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The Pointwise Densities of the Cantor Measure[1]

De-Jun Feng

Department of Mathematical Science and Center for Advanced Study, Tsinghua University, Beijing 100084, People's Republic of China
E-mail: dfeng@math.tsinghua.edu.cn

Su Hua

Department of Mathematical Science, Tsinghua University, Beijing 100084, People's Republic of China
E-mail: shua@math.tsinghua.edu.cn

and

Zhi-Ying Wen

Department of Mathematical Science, Tsinghua University, Beijing 100084, People's Republic of China; and Department of Mathematics, Wuhan University, Wuhan 430072, People's Republic of China
E-mail: wenzy@tsinghua.edu.cn

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Let $\mathscr{C}$ be the classical middle-third Cantor set and let $\mu_{\mathscr{C}}$ be the Cantor measure. Set $s = \log 2/\log 3$. We will determine by an explicit formula for every point $x \in \mathscr{C}$ the upper and lower s-densities $\Theta^{*s}(\mu_{\mathscr{C}}, x)$, $\Theta_*^s(\mu_{\mathscr{C}}, x)$ of the Cantor measure at the point x, in terms of the 3-adic expansion of x. We show that there exists a countable set $F \subset \mathscr{C}$ such that $9(\Theta^{*s}(\mu_{\mathscr{C}}, x))^{-1/s} + (\Theta_*^s(\mu_{\mathscr{C}}, x))^{-1/s} = 16$ holds for $x \in \mathscr{C} \setminus F$. Furthermore, for μ_C almost all x, $\Theta^{*s}(\mu_{\mathscr{C}}, x) = 2 \cdot 4^{-s}$ and $\Theta_*^s(\mu_{\mathscr{C}}, x) = 4^{-s}$. As an application, we will show that the s-dimensional packing measure of the middle-third Cantor set $\mathscr{C}$ is 4^s. © 2000 Academic Press

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1. INTRODUCTION

In this paper, we adopt the following terminologies and notations:

Let $0 \le t < \infty$ and let ν be a measure on $\mathbb{R}^n$. The *upper and lower t-densities* of ν at $a \in \mathbb{R}^n$ are defined respectively by

$$\Theta^{*t}(\nu, a) = \limsup_{r \downarrow 0} (2r)^{-t} \nu(B(a,r)),$$

$$\Theta^{t}_{*}(\nu, a) = \liminf_{r \downarrow 0} (2r)^{-t} \nu(B(a,r)),$$

where $B(a,r)$ denotes the closed ball with diameter $2r$ and center a.

We denote by $\mathscr{C}$ the middle-third Cantor set. That is, $\mathscr{C} = \{x = \sum_{i=1}^{\infty} x_i 3^{-i} : \forall i \ge 1,\ x_i = 0 \text{ or } 2\}$. Let $\mathscr{H}^t$ and $\mathscr{P}^t$ denote respectively the t-dimensional Hausdorff measure and packing measure; $\dim_H E$ and $\dim_P E$ denote respectively the Hausdorff and packing dimension of E.

It is known that $\dim_H \mathscr{C} = \dim_P \mathscr{C} = s$ where $s = \log 2/\log 3$. In what follows, we always assume $s = \log 2/\log 3$.

For the above definitions and related properties, we refer to [3].

Now consider similarity contractions ϕ_0, ϕ_1: $\mathbb{R} \to \mathbb{R}$ defined by $\phi_0(x) = \frac{x}{3}$ and $\phi_1(x) = \frac{2}{3} + \frac{x}{3}$. By [4] there exists a unique Borel probability measure $\mu_{\mathscr{C}}$ such that

$$\mu_{\mathscr{C}}(A) = \tfrac{1}{2}\mu_{\mathscr{C}}\big(\phi_0^{-1}(A)\big) + \tfrac{1}{2}\mu_C\big(\phi_1^{-1}(A)\big) \qquad (*)$$

for all Borel set A.

The measure $\mu_{\mathscr{C}}$ is a self-similar measure which we call a *Cantor measure*.

We summarize some properties of the Cantor measure $\mu_{\mathscr{C}}$ used later which can be found in Falconer [4].

1°. The support of $\mu_{\mathscr{C}}$ is $\mathscr{C}$, $\phi_0(\mathscr{C}) \cup \phi_1(\mathscr{C}) = \mathscr{C}$.

2°. $1 = \mathscr{H}^s(\mathscr{C}) \le \mathscr{P}^s(\mathscr{C}) < \infty$.

3°. $\mu_{\mathscr{C}} = \mathscr{H}^s|_{\mathscr{C}}$, where $\mathscr{H}^s|_{\mathscr{C}}$ is the restriction of the Hausdorff measure $\mathscr{H}^s$ over the set $\mathscr{C}$ (defined by $\mathscr{H}^s|_{\mathscr{C}}(A) = \mathscr{H}^s(A \cap \mathscr{C})$ for all $A \subset \mathbb{R}$).

4°. There exist $0 < d_* < d^* \le 1$ such that for $\mu_{\mathscr{C}}$-almost all $x \in \mathscr{C}$,

$$\Theta^{s}_{*}(\mu_{\mathscr{C}}, x) = d_* \qquad \text{and} \qquad \Theta^{*s}(\mu_{\mathscr{C}}, x) = d^*.$$

Notice that $d_* < d^*$ means that the ratio $\mu_{\mathscr{C}}(B(x,r))/(2r)^s$ oscillates between d_* and d^* when r is small. It is natural to try and describe this oscillation: the size of this oscillation, in particular, the exact value of these densities. Even if for the Cantor measure, the simplest self-similar measure, to our knowledge, the above questions are still open.

