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LYAPUNOV EXPONENTS FOR PRODUCTS OF MATRICES AND MULTIFRACTAL ANALYSIS. PART I: POSITIVE MATRICES

DE-JUN FENG

ABSTRACT. Let (Σ, σ) be a full shift space on an alphabet consisting of m symbols and let $M: \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ be a continuous function taking values in the set of $d \times d$ positive matrices. Denote by $\lambda_M(x)$ the upper Lyapunov exponent of M at x. The set of possible Lyapunov exponents is just an interval. For any possible Lyapunov exponent α, we prove the following variational formula

$$\begin{aligned}\dim\{x \in \Sigma\colon \lambda_M(x) = \alpha\} &= \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P_M(q)\} \\ &= \frac{1}{\log m} \max_{\mu} \{h(\mu)\colon\ M_*(\mu) = \alpha\},\end{aligned}$$

where dim is the Hausdorff dimension or the packing dimension, $P_M(q)$ is the pressure function of M, μ is a σ-invariant Borel probability measure on Σ, $h(\mu)$ is the entropy of μ, and $M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \|M(y)M(\sigma y)\ldots M(\sigma^{n-1}y)\| d\mu(y)$.

1. Introduction

Let σ be the shift map on $\Sigma = \{1, 2, \ldots, m\}^{\mathbb{N}}$ ($m \geq 2$ an integer). Let M be a continuous function defined on Σ taking values in $L^+(\mathbb{R}^d, \mathbb{R}^d)$, the set of $d \times d$ matrices with positive entries. We define the **upper Lyapunov exponent** $\lambda_M(x)$ of M by

$$\lambda_M(x) = \lim_{n\to\infty} \frac{1}{n} \log \|M(x)M(\sigma x)\ldots M(\sigma^{n-1}x)\|, \tag{1.1}$$

when the limit exists. Here $\|\cdot\|$ denotes the matrix norm defined by $\|A\| := \mathbf{1}^\tau A \mathbf{1}$, where $\mathbf{1}$ is the d-dimensional column vector each coordinate of which is 1.

Let L_M be the set of point $\alpha \in \mathbb{R}$ such that $\alpha = \lambda_M(x)$ for some $x \in \Sigma$. By using the specification property of Σ and the continuity of M, we show that L_M is a non-empty closed interval (see Proposition 2.2).

Key words and phrases. Matrix products, Lyapunov exponents, Symbolic dynamics, Hausdorff dimensions, Packing dimensions, Entropies, Pressure functions, Multifractals.

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For any $q \in \mathbb{R}$, define

$$P_M(q) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\Sigma_n} \sup_{x\in[\omega]} \|M(x)M(\sigma x)\dots M(\sigma^{n-1}x)\|^q,$$

where Σ_n denotes the set of all words of length n over $\{1,\dots,m\}$; for $\omega = \omega_1\dots\omega_n \in \Sigma_n$, $[\omega]$ denotes the cylinder set $\{x = (x_i) \in \Sigma\colon\ x_i = \omega_i,\ 1 \le i \le n\}$. An subadditive argument shows that the limit in the above definition exists. We call $P_M(q)$ the **pressure function** of M.

Let $\mathcal{M}_\sigma(\Sigma)$ be the set of all σ-invariant Borