fact.

Lemma 5.7. *Let (X,σ_X), (Y,σ_Y) be one-sided subshifts over finite alphabets $\mathcal{A},\mathcal{A}'$, respectively. Assume that Y is a factor of X with a one-block factor map $\pi: X\to Y$. Let $p\in\mathbb{N}$ and $a>0$. For $\phi\in\mathcal{D}_w(X,p)$, define $\phi^a: \mathcal{L}(X)\to[0,\infty)$ and $\psi: \mathcal{L}(Y)\to[0,\infty)$ by*

$$\phi^a(I) = \phi(I)^a \text{ for } I\in\mathcal{L}(X), \quad \psi(J) = \sum_{I\in\mathcal{L}(X):\, \pi I = J} \phi(I) \text{ for } J\in\mathcal{L}(Y).$$

Then $\phi^a\in\mathcal{D}_w(X,p)$ and $\psi\in\mathcal{D}_w(Y,p)$. Furthermore if $\phi\in\mathcal{D}(X,p)$, then $\phi^a\in\mathcal{D}(X,p)$ and $\psi\in\mathcal{D}(Y,p)$.

Proof. Let $\phi\in\mathcal{D}_w(X,p)$ with the corresponding constant $c\in(0;1]$. Clearly $\phi^a\in\mathcal{D}_w(X,p)$. Here we show $\psi\in\mathcal{D}_w(Y,p)$. Observe that for $J_1J_2\in\mathcal{L}(Y)$,

$$\begin{aligned}\psi(J_1J_2) &= \sum_{I_1I_2\in\mathcal{L}(X):\, \pi I_1=J_1,\, \pi I_2=J_2} \phi(I_1I_2)\\ &\le \sum_{I_1I_2\in\mathcal{L}(X):\, \pi I_1=J_1,\, \pi I_2=J_2} c^{-1}\phi(I_1)\phi(I_2)\\ &\le \sum_{I_1\in\mathcal{L}(X):\, \pi I_1=J_1}\ \sum_{I_2\in\mathcal{L}(X):\, \pi I_2=J_2} c^{-1}\phi(I_1)\phi(I_2) = c^{-1}\psi(J_1)\psi(J_2).\end{aligned}$$

Furthermore for any $J_1, J_2 \in \mathcal{L}(Y)$,

$$\begin{aligned}
&\sum_{W \in \bigcup_{i=0}^{p} \mathcal{L}_i(Y):\ J_1 W J_2 \in \mathcal{L}(Y)} \psi(J_1 W J_2) \\
&= \sum_{I_1 \in \mathcal{L}(X):\ \pi I_1 = J_1} \ \sum_{I_2 \in \mathcal{L}(X):\ \pi I_2 = J_2} \ \sum_{K \in \bigcup_{i=0}^{p} \mathcal{L}_i(X):\ I_1 K I_2 \in \mathcal{L}(X)} \phi(I_1 K I_2) \\
&\geq \sum_{I_1 \in \mathcal{L}(X):\ \pi I_1 = J_1} \ \sum_{I_2 \in \mathcal{L}(X):\ \pi I_2 = J_2} c\phi(I_1)\phi(I_2) = c\psi(J_1)\psi(J_2).
\end{aligned}$$

Therefore there exists $W \in \bigcup_{i=0}^{p} \mathcal{L}_i(Y)$, such that $J_1 W J_2 \in \mathcal{L}(Y)$, and $\psi(J_1 W J_2) \geq \frac{c}{L}\psi(J_1)\psi(J_2)$, where L denotes the cardinality of $\bigcup_{i=0}^{p} \mathcal{L}_i(Y)$. Hence $\psi \in \mathcal{D}_w(Y, p)$. A similar argument shows that $\psi \in \mathcal{D}(Y, p)$ whenever $\phi \in \mathcal{D}(X, p)$. $\square$

6. Uniqueness of weighted equilibrium states: $k = 2$

Assume that (X, σ_X) is a one-sided subshift over a finite alphabet $\mathcal{A}$. Let (Y, σ_Y) be a one-sided subshift factor of X with a one-block factor map $\pi:\ X \to Y$.