Bedford and Fisher [2] introduced another way, the average densities of a measure, to describe the oscillation by an average number; as an particular case, $\mu_{\mathscr{C}}$ was studied by Bedford and Fisher [2], Patzschke and Zähle [10], and Falconer [5]. Graf [7] and Krieg and Moerters [8] dealt with generalizations of average density approach.

Another type of density of $\mu_{\mathscr{C}}$, *maximum density*, was introduced and studied by Strichartz et al. [11] and Ayer and Strichartz [1].

In this paper, we will determine d_*, d^*, and the values of $\Theta^s_*(\mu_{\mathscr{C}}, x)$, $\Theta^{*s}(\mu_{\mathscr{C}}, x)$ for any $x \in \mathscr{C}$. As an application, we will prove that the s-dimensional packing measure of $\mathscr{C}$ is equal to 4^s.

For $x \in \mathscr{C}$, let $x = \sum_{i=1}^{\infty} x_i 3^{-i}$ ($x_i = 0$ or 2) be the 3-adic decimal expansion of x. We say that x is a *finite* 3*-adic decimal* if $x_i \equiv 0$ or $x_i \equiv 2$ for all large enough i. Define $\hat{\tau}(x) = \liminf_{k\to\infty} \sum_{i=1}^{\infty} x_{i+k} 3^{-i}$ and $\tau(x) = \min\{\hat{\tau}(x), \hat{\tau}(1-x)\}$. Then we can formulate our results as follows:

THEOREM 1.1. (i) *For any* $x \in \mathscr{C}$,

$$\Theta^s_*(\mu_{\mathscr{C}}, x) = (4 - 6\tau(x))^{-s}.$$

(ii) *For any* $x \in \mathscr{C}$

$$\Theta^{*s}(\mu_{\mathscr{C}}, x) = \begin{cases} 2^{-s} & \text{if } x \text{ is a finite 3-adic decimal}, \\ \left(\dfrac{4 + 2\tau(x)}{3}\right)^{-s} & \text{otherwise} \end{cases}$$

(iii) *If* $x \in \mathscr{C}$ *is not a finite* 3*-adic decimal, then*

$$9(\Theta^{*s}(\mu, x))^{-1/s} + (\Theta^s_*(\mu, x))^{-1/s} = 16.$$

(iv) $\sup\{\Theta^{*s}(\mu_{\mathscr{C}}, x) - \Theta^s_*(\mu_{\mathscr{C}}, x) : x \in \mathscr{C}\} = 4^{-s} \approx 0.41701$, *where the supremum can be attained at* $\{x \in \mathscr{C} : \tau(x) = 0\}$, *and* $\inf\{\Theta^{*s}(\mu_{\mathscr{C}}, x) - \Theta^s_*(\mu_{\mathscr{C}}, x) : x \in \mathscr{C}\} = (\frac{3}{2})^{-s} - (\frac{5}{2})^{-s} \approx 0.21333$, *where the infimum can be attained at* $\{x \in \mathscr{C} : \tau(x) = 1/4\}$.

(v) *For* $\mu_{\mathscr{C}}$*-almost all* $x \in \mathscr{C}$,

$$\Theta^s_*(\mu_{\mathscr{C}}, x) = 4^{-s}, \qquad \Theta^{*s}(\mu_{\mathscr{C}}, x) = 2 \cdot 4^{-s}.$$

THEOREM 1.2. $\mathscr{P}^s(\mathscr{C}) = 4^s$.

We should point out that our method can be used to determine the upper and lower densities of the center Cantor-type measure μ_ρ $(0 < \rho < 1/3)$ at every point $x \in [0,1]$, where μ_ρ satisfies the equation

$$\mu_\rho(A) = \tfrac{1}{2}\mu_\rho\big(\phi_{0,\rho}^{-1}(A)\big) + \tfrac{1}{2}\mu_\rho\big(\phi_{1,\rho}^{-1}(A)\big), \qquad \forall A \subset [0,1],$$

where $\phi_{0,\rho}, \phi_{1,\rho}: \mathbb{R} \to \mathbb{R}$ are defined by $\phi_{0,\rho}(x) = \rho x$, and $\phi_{1,\rho}(x) = \rho x + (1-\rho)$.

2. PROOF OF THEOREM 1.1

At first we prove some lemmas.

LEMMA 2.1. *For any Borel set $A \subset (-1,2)$, and $i_1, \dots, i_k \in \{0,1\}$, we have*

$$\mu_{\mathscr{C}}\big(\phi_{i_1} \circ \cdots \circ \phi_{i_k}(A)\big) = 2^{-k}\mu_{\mathscr{C}}(A),$$

where $\mu_{\mathscr{C}}, \phi_0, \phi_1$ are defined as in Section 1:

Proof. Since $A \subset (-1,2)$, $\phi_0(A) \subset (\frac{-1}{3}, \frac{2}{3})$, $\phi_1(A) \subset (\frac{1}{3}, \frac{4}{3})$. So $\phi_0(A) \cap \phi_1(\mathscr{C}) = \emptyset$ and $\phi_1(A) \cap \phi_0(\mathscr{C}) = \emptyset$. Therefore $\phi_1^{-1}(\phi_0(A)) \cap \mathscr{C} = \emptyset$ and $\phi_0^{-1}(\phi_1(A)) \cap \mathscr{C} = \emptyset$. By $(*)$, we see that

$$\mu_{\mathscr{C}}\big(\phi_0(A)\big) = \tfrac{1}{2}\Big(\mu_{\mathscr{C}}(A) + \mu_{\mathscr{C}}\big(\phi_1^{-1}(\phi_0(A))\big)\Big) = \tfrac{1}{2}\mu_{\mathscr{C}}(A),$$

$$\mu_{\mathscr{C}}\big(\phi_1(A)\big) = \tfrac{1}{2}\Big(\mu_{\mathscr{C}}(A) + \mu_{\mathscr{C}}\big(\phi_0^{-1}(\phi_1(A))\big)\Big) = \tfrac{1}{2}\mu_{\mathscr{C}}(A);$$

then by induction, we get the conclusion of the lemma. ∎

LEMMA 2.2. *For any* $0 \le t \le 1$, *we have* $\mu_{\mathscr{C}}([0,t]) \ge \frac{1}{2}t^s$.

Proof. (1) By $(*)$, $\mu_{\mathscr{C}}([0,\frac{1}{3}]) = \frac{1}{2}$; thus if $1 \ge t \ge \frac{1}{3}$, then

$$\mu_{\mathscr{C}}([0,t]) \ge \mu_{\mathscr{C}}\big([0,\tfrac{1}{3}]\big) = \tfrac{1}{2} \ge \tfrac{1}{2}t^s.$$

(2) If $0 < t < \frac{1}{3}$, take $k \in \mathbb{N}$ such that $3^{-k-1} \le t < 3^{-k}$; then $[0, 3^k t] \subset (-1,2)$. Notice that $\phi_0^k([0,3^k t]) = [0,t]$; by Lemma 2.1, $\mu_{\mathscr{C}}([0,t]) = 2^{-k}\mu_{\mathscr{C}}([0,3^k t])$. We get thus, from (1) and the fact that $3^k t \ge \frac{1}{3}$,

$$\mu_{\mathscr{C}}([0,t]) = 2^{-k}\mu_{\mathscr{C}}\big([0,3^k t]\big) \ge 2^{-k-1}(3^k t)^s = \tfrac{1}{2}t^s.$$

∎

LEMMA 2.3. *For any* $0 \le t \le 1$, $\mu_{\mathscr{C}}([0,t]) \le t^s$.

Proof. Since $\mu_{\mathscr{C}}$ is supported by $\mathscr{C}$, we only need to prove the above inequality holds for $t \in \mathscr{C}$. In fact, if $t \notin \mathscr{C}$, let $t^* := \sup\{x \in \mathscr{C}, x \le t\}$. Then $t^* \in \mathscr{C}$; thus $\mu_{\mathscr{C}}([0,t]) = \mu_{\mathscr{C}}([0,t^*]) \le t^{*s} < t^s$.

Let $t \in \mathscr{C}$; take $k \in \mathbb{N}$ such that $3^{-k-1} \le t < 3^{-k}$.

(1) If $t = 3^{-k-1}$, by a simple calculation, we have $\mu_{\mathscr{C}}([0,t]) = t^s$.

Now assume $3^{-k-1} < t < 3^{-k}$; by the construction of $\mathscr{C}$, $t \geq 2 \cdot 3^{-k-1}$. We have thus

$$\begin{aligned}\mu_{\mathscr{C}}([0,t]) &= \mu_{\mathscr{C}}([0,3^{-k-1}]) + \mu_{\mathscr{C}}([2\cdot 3^{-k-1}, t]) \\ &= \mu_{\mathscr{C}}([0,3^{-k-1}]) + \mu_{\mathscr{C}}([0, t - 2\cdot 3^{-k-1}]) \\ &= 3^{(-k-1)s} + \mu_{\mathscr{C}}([0, t-2\cdot 3^{-k-1}]).\end{aligned}$$

Let $t_1 = t - 2\cdot 3^{(-k-1)}$; we have $t^s \geq 3^{(-k-1)s} + (t - 2\cdot 3^{(-k-1)})^s$ (in general, if $x \geq y > 0$, then $(2x+y)^s \geq x^s + y^s$ holds); thus $t^s - \mu_{\mathscr{C}}([0,t]) \geq t_1^s - \mu_{\mathscr{C}}([0,t_1])$.

(2) Since $t \in \mathscr{C}$, $t_1 \in \mathscr{C}$. Take $k_1 \in \mathbb{N}$ such that $3^{-k_1-1} \leq t_1 < 3^{-k_1}$. Clearly $k_1 > k$.