Let $\mathbf{a} = (a_1, a_2) \in \mathbb{R}^2$ so that $a_1 > 0$ and $a_2 \geq 0$. Assume that $\mathcal{D}_w(X, p) \neq \emptyset$ for some $p \in \mathbb{N}$, equivalently, X satisfies weak p-specification. Let $\phi \in \mathcal{D}_w(X, p)$. Define $\phi^{(2)}:\ \mathcal{L}(Y) \to [0, \infty)$ by

$$\phi^{(2)}(J) = \Big(\sum_{I \in \mathcal{L}_n(X):\ \pi I = J} \phi(I)^{\frac{1}{a_1}} \Big)^{a_1} \text{ for } J \in \mathcal{L}_n(Y),\ n \in \mathbb{N}. \tag{6.1}$$

Furthermore, define $\phi^{(3)}:\ \mathbb{N} \to [0, \infty)$ by

$$\phi^{(3)}(n) = \sum_{J \in \mathcal{L}_n(Y)} \phi^{(2)}(J)^{\frac{1}{a_1 + a_2}}, \quad n \in \mathbb{N}. \tag{6.2}$$

The main result of this section is the following.

Theorem 6.1. *Let $\phi \in \mathcal{D}_w(X, p)$. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{sa}(X, \sigma_X)$ be generated by by ϕ, i.e. $\phi_n(x) = \phi(x_1 \cdots x_n)$ for $x = (x_i)_{i=1}^{\infty} \in X$. Then Φ has a unique $\mathbf{a}$-weighted equilibrium state μ. Furthermore, μ is ergodic and has the following properties:*

(i) *$\mu(I) \approx \widetilde{\phi}^*(I) \succcurlyeq \widetilde{\phi}(I)$ for $I \in \mathcal{L}(X)$, where $\widetilde{\phi}, \widetilde{\phi}^*:\ \mathcal{L}(X) \to [0, \infty)$ are defined by*

$$\widetilde{\phi}(I) = \frac{\phi(I)^{\frac{1}{a_1}}}{\phi^{(2)}(\pi I)^{\frac{1}{a_1}}} \cdot \frac{\phi^{(2)}(\pi I)^{\frac{1}{a_1 + a_2}}}{\phi^{(3)}(n)}, \quad I \in \mathcal{L}_n(X),\ n \in \mathbb{N} \tag{6.3}$$

and

$$\widetilde{\phi}^*(I) = \sup_{m,n \geq 0} \sum_{I_1 \in \mathcal{L}_m(X),\ I_2 \in \mathcal{L}_n(X):\ I_1 I I_2 \in \mathcal{L}(X)} \widetilde{\phi}(I_1 I I_2), \quad I \in \mathcal{L}(X),$$

where in (6.3) we take the convention $0/0 = 0$.

(ii) $\liminf_{n\to\infty}\sum_{i=0}^{p}\mu\left(A\cap\sigma_X^{-n-i}(B)\right)\succcurlyeq\mu(A)\mu(B)$ *for Borel sets* $A,B\subseteq X$.

(iii) *We have the estimates:*

$$\sum_{I\in\mathcal{L}_n(X)}\mu(I)\log\mu(I)=\sum_{I\in\mathcal{L}_n(X)}\mu(I)\log\widetilde{\phi}(I)+O(1)=-nh_\mu(\sigma_X)+O(1),$$

$$\sum_{I\in\mathcal{L}_n(X)}\mu(I)\log\phi(I)=n\Phi_*(\mu)+O(1).$$

Moreover, if $\phi\in\mathcal{D}(X,p)$*, then instead of (ii) we have*

(iv) $\liminf_{n\to\infty}\mu\left(A\cap\sigma_X^{-n}(B)\right)\succcurlyeq\mu(A)\mu(B)$ *for Borel sets* $A,B\subseteq X$.

Proof. By (6.3), we have

$$\widetilde{\phi}(I)=\frac{\phi(I)^{\frac{1}{a_1}}}{\theta(I)},\quad I\in\mathcal{L}_n(X),\ n\in\mathbb{N},$$

where $\theta(I)$ is given by

$$\theta(I)=\phi^{(3)}(n)\phi^{(2)}(\pi I)^{\frac{1}{a_1}-\frac{1}{a_1+a_2}},\quad I\in\mathcal{L}_n(X),\ n\in\mathbb{N}.$$

We claim that $\widetilde{\phi}$ and θ satisfy the following properties:

(a) $\sum_{I\in\mathcal{L}_n(X)}\widetilde{\phi}(I)=1$ for each $n\in\mathbb{N}$.
(b) For any $I\in\mathcal{L}(X)$, if $\phi(I)>0$ then $\theta(I)>0$.
(c) $\theta(I_1I_2)\preccurlyeq\theta(I_1)\theta(I_2)$ for $I_1I_2\in\mathcal{L}(X)$.

Property (a) follows immediately from the definition of $\widetilde{\phi}$. To see (b), one observes that if $\phi(I)>0$ for some $I\in\mathcal{L}_n(X)$, then so are $\phi^{(2)}(\pi I)$ and $\phi^{(3)}(n)$, hence $\theta(I)>0$. To see (c), by Lemma 5.7, $\phi^{(2)}\in\mathcal{D}_w(Y,p)$ and thus

$$\phi^{(2)}(\pi(I_1I_2))\preccurlyeq\phi^{(2)}(\pi I_1)\phi^{(2)}(\pi I_2),\qquad I_1I_2\in\mathcal{L}(X).$$

Furthermore by Lemma 5.2, $\phi^{(3)}(n+m)\approx\phi^{(3)}(n)\phi^{(3)}(m)$. Hence (c) follows.

Extend $\widetilde{\phi},\ \widetilde{\phi}^*:\ \mathcal{A}^*\to[0,\infty)$ by setting $\widetilde{\phi}(I)=\widetilde{\phi}^*(I)=0$ for $I\in\mathcal{A}^*\backslash\mathcal{L}(X)$. By (a), (b), (c) and Lemma 5.2(i), we see that $\widetilde{\phi}\in\Omega_w(\mathcal{A}^*,p)$. Hence by Proposition 4.3, there exists an ergodic measure $\mu\in\mathcal{M}(\mathcal{A}^{\mathbb{N}},\sigma)$ such that

$$\mu(I)\approx\widetilde{\phi}^*(I)\succcurlyeq\widetilde{\phi}(I),\quad I\in\mathcal{A}^{\mathbb{N}}.\tag{6.4}$$

Moreover, μ satisfies

$$\liminf_{n\to\infty}\sum_{i=0}^{p}\mu\left(A\cap\sigma^{-n-i}(B)\right)\succcurlyeq\mu(A)\mu(B)\text{ for Borel sets }A,B\subseteq\mathcal{A}^{\mathbb{N}}.\tag{6.5}$$

By (6.4), μ is supported on X and $\mu\in\mathcal{M}(X,\sigma_X)$.

Let $\Phi^{(2)}=(\log\phi_n^{(2)})_{n=1}^\infty\in\mathcal{C}_{sa}(Y,\sigma_Y)$ be generated by $\phi^{(2)}$, i.e.

$$\phi_n^{(2)}(y)=\phi^{(2)}(y_1\cdots y_n)\text{ for }y=(y_i)_{i=1}^\infty\in Y.$$

Define $\widetilde{\psi}:\ \mathcal{L}(Y)\to[0,\infty)$ by

$$\widetilde{\psi}(J)=\frac{\phi^{(2)}(J)^{\frac{1}{a_1+a_2}}}{\phi^{(3)}(n)},\quad J\in\mathcal{L}_n(Y),\ n\in\mathbb{N}.$$

By the definitions of $\widetilde{\phi}$ and $\widetilde{\psi}$, we have

$$\widetilde{\phi}(I)=\frac{\phi(I)^{\frac{1}{a_1}}}{\phi^{(2)}(\pi I)^{\frac{1}{a_1}}}\cdot\widetilde{\psi}(\pi I),\quad I\in\mathcal{L}(X). \tag{6.6}$$

Since $\phi^{(2)}\in\mathcal{D}_w(Y,p)$, by Lemma 5.7, $(\phi^{(2)})^{1/(a_1+a_2)}\in\mathcal{D}_w(Y,p)$. Hence by Theorem 5.5, $\frac{1}{a_1+a_2}\Phi^{(2)}$ has a unique equilibrium state $\nu\in\mathcal{M}(Y,\sigma_Y)$ and ν satisfies the properties

$$\sum_{J\in\mathcal{L}_n(Y)}\nu(J)\log\nu(J)=\sum_{J\in\mathcal{L}_n(Y)}\nu(J)\log\widetilde{\psi}(J)+O(1)=-nh_\nu(\sigma_Y)+O(1), \tag{6.7}$$

and

$$\sum_{J\in\mathcal{L}_n(Y)}\nu(J)\log\phi^{(2)}(J)=n\Phi_*^{(2)}(\nu)+O(1). \tag{6.8}$$

Assume that η is an ergodic **a**-equilibrium state of Φ. By Corollary 3.12(i), $\eta\circ\pi^{-1}=\nu$ and η is a conditional equilibrium state of $\frac{1}{a_1}\Phi$ with respect to ν, that is,

$$\frac{1}{a_1}\Phi_*(\eta)+h_\eta(\sigma_X)-h_\nu(\sigma_Y)=\frac{1}{a_1}\Phi_*^{(2)}(\nu). \tag{6.9}$$