By the same discussion as in (1), we see that, if $t_1 = 3^{-k_1-1}$, then $\mu_{\mathscr{C}}([0,t_1]) = t_1^s$ and in this case $t_1^s - \mu_{\mathscr{C}}([0,t_1]) = 0$. If $t_1 > 3^{-k_1-1}$, let $t_2 = t_1 - 2\cdot 3^{-k_1-1}$; then

$$t^s - \mu_{\mathscr{C}}([0,t]) \geq t_1^s - \mu_{\mathscr{C}}([0,t_1]) \geq t_2^s - \mu_{\mathscr{C}}([0,t_2]).$$

(3) Repeat the above discussions. We see that either $t^s - \mu_{\mathscr{C}}([0,t[)$ ≥ 0 or, for any $m \in \mathbb{N}$,

$$t^s - \mu_{\mathscr{C}}([0,t]) \geq t_m^s - \mu_{\mathscr{C}}([0,t_m]) \geq -\mu_{\mathscr{C}}([0,t_m]).$$

Since $t_m \to 0$, $\mu_{\mathscr{C}}([0,t_m]) \to 0$ when $m \to \infty$, we get finally $t^s - \mu_{\mathscr{C}}([0,t])$ ≥ 0. ∎

LEMMA 2.4. *For any $x \in [0, \frac{1}{3}]$ and $a > 0$, define*

$$f_{x,a}(t) = \frac{\frac{1}{2} + at^s}{\left(\frac{2}{3} - x + t\right)^s}.$$

Then

(1) *On the interval $[0, \frac{1}{3}]$, the function $f_{x,a}(t)$ attains its minimum at $t = 0$ or $t = \frac{1}{3}$. In particular, $f_{x,a}(t)$ attains its minimum at $t = 0$ if $a \geq 2^{-s}$.*

(2) *If $a = 1$, the function $f_{x,a}(t)$ increases strictly on the interval $[0, \frac{1}{3}]$.*

(3) *If $x = 0$ and $a \geq \frac{1}{2}$, the function $f_{x,a}(t)$ increases strictly on the interval $[0, \frac{1}{3}]$.*

Proof. The conclusions of the lemma can be obtained by an elementary discussion. ∎

LEMMA 2.5. *For any* $r \in [0,1]$, $\mu_{\mathscr{C}}([0,r]) \geq 2^{-s} r^s$.

Proof. (1) If $\frac{1}{3} \leq r \leq \frac{2}{3}$, $\mu_{\mathscr{C}}([0,r]) = \mu_{\mathscr{C}}([0,\frac{1}{3}]) = \frac{1}{2}$; consequently

$$\frac{\mu_{\mathscr{C}}([0,r])}{r^s} \geq \frac{1}{2}\left(\frac{2}{3}\right)^{-s} = 2^{-s}.$$

(2) If $\frac{2}{3} < r \leq 1$, let $t = r - \frac{2}{3}$; then $0 \leq t \leq \frac{1}{3}$. Thus

$$\frac{\mu_{\mathscr{C}}([0,r])}{r^s} = \frac{\mu_{\mathscr{C}}\left([0,\frac{1}{3}]\right) + \mu_{\mathscr{C}}\left([\frac{2}{3},\frac{2}{3}+t]\right)}{\left(\frac{2}{3}+t\right)^s}$$

$$= \frac{\mu_{\mathscr{C}}\left([0,\frac{1}{3}]\right) + \mu_{\mathscr{C}}([0,t])}{\left(\frac{2}{3}+t\right)^s}.$$

By Lemmas 2.2 and 2.4(3), we have

$$\frac{\mu_{\mathscr{C}}([0,r])}{r^s} \geq \frac{\frac{1}{2}+\frac{1}{2}t^s}{\left(\frac{2}{3}+t\right)^s} \geq \frac{1}{2}\left(\frac{2}{3}\right)^{-s} = 2^{-s}.$$

(3) If $3^{-k-1} \leq r \leq 3^{-k}$ holds for some positive integer k, then by Lemma 2.1, we have

$$\frac{\mu_{\mathscr{C}}([0,r])}{r^s} = \frac{\mu_{\mathscr{C}}([0,3^k r])}{(3^k r)^s},$$

thus by (1), (2), we get

$$\frac{\mu_{\mathscr{C}}([0,r])}{r^s} \geq 2^{-s}.$$

∎

LEMMA 2.6. *Let* $x \in [0,1/3]$; *then for any r with* $\max\{x, \frac{1}{3}-x\} \leq r \leq 1 - x$, *we have*

$$\frac{\mu_{\mathscr{C}}([x-r,x+r])}{(2r)^s} \geq (4-6x)^{-s},$$

where the equality holds at $r = \frac{2}{3} - x$.

Proof. (1) If $\max\{x, \frac{1}{3}-x\} \leq r \leq \frac{2}{3} - x$, then $[0,\frac{1}{3}] \subset [x-r, x+r] \subset [-\frac{2}{3}, \frac{2}{3}]$, so $\mu_{\mathscr{C}}([x-r,x+r]) = \mu_{\mathscr{C}}([0,\frac{1}{3}]) = \frac{1}{2}$; hence

$$\frac{\mu_{\mathscr{C}}([x-r,x+r])}{(2r)^s} = \frac{1}{2}(2r)^{-s} = (6r)^{-s} \geq (4-6x)^{-s},$$

where the equality holds at $r = \frac{2}{3} - x$.

(2) If $\frac{2}{3} - x < r \leq 1 - x$, let $r - (\frac{2}{3} - x)$; then $0 < t \leq \frac{1}{3}$. By Lemmas 2.5 and 2.4(1), we have

$$\frac{\mu_{\mathscr{C}}([x - r, x + r])}{(2r)^s} = \frac{\frac{1}{2} + \mu_{\mathscr{C}}([0, t])}{2^s(\frac{2}{3} - x + t)^s}$$

$$\geq \frac{\frac{1}{2} + 2^{-s}t^s}{2^s(\frac{2}{3} - x + t)^s} > \frac{\frac{1}{2}}{2^s(\frac{2}{3} - x)^s} = (4 - 6x)^{-s}.$$

∎

DEFINITION 2.7. Define $T: \mathscr{C} \to \mathscr{C}$ by

$$T(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq \frac{1}{3} \\ 3x - 2 & \text{if } \frac{2}{3} \leq x \leq 1. \end{cases}$$

For any $x \in \mathscr{C}$, define $\hat{\tau}(x) := \liminf_{k \to \infty} T^k(x)$ and $\tau := \min\{\hat{\tau}(x), \hat{\tau}(1 - x)\}$, where T^k is the kth iteration of T.

Remark 2.8. Since any $x \in \mathscr{C}$ can be written as $x = \sum_{i=1}^{\infty} x_i 3^{-i}$ ($x_i = 0$ or 2), it follows that under the above definition, we have $T(x) = \sum_{i=1}^{\infty} x_{i+1} 3^{-i}$.

PROPOSITION 2.9. *For any* $x \in \mathscr{C}$, $0 \leq \tau(x) \leq 1/4$, *and* $\tau(y) = 1/4$ *for* $y \in V$, *where*

$$V = \left\{ x = \sum_{i=1}^{\infty} x_i 3^{-i} \in \mathscr{C} : \exists l \geq 0,\ x_{l+2k} = 0,\ x_{l+2k+1} = 2 \text{ for any } k \geq 0 \right\}.$$

Proof. By the definition of T and a direct check, $\tau(y) = 1/4$ for $y \in V$.

If $x = \sum_{i=1}^{\infty} x_i 3^{-i} \notin V$ ($x_i = 0$ or 2), then there exist finitely many blocks 00 or 22 in the sequence (x_i). Suppose that $x_j, x_{j+1} = 0$ for some $j > 1$; by Remark 2.8,

$$T^{j-1}(x) = \sum_{i=1}^{\infty} x_{i+j-1} 3^{-i} \leq \sum_{i=3}^{\infty} 2 \cdot 3^{-i} = 1/9.$$

Similarly, $T^{j-1}(1 - x) \leq 1/9$ if $x_j, x_{j+1} = 2$ for some $j > 1$. Therefore, $\tau(x) \leq 1/9$ when $x \notin V$. ∎

PROPOSITION 2.10. *For* $\mu_{\mathscr{C}}$*-almost all* $x \in \mathscr{C}$, $\tau(x) = 0$.

We remark that this proposition follows easily by using the law of large numbers. In the following we will prove it in another direct way.

Proof. Let $l \geq 2$ be an integer. For any $i_1, \ldots, i_l \in \{0, 1\}$, denote $S_{i_1 \cdots i_l} = \phi_{i_1} \circ \cdots \circ \phi_{i_l}$. It is clear that $S_{i_1 \cdots i_l}$ is a contracting similarity with ratio 3^{-l}. Moreover, these 2^l contracting similarities satisfy the open set condition (in fact, they generate the Cantor set $\mathscr{C}$).

Set

$$B_l = \left\{ x = \sum_{i=1}^{\infty} x_i 3^{-i} \in \mathscr{C} : \forall m \geq 0,\ x_{ml+1} \cdots x_{(m+1)l} \neq \underbrace{0 \cdots 0}_{l} \right\};$$

then $B_l \subset \mathscr{C}$ is the self-similar set generated by $2^l - 1$ contracting similarities $S_{i_1 \cdots i_l} : i_1 \cdots i_l \neq 0 \cdots 0$. Thus, by [3], the Hausdorff dimension of the set B_l is

$$\dim_H B_l = \frac{\log(2^l - 1)}{\log(3^l)} < \frac{\log 2}{\log 3} = s,$$

from which it follows that $\mathscr{H}^s(B_l) = 0$. Thus $\mu_{\mathscr{C}}(B_l) = \mathscr{H}^s(B_l \cap \mathscr{C}) = 0$; consequently $\mu_{\mathscr{C}}(\bigcup_{l \geq 1} B_l) = 0$. On the other hand, for any $x \in \mathscr{C} \setminus \bigcup_{l \geq 1} B_l$, $x \in \mathscr{C}$, it is ready to verify $\liminf_{k \to \infty} T^k(x) = 0$; thus $\tau(x) = 0$. We thus finish the proof of the proposition. ∎

Proof of Theorem 1.1(i). Given $x \in \mathscr{C}$ and $0 < r < \frac{1}{3}$, then there exists a sequence $\{i_k\}_{k \geq 1}$ taking the values 0 and 1 such that

$$x = \lim_{k \to \infty} \phi_{i_1} \circ \cdots \circ \phi_{i_k}([0, 1]).$$

Choose the positive integer k such that $[x - r, x + r]$ contains the interval $\phi_{i_1} \circ \cdots \circ \phi_{i_k}([0, 1])$, but does not contain the interval $\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}}([0, 1])$. Thus $(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}([x - r, x + r])$ contains the interval $\phi_{i_k}([0, 1])$, but does not contain $[0, 1]$; therefore

$$(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}([x - r, x + r]) \subset (-1, 2).$$

Let $y = (\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}(x)$; then by the definition of T, $y = T^{k-1}(x)$. Let $r' = 3^{k-1} r$; then $0 < r' < 1$. Moreover

$$(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}([x - r, x + r]) = [y - r', y + r'].$$