By (6.7) and (6.8), we have

$$nh_\nu(\sigma_Y)+\frac{n}{a_1}\Phi_*^{(2)}(\nu)=-\sum_{J\in\mathcal{L}_n(Y)}\nu(J)\log\frac{\widetilde{\psi}(J)}{\phi^{(2)}(J)^{\frac{1}{a_1}}}+O(1). \tag{6.10}$$

By Lemma 3.2(ii) and (3.2), we have

$$\sum_{I\in\mathcal{L}_n(X)}\eta(I)\log\phi(I)\geq n\Phi_*(\eta)+O(1),\quad -\sum_{I\in\mathcal{L}_n(X)}\eta(I)\log\eta(I)\geq nh_\eta(\sigma_X). \tag{6.11}$$

Combining (6.9), (6.10) and (6.11), we obtain

$$
\begin{aligned}
O(1) &\leq \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log(\phi(I)^{\frac{1}{a_1}}) - \eta(I)\log\eta(I)\Big) - \frac{n}{a_1}\Phi_*(\eta) - nh_\eta(\sigma_X)\\
&= \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log(\phi(I)^{\frac{1}{a_1}}) - \eta(I)\log\eta(I)\Big) - nh_\nu(\sigma_{X_2}) - \frac{n}{a_1}\Phi_*^{(2)}(\nu)\\
&= \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log(\phi(I)^{\frac{1}{a_1}}) - \eta(I)\log\eta(I)\Big)\\
&\qquad\qquad + \sum_{J\in\mathcal{L}_n(Y)} \nu(J)\log\frac{\widetilde{\psi}(J)}{\phi^{(2)}(J)^{\frac{1}{a_1}}} + O(1) \qquad (6.12)\\
&= \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\frac{\phi(I)^{\frac{1}{a_1}}\widetilde{\psi}(\pi I)}{\phi^{(2)}(\pi I)^{\frac{1}{a_1}}} - \eta(I)\log\eta(I)\Big) + O(1)\\
&= \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\widetilde{\phi}(I) - \eta(I)\log\eta(I)\Big) + O(1) \qquad \text{(by (6.6))}.
\end{aligned}
$$

That is,

$$
\sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\widetilde{\phi}(I) - \eta(I)\log\eta(I)\Big) \geq O(1). \tag{6.13}
$$

Combining (6.13) and (6.4) yields

$$
\begin{aligned}
&\sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\mu(I) - \eta(I)\log\eta(I)\Big)\\
&\qquad \geq \sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\widetilde{\phi}(I) - \eta(I)\log\eta(I)\Big) + O(1) \geq O(1).
\end{aligned} \tag{6.14}
$$

By (6.14) and Lemma 5.4, η is absolutely continuous with respect to μ. Since both μ and η are ergodic, we have $\eta = \mu$ (cf. [35, Theorem 6.10(iv)]). This implies that μ is the unique ergodic $\mathbf{a}$-weighted equilibrium state of Φ. By Corollary 3.12(iii), μ is the unique $\mathbf{a}$-weighted equilibrium state of Φ. Now parts (i), (ii) of the theorem follow from (6.4)-(6.5).

To show (iii), due to $\eta = \mu$, the left hand side of (6.14) equals 0. Hence by (6.14),

$$
\sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log\widetilde{\phi}(I) - \eta(I)\log\eta(I)\Big) = O(1). \tag{6.15}
$$

Combining (6.15) and (6.12) yields

$$
\sum_{I\in\mathcal{L}_n(X)} \Big(\eta(I)\log(\phi(I)^{\frac{1}{a_1}}) - \eta(I)\log\eta(I)\Big) - \frac{n}{a_1}\Phi_*(\eta) - nh_\eta(\sigma_X) = O(1). \tag{6.16}
$$

However (6.16) and (6.11) imply

(6.17)
$$\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\phi(I) = n\Phi_*(\eta)+O(1),\ -\sum_{I\in\mathcal{L}_n(X)} \eta(I)\log\eta(I) = nh_\eta(\sigma_X)+O(1).$$

Now part (iii) follows from (6.17) and (6.15). To see (iv), note that whenever $\phi\in\mathcal{D}(X,p)$, we have $\widetilde{\phi}\in\Omega(\mathcal{A}^*,p)$, following from (a)-(c). Now (iv) follows from Proposition 4.3(vi). This finishes the proof of the theorem. $\square$