By Lemma 2.1, $\mu_{\mathscr{C}}([x - r, x + r]) = 3^{-(k-1)s} \mu_{\mathscr{C}}([y - r', y + r'])$; consequently

$$\frac{\mu_{\mathscr{C}}([x - r, x + r])}{(2r)^s} = \frac{\mu_{\mathscr{C}}([y - r', y + r'])}{(2r')^s}. \tag{1}$$

(1) If $y \in [0, \frac{1}{3}]$, then the interval $[y - r', y + r']$ contains $\phi_{i_k}([0, 1]) = [0, \frac{1}{3}]$, but does not contain $[0, 1]$, so $\max\{y, \frac{1}{3} - y\} \leq r' \leq 1 - y$. By

Lemma 2.6, we have

$$\frac{\mu_{\mathscr{C}}([y-r',y+r'])}{(2r')^s} \geq (4-6y)^{-s}, \tag{2}$$

where the equality holds if $r' = \frac{2}{3} - y$.

(2) If $y \in [\frac{2}{3}, 1]$, by the symmetry of $\mathscr{C}$, we have always

$$\mu_{\mathscr{C}}([y-r',y+r']) = \mu_{\mathscr{C}}([1-y-r',1-y+r']);$$

thus by the inequality (2), we have

$$\frac{\mu_{\mathscr{C}}([y-r',y+r'])}{(2r')^s} \geq (4-6(1-y))^{-s}, \tag{3}$$

where the equality holds if $r' = \frac{2}{3} - (1-y)$.

Notice that in both cases (1) and (2), $y = T^{k-1}(x)$ and $1 - y = T^{k-1}(1 - x)$; thus from (1), (2), and (3),

$$\begin{aligned}
\liminf_{r\to 0} &\frac{\mu_{\mathscr{C}}([x-r,x+r])}{(2r)^s} \\
&\geq \left(4 - 6\min\left\{\liminf_{k\to\infty} T^k(x),\ \liminf_{k\to\infty} T^k(1-x)\right\}\right)^{-s} \\
&= (4-6\tau(x))^{-s}.
\end{aligned}$$

Since the equalities can hold in (2) and (3), we get finally

$$\liminf_{r\to 0} \frac{\mu_{\mathscr{C}}([x-r,x+r])}{(2r)^s} = (4-6\tau(x))^{-s},$$

which implies immediately the conclusion of Theorem 1.1(i). ∎

Now we are going to prove Theorem 1.1(ii). We first prove some lemmas.

LEMMA 2.11. *Given* $x \in [0, \frac{1}{3}]$, *then on the interval* $[\max\{x, \frac{1}{3} - x\}, 1 - x]$, *the function* $\mu_{\mathscr{C}}([x-r, x+r])(2r)^{-s}$ *attains its maximum either at* $r = \max\{x, \frac{1}{3} - x\}$ *or at* $r = 1 - x$. *Moreover, the maximum is*

$$\max\left\{\frac{1}{2^s(\max\{3x, 1-3x\})^s}, \frac{1}{2^s(1-x)^s}\right\}.$$

Proof. (1) If $\max\{x, \frac{1}{3} - x\} \leq r \leq \frac{2}{3} - x$, we have $\mu_{\mathscr{C}}([x-r, x+r]) = \mu_{\mathscr{C}}([0, \frac{1}{3}]) = \frac{1}{2}$; thus the function $\mu_{\mathscr{C}}([x-r, x+r])(2r)^{-s}$ decreases strictly on the interval $[\max\{x, \frac{1}{3} - x\}, \frac{2}{3} - x]$, so the function attains its maximum at $r = \max\{x, \frac{1}{3} - x\}$ with maximum $(2^s(\max\{3x, 1 - 3x\})^s)^{-1}$.

(2) If $\frac{2}{3} - x < r \le 1 - x$, let $t = r - (\frac{2}{3} - x)$; then $0 < t \le \frac{1}{3}$. By Lemma 2.3, we have

$$\mu_{\mathscr{C}}([x - r, x + r]) = \mu_{\mathscr{C}}\left([0, \tfrac{1}{3}]\right) + \mu_{\mathscr{C}}([0, t]) \le \tfrac{1}{2} + t^s;$$

thus by Lemma 2.4(2), we have

$$\frac{\mu_{\mathscr{C}}([x - r, x + r])}{(2r)^s} \le \frac{\frac{1}{2} + t^s}{2^s\left(\frac{2}{3} - x + t\right)^s} \le \frac{\frac{1}{2} + 3^{-s}}{2^s\left(\frac{2}{3} - x + \frac{1}{3}\right)^s}$$
$$= \frac{1}{2^s(1 - x)^s},$$

where the equality holds at $r = 1 - x$.

From (1) and (2), we get the lemma. ∎

DEFINITION 2.12. We define the function $p\colon \mathscr{C} \to \mathbb{R}$ by

$$p(x) = \max\left\{\frac{1}{2^s(\max\{3x, 1 - 3x\})^s}, \frac{1}{2^s(\max\{x, 1 - x\})^s}\right\}$$

if $x \in [0, 1/3]$, and $p(x) = p(1 - x)$ if $x \in [\frac{2}{3}, 1]$.

LEMMA 2.13. *For any* $x \in \mathscr{C}$, $\Theta^{*s}(\mu_{\mathscr{C}}, x) = \limsup_{k \to \infty} p(T^k x)$.