7. Uniqueness of weighted equilibrium states: $k\geq 2$

Let $k\geq 2$ be an integer. Assume that (X_i,σ_{X_i}) $(i=1,\ldots,k)$ are one-sided subshifts over finite alphabets so that X_{i+1} is a factor of X_i with a one-block factor map $\pi_i:\ X_i\to X_{i+1}$ for $i=1,\ldots,k-1$. For convenience, we use π_0 to denote the identity map on X_1. Define $\tau_i:\ X_1\to X_{i+1}$ by $\tau_i=\pi_i\circ\pi_{i-1}\circ\cdots\circ\pi_0$ for $i=0,1,\ldots,k-1$.

Let $\mathbf{a}=(a_1,\ldots,a_k)\in\mathbb{R}^k$ so that $a_1>0$ and $a_i\geq 0$ for $i>1$. Let $\phi\in\mathcal{D}_w(X_1,p)$. Set $\phi^{(1)}=\phi$ and define $\phi^{(i)}:\ \mathcal{L}(X_i)\to[0,\infty)$ $(i=2,\ldots,k)$ recursively by

$$\phi^{(i)}(J)=\Big(\sum_{I\in\mathcal{L}_n(X_{i-1}):\ \pi_{i-1}I=J}\phi^{(i-1)}(I)^{\frac{1}{a_1+\cdots+a_{i-1}}}\Big)^{a_1+\cdots+a_{i-1}}$$

for $n\in\mathbb{N}$, $J\in\mathcal{L}_n(X_i)$. Furthermore, define $\phi^{(k+1)}:\ \mathbb{N}\to[0,\infty)$ by

$$\phi^{(k+1)}(n)=\sum_{I\in\mathcal{L}_n(X_k)}\phi^{(k)}(I)^{\frac{1}{a_1+\cdots+a_k}}.$$

Definition 7.1. Let $\Phi=(\log\phi_n)_{n=1}^\infty\in\mathcal{C}_{sa}(X_1,\sigma_{X_1})$ be generated by ϕ. Say that $\mu\in\mathcal{M}(X_1,\sigma_{X_1})$ is an $\mathbf{a}$-*weighted equilibrium state of* Φ if

$$\Phi_*(\mu)+h_\mu^{\mathbf{a}}(\sigma_{X_1})=\sup\left\{\Phi_*(\eta)+h_\eta^{\mathbf{a}}(\sigma_{X_1}):\ \eta\in\mathcal{M}(X,\sigma_{X_1})\right\},$$

where $h_\mu^{\mathbf{a}}(\sigma_{X_1}):=\sum_{i=1}^k a_ih_{\mu\circ\tau_{i-1}^{-1}}(\sigma_{X_i})$. Let $\mathcal{I}(\Phi,\mathbf{a})$ be the collection of all $\mathbf{a}$-*weighted equilibrium states of* Φ.

Let $\Phi^{(2)}\in\mathcal{C}_{sa}(X_2,\sigma_{X_2})$ be generated by $\phi^{(2)}$. By a proof essentially identical to that of Corollary 3.12, we have

Lemma 7.2. (i) *$\mathcal{I}(\Phi,\mathbf{a})$ is a non-empty compact convex subset of $\mathcal{M}(X_1,\sigma_{X_1})$. Each extreme point of $\mathcal{I}(\Phi,\mathbf{a})$ is ergodic.*

(ii) *$\mu\in\mathcal{I}(\Phi,\mathbf{a})$ if and only if $\mu\in\mathcal{I}_{\mu\circ\pi_1^{-1}}(\frac{1}{a_1}\Phi)$ together with $\mu\circ\pi^{-1}\in\mathcal{I}(\Phi^{(2)},\mathbf{b})$, where $\mathbf{b}=(a_1+a_2,a_3,\ldots,a_k)\in\mathbb{R}^{k-1}$.*

(iii) *$\mathcal{I}(\Phi,\mathbf{a})$ is a singleton if and only if $\mathcal{I}(\Phi^{(2)},\mathbf{b})$ is a singleton $\{\nu\}$ and, $\mathcal{I}_\nu(\frac{1}{a_1}\Phi)$ contains a unique ergodic measure.*

As the high dimensional version of Theorem 6.1, we have

Theorem 7.3. *Let $\phi \in \mathcal{D}_w(X_1, p)$. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{sa}(X_1, \sigma_{X_1})$ be generated by ϕ. Then Φ has a unique* **a***-weighted equilibrium state μ. Furthermore, μ is ergodic and has the following properties:*

(i) *$\mu(I) \approx \widetilde{\phi}^*(I) \succcurlyeq \widetilde{\phi}(I)$ for $I \in \mathcal{L}(X_1)$, where $\widetilde{\phi},\ \widetilde{\phi}^* :\ \mathcal{L}(X_1) \to [0, \infty)$ are defined respectively by*

$$\widetilde{\phi}(I) = \left(\prod_{i=1}^{k-1} \frac{\phi^{(i)}(\tau_{i-1} I)^{\frac{1}{a_1+\cdots+a_i}}}{\phi^{(i+1)}(\tau_i I)^{\frac{1}{a_1+\cdots+a_i}}} \right) \cdot \frac{\phi^{(k)}(\tau_{k-1} I)^{\frac{1}{a_1+\cdots+a_k}}}{\phi^{(k+1)}(n)} \tag{7.1}$$

for $I \in \mathcal{L}_n(X_1)$, $n \in \mathbb{N}$, and

$$\widetilde{\phi}^*(I) = \sup_{m,n \geq 0} \sum_{I_1 \in \mathcal{L}_m(X_1),\ I_2 \in \mathcal{L}_n(X_1):\ I_1 I I_2 \in \mathcal{L}(X_1)} \widetilde{\phi}(I_1 I I_2), \quad I \in \mathcal{L}(X_1).$$

(ii) *$\liminf_{n\to\infty} \sum_{i=0}^{p} \mu\left(A \cap \sigma_{X_1}^{-n-i}(B)\right) \succcurlyeq \mu(A)\mu(B)$ for Borel sets $A, B \subseteq X_1$.*

(iii) *We have the estimates:*

$$\sum_{I \in \mathcal{L}_n(X_1)} \mu(I) \log \mu(I) = \sum_{I \in \mathcal{L}_n(X_1)} \mu(I) \log \widetilde{\phi}(I) + O(1) = -n h_\mu(\sigma_{X_1}) + O(1),$$

$$\sum_{I \in \mathcal{L}_n(X_1)} \mu(I) \log \phi(I) = n\Phi_*(\mu) + O(1).$$

Moreover, if $\phi \in \mathcal{D}(X_1, p)$, then instead of (ii) we have

(iv) *$\liminf_{n\to\infty} \mu\left(A \cap \sigma_{X_1}^{-n}(B)\right) \succcurlyeq \mu(A)\mu(B)$ for Borel sets $A, B \subseteq X_1$.*

Proof. We prove the theorem by induction on the dimension k. By Theorem 6.1, Theorem 7.3 is true when the dimension equals 2. Now assume that the theorem is true when the dimension equals $k-1$. In the following we prove that the theorem is also true when the dimension equals k.

By (7.1), we have

$$\widetilde{\phi}(I) = \frac{\phi(I)^{\frac{1}{a_1}}}{\theta(I)}, \quad I \in \mathcal{L}_n(X_1),\ n \in \mathbb{N},$$

where $\theta(I)$ is given by

$$\theta(I) = \phi^{(k+1)}(n) \prod_{i=2}^{k} \phi^{(i)}(\tau_{i-1} I)^{\frac{1}{a_1+\cdots+a_{i-1}} - \frac{1}{a_1+\cdots+a_i}}, \quad I \in \mathcal{L}_n(X_1),\ n \in \mathbb{N}.$$

By Lemma 5.7 and Lemma 5.2(ii), we have $\phi^{(i)} \in \mathcal{D}_w(X_i, p)$ for $i = 2, \ldots, k$, and $\phi^{(k+1)}(n+m) \approx \phi^{(k+1)}(n)\phi^{(k+1)}(m)$. Similar to the proof of Theorem 6.1, we can show that $\widetilde{\phi}$ and θ satisfy the following properties:

(a) $\sum_{I \in \mathcal{L}_n(X_1)} \widetilde{\phi}(I) = 1$ for each $n \in \mathbb{N}$.

(b) For any $I \in \mathcal{L}(X_1)$, if $\phi(I) > 0$ then $\theta(I) > 0$.

(c) $\theta(I_1 I_2) \preccurlyeq \theta(I_1)\theta(I_2)$ for $I_1 I_2 \in \mathcal{L}(X_1)$.