Proof. Given $x \in \mathscr{C}$ and $0 < r < \frac{1}{3}$, choose $k \in \mathbb{N}$ such that $[x - r, x + r]$ contains an interval $\phi_{i_1} \circ \cdots \circ \phi_{i_k}([0, 1])$, but does not contain the interval $\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}}([0, 1])$. Then $(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}([x - r, x + r])$ contains $\phi_{i_k}([0, 1])$ and does not contain $[0, 1]$, which implies that

$$\left(\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}}\right)^{-1}([x - r, x + r]) \subset (-1, 2).$$

Let $y = (\phi_{i_1} \circ \cdots \circ \phi_{i_{k-1}})^{-1}(x)$ and $r' = 3^{k-1}r$; then $y = T^{k-1}(x)$ and $0 < r' < 1$. By Lemma 2.1 and a direct calculation, we have

$$\frac{\mu_{\mathscr{C}}([x - r, x + r])}{(2r)^s} = \frac{\mu_{\mathscr{C}}([y - r', y + r'])}{(2r')^s}.$$

(1) If $y \in [0, \frac{1}{3}]$, then $[y - r', y + r']$ contains $\phi_{i_k}([0, 1]) = [0, \frac{1}{3}]$, but does not contain $[0, 1]$, so $\max\{y, \frac{1}{3} - y\} \le r' \le 1 - y$. By Lemma 2.11 and

the definition of the function $p(\cdot)$, we have

$$\frac{\mu_{\mathscr{C}}([y-r', y+r'])}{(2r')^s} \leq p(y), \tag{4}$$

where the equality holds for $r' = \max\{y, \frac{1}{3} - y\}$ or $1 - y$.

(2) If $y \in [\frac{2}{3}, 1]$, by the symmetry of $\mathscr{C}$, we have always

$$\mu_{\mathscr{C}}([y-r', y+r']) = \mu_{\mathscr{C}}([1-y-r', 1-y+r']);$$

thus by the inequality (4), we have

$$\frac{\mu_{\mathscr{C}}([y-r', y+r'])}{(2r')^s} \leq p(1-y) = p(y), \tag{5}$$

where the equality holds at $r' = \max\{1-y, \frac{1}{3} - (1-y)\}$ or y.

From (1), (2), and the fact that $y = T^k(x)$ we get the conclusion of the lemma. ∎

Proof of Theorem 1.1(ii). (1) If $x \in \mathscr{C}$ is a finite 3-adic decimal, then there exists $l \in \mathbb{N}$ such that $T^k(x) = 0$ or 1 when $k \geq l$; consequently $p(T^k(x)) = 2^{-s}$ when $k \geq l$, so by Lemma 2.13, we have $\Theta^{*s}(\mu_{\mathscr{C}}, x) = 2^{-s}$.

(2) In the other case, there exist infinitely many $k \in \mathbb{N}$ such that $T^k(x) \in [0, 1/3]$, and infinitely many $l \in \mathbb{N}$ such that $T^l(x) \in [2/3, 1]$. From the definition of $p(x)$, we have

$$\begin{aligned}\limsup_{k\to\infty} p\big(T^k(x)\big) &= \limsup_{k\to\infty} \frac{1}{2^s\big(\max\{T^k(x), T^k(1-x)\}\big)^s} \\ &= 2^{-s}\Big(\liminf_{k\to\infty} \max\{T^k(x), T^k(1-x)\}\Big)^{-s}.\end{aligned}$$

Let

$$\Omega_1 = \big\{k \in \mathbb{N} : T^k(x) \in \big[0, \tfrac{1}{3}\big]\big\}, \qquad \Omega_2 = \big\{k \in \mathbb{N} : T^k(x) \in \big[\tfrac{2}{3}, 1\big]\big\};$$

then

$$\begin{aligned}&\liminf_{k\to\infty} \max\{T^k(x), T^k(1-x)\} \\ &\quad = \min\Big\{\liminf_{k\in\Omega_1, k\to\infty} T^k(1-x), \liminf_{k\in\Omega_2, k\to\infty} T^k(x)\Big\}.\end{aligned}$$

Notice that $T^k(1-x) = \frac{2}{3} + \frac{1}{3}T^{k+1}(1-x)$ if $k \in \Omega_1$, and $T^k(x) = \frac{2}{3} + \frac{1}{3}T^{k+1}(x)$ if $k \in \Omega_2$; thus

$$\liminf_{k\in\Omega_1, k\to\infty} T^k(1-x) = \frac{1}{3}\Big(\liminf_{k\in\Omega_1, k\to\infty} T^{k+1}(1-x)\Big) + \frac{2}{3}$$
$$= \frac{1}{3}\Big(\liminf_{k\to\infty} T^k(1-x)\Big) + \frac{2}{3}.$$

By the same way, we have

$$\liminf_{k\in\Omega_2, k\to\infty} T^k(x) = \frac{1}{3}\Big(\liminf_{k\to\infty} T^k(x)\Big) + \frac{2}{3}.$$

We get therefore

$$\min\Big\{\liminf_{k\in\Omega_1, k\to\infty} T^k(1-x), \liminf_{k\in\Omega_2, k\to\infty} T^k(x)\Big\}$$
$$= \frac{1}{3}\min\Big\{\liminf_{k\to\infty} T^k(x), \liminf_{k\to\infty} T^k(1-x)\Big\} + \frac{2}{3}$$
$$= \frac{1}{3}\min\{\hat{\tau}(x), \hat{\tau}(1-x)\} + \frac{2}{3} = \frac{1}{3}\tau(x) + \frac{2}{3}.$$