Extend $\widetilde{\phi},\ \widetilde{\phi}^*:\ \mathcal{A}_1^* \to [0,\infty)$ by setting $\widetilde{\phi}(I) = \widetilde{\phi}^*(I) = 0$ for $I \in \mathcal{A}_1^* \backslash \mathcal{L}(X_1)$. By (a), (b), (c) and Lemma 5.2(i), we see that $\widetilde{\phi} \in \Omega_w(\mathcal{A}_1^*, p)$. Hence by Proposition 4.3, there exists an ergodic measure $\mu \in \mathcal{M}(\mathcal{A}_1^{\mathbb{N}}, \sigma)$ such that

$$\mu(I) \approx \widetilde{\phi}^*(I) \succcurlyeq \widetilde{\phi}(I), \quad I \in \mathcal{A}_1^{\mathbb{N}}. \tag{7.2}$$

Moreover, μ satisfies

$$\liminf_{n\to\infty} \sum_{i=0}^{p} \mu\left(A \cap \sigma^{-n-i}(B)\right) \succcurlyeq \mu(A)\mu(B) \text{ for Borel sets } A, B \subseteq \mathcal{A}_1^{\mathbb{N}}.$$

By (7.2), μ is supported on X_1 and $\mu \in \mathcal{M}(X_1, \sigma_{X_1})$.

Let $\Phi^{(2)} = (\log \phi_n^{(2)})_{n=1}^{\infty} \in \mathcal{C}_{sa}(X_2, \sigma_{X_2})$ be generated by $\phi^{(2)}$, i.e.

$$\phi_n^{(2)}(x) = \phi^{(2)}(x_1 \cdots x_n) \text{ for } x = (x_i)_{i=1}^{\infty} \in X_2.$$

Let $\mathbf{b} = (a_1 + a_2, a_3, \ldots, a_k) \in \mathbb{R}^{k-1}$. Define $\widetilde{\psi}:\ \mathcal{L}(X_2) \to [0,\infty)$ by

$$\widetilde{\psi}(J) = \left(\prod_{i=2}^{k-1} \frac{\phi^{(i)}(\xi_{i-1}J)^{\frac{1}{a_1+\cdots+a_i}}}{\phi^{(i+1)}(\xi_i J)^{\frac{1}{a_1+\cdots+a_i}}}\right) \cdot \frac{\phi^{(k)}(\xi_{k-1}J)^{\frac{1}{a_1+\cdots+a_k}}}{\phi^{(k+1)}(n)}, \quad J \in \mathcal{L}(X_2),\ n \in \mathbb{N},$$

where $\xi_1 := Id$, and $\xi_i = \pi_i \circ \cdots \circ \pi_2$ for $i \geq 2$. By the definitions of $\widetilde{\phi}$ and $\widetilde{\psi}$, we have

$$\widetilde{\phi}(I) = \frac{\phi^{(1)}(I)^{\frac{1}{a_1}}}{\phi^{(2)}(\pi_1 I)^{\frac{1}{a_1}}} \cdot \widetilde{\psi}(\pi_1 I), \quad I \in \mathcal{L}(X_1). \tag{7.3}$$

Since $\phi^{(2)} \in \mathcal{D}_w(X_2, p)$, by the assumption of the induction, $\Phi^{(2)}$ has a unique $\mathbf{b}$-weighted equilibrium state $\nu \in \mathcal{M}(X_2, \sigma_{X_2})$ and ν satisfies the properties

$$\sum_{J\in\mathcal{L}_n(X_2)} \nu(J)\log\nu(J) = \sum_{J\in\mathcal{L}_n(X_2)} \nu(J)\log\widetilde{\psi}(J) + O(1) = -nh_\nu(\sigma_{X_2}) + O(1), \tag{7.4}$$

and

$$\sum_{J\in\mathcal{L}_n(X_2)} \nu(J)\log\phi^{(2)}(J) = n\Phi_*^{(2)}(\nu) + O(1). \tag{7.5}$$

Assume that η is an ergodic $\mathbf{a}$-equilibrium state of Φ. By Lemma 7.2, $\eta\circ\pi_1^{-1} = \nu$ and η is a conditional equilibrium state of $\frac{1}{a_1}\Phi$ with respect to ν, that is,

$$\frac{1}{a_1}\Phi_*(\eta) + h_\eta(\sigma_{X_1}) - h_\nu(\sigma_{X_2}) = \frac{1}{a_1}\Phi_*^{(2)}(\nu). \tag{7.6}$$

Using (7.2), (7.3), (7.4)-(7.6), and taking a process the same as in the proof of Theorem 6.1, we prove Theorem 7.3 when the dimension equals k. $\square$

Remark 7.4. Let $\widetilde{\phi}$ be defined as in (7.1), and let (η_n) be a sequence in $\mathcal{M}(X)$ so that $\eta_n(I) = \widetilde{\phi}(I)$ for each $I \in \mathcal{L}_n(X_1)$. Then by Proposition 4.3(v) and the above proof, the measure μ in Theorem 7.3 satisfies

$$\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \eta_n \circ \sigma_{X_1}^{-i}.$$

Proof of Theorem 1.2. We first consider the case that X_i $(i = 1, \ldots, k)$ are one-sided subshifts. Recoding $X_{k-1}, X_{k-1}, \ldots, X_1$ recursively through their higher block representations (cf. Proposition 1.5.12 in [25]), if necessary, we may assume that $\pi_i : X_i \to X_{i+1}$ $(i = 1, \ldots, k-1)$ are all one-block factor maps. Recall that X_1 satisfies weak specification. (Notice that this property is preserved by recoding via higher block representations). Let $f \in V(\sigma_{X_1})$ (see (1.3) for the definition). Define $\phi : \mathcal{L}(X_1) \to [0, \infty)$ by

$$\phi(I) = \sup_{x \in X_1 \cap [I]} \exp(S_n f(x)), \quad I \in \mathcal{L}_n(X_1),\ n \in \mathbb{N},$$

where $S_n f$ is defined as in (1.2). Since $f \in V(\sigma_{X_1})$, it is direct to check that $\phi \in \mathcal{D}_w(X_1, p)$, where p is any integer so that X_1 satisfies weak p-specification. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{sa}(X_1, \sigma_{X_1})$ be generated by ϕ. Again by $f \in V(\sigma_{X_1})$, we have $\Phi_*(\mu) = \int f \, d\mu$ for any $\mu \in \mathcal{M}(X_1, \sigma_{X_1})$. It follows that μ is an $\mathbf{a}$-weighted equilibrium state of f if and only if that, μ is an $\mathbf{a}$-weighted equilibrium state of Φ. Now the theorem follows from Theorem 7.3.

Next we consider the case that X_i's are two-sided subshifts over finite alphabets $\mathcal{A}_i$'s. Again we may assume that π_i's are one-block factor maps. Define for $i = 1, \ldots, k$,

$$X_i^+ := \left\{ (x_j)_{j=1}^{\infty} \in \mathcal{A}_i^{\mathbb{N}} :\ \exists\, (y_j)_{j\in\mathbb{Z}} \in X_i \text{ such that } x_j = y_j \text{ for } j \geq 1 \right\}.$$

Then $(X_i^+, \sigma_{X_i^+})$ becomes a one-sided subshift for each i. Furthermore define $\Gamma_i : X_i \to X_i^+$ by $(x_j)_{j\in\mathbb{Z}} \mapsto (x_j)_{j\in\mathbb{N}}$. Then for each $1 \leq i \leq k$, the mapping $\mu \mapsto \mu \circ {\Gamma_i}^{-1}$ is a homeomorphism from $\mathcal{M}(X_i, \sigma_{X_i})$ to $\mathcal{M}(X_i^+, \sigma_{X_i^+})$ which preserves the measure theoretic entropy. Now $\pi_i : X_i^+ \to X_{i+1}^+$ becomes a one-block factor between one-sided subshifts for $i = 1, \ldots, k-1$. Let $f \in V(\sigma_{X_1})$. Define $\phi : \mathcal{L}(X_1^+) \to [0, \infty)$ by

$$\phi(I) = \sup_{x \in X_1 :\, x_1 \ldots x_n = I} \exp(S_n f(x)), \quad I \in \mathcal{L}_n(X_1^+),\ n \in \mathbb{N}.$$

Similarly, $\phi \in \mathcal{D}_w(X_1, p)$ for some $p \in \mathbb{N}$. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{sa}(X_1^+, \sigma_{X_1^+})$ be generated by ϕ. Due to $f \in V(\sigma_{X_1})$, we have

$$\int f \, d\mu = \Phi_*(\mu \circ \Gamma_1^{-1}), \qquad \mu \in \mathcal{M}(X_1, \sigma_{X_1}).$$

It follows that μ is an $\mathbf{a}$-weighted equilibrium state of f if and only if that, $\mu \circ \Gamma_1^{-1}$ is an $\mathbf{a}$-weighted equilibrium state of Φ. Thus the results of the theorem follow from Theorem 7.3. □

After this work, Yayama [39] independently obtained Proposition 3.7 and the formula of the $\mathbf{a}$-weighted topological pressure for $f = 0$.

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