By the above discussions, we get

$$\limsup_{k\to\infty} p(T^k(x) = 2^{-s}\left(\frac{\tau(x)+2}{3}\right)^{-s} = \left(\frac{2\tau(x)+4}{3}\right)^{-s},$$

which yields finally from Lemma 2.13

$$\Theta^{*s}(\mu_{\mathscr{C}}, x) = \left(\frac{4+2\tau(x)}{3}\right)^{-s}.$$

∎

Proof of Theorem 1.1(iii), (iv), (v). It is clear that part (iii) of Theorem 1.1 is the direct corollary of the parts (i) and (ii), and part (iv) is the corollary of Proposition 2.9, parts (i) and (ii); part (v) is the corollary of Proposition 2.10, parts (i) and (ii). ∎

3. PROOF OF THEOREM 1.2

LEMMA 3.1. *For any Borel set $A \subset \mathbb{R}$, we have*

$$\mathscr{P}^s|_{\mathscr{C}}(A) = \mathscr{P}^s(\mathscr{C})\mu_{\mathscr{C}}(A).$$

Proof. Let $\mathscr{A} = \{\text{Borel set } A \subset \mathscr{C} : \mathscr{P}^s|_{\mathscr{C}}(A) = \mathscr{P}^s(\mathscr{C})\mu_{\mathscr{C}}(A)\}$. From the scaling property of $\mathscr{P}^s$ and $\mathscr{H}^s$ (that is, for any $\lambda > 0$ and $E \subset \mathbb{R}^n$, $\mathscr{P}^s(\lambda E) = \lambda^s \mathscr{P}^s(E)$, $\mathscr{H}^s(\lambda E) = \lambda^s \mathscr{H}^s(E)$), and the facts $\mu_{\mathscr{C}} = \mathscr{H}^s|_{\mathscr{C}}$, $\mu_{\mathscr{C}}(\mathscr{C}) = 1$, it is easy to prove that for any $k \in \mathbb{N}$ and $i_1, \dots, i_k \in \{0, 1\}$,

$$\phi_{i_1} \circ \cdots \circ \phi_{i_k}([0,1]) \cap \mathscr{C} \in \mathscr{A}.$$

Now set

$$\mathscr{E} = \{\emptyset\} \cup \Big\{\text{the finite union of sets of form } \phi_{i_1} \circ \cdots \circ \phi_{i_k}([0,1]) \cap \mathscr{C} : k \in \mathbb{N}\Big\}.$$

Then $\mathscr{E}$ has *finite intersection property*; that is, $A, B \in \mathscr{E} \Rightarrow A \cap B \in \mathscr{E}$. Moreover, the least σ-algebra generated by $\mathscr{E}$, denoted by $\sigma(\mathscr{E})$, contains all Borel subsets of $\mathscr{C}$.

On the other hand, it is easy to verify that $\mathscr{A}$ is a λ-class (i.e., $A, B \in \mathscr{A}$, $B \subset A \Rightarrow A \setminus B \in \mathscr{A}$, and $A_i \in \mathscr{A}$, $A_i \uparrow A$ or $A_i \downarrow A \Rightarrow A \in \mathscr{A}$). Thus from the monotone class theorem (see, for example, Feller [6]), $A \supset \sigma(\mathscr{E})$. Since $\sigma(\mathscr{E}) \supset \mathscr{A}$, we get $\mathscr{A} = \sigma(\mathscr{E})$, which contains all Borel subsets of $\mathscr{C}$. ∎

LEMMA 3.2 [9]. *Let $A \subset \mathbb{R}^n$ be a Borel set. If $\mathscr{P}^t(A) < \infty$, then for $\mathscr{P}^t|_A$-almost all $x \in \mathbb{R}^n$, we have $\Theta^t_*(\mathscr{P}^t|_A, x) = 1$.*

Proof of Theorem 1.2. From Lemma 3.1, for any $x \in \mathscr{C}$,

$$\Theta^s_*(\mathscr{P}^s|_{\mathscr{C}}, x) = \mathscr{P}^s(\mathscr{C})\Theta^s_*(\mu_{\mathscr{C}}, x);$$

thus from Theorem 1.1(v), for $\mu_{\mathscr{C}}$-almost all $x \in \mathbb{R}$, we have

$$\Theta^s_*(\mathscr{P}^s|_{\mathscr{C}}, x) = 4^{-s}\mathscr{P}^s(\mathscr{C}).$$

Consequently for $\mathscr{P}^s|_{\mathscr{C}}$-almost all $x \in \mathbb{R}$,

$$\Theta^s_*(\mathscr{P}^s|_{\mathscr{C}}, x) = 4^{-s}\mathscr{P}^s(\mathscr{C}).$$

On the other hand, since $\mathscr{P}^s(\mathscr{C}) < \infty$, by Lemma 3.2, we have for $\mathscr{P}^s|_{\mathscr{C}}$-almost all $x \in \mathbb{R}$, $\Theta^s_*(\mathscr{P}^s|_{\mathscr{C}}, x) = 1$; thus $4^{-s}\mathscr{P}^s(\mathscr{C}) = 1$, which yields the conclusion of the theorem. ∎

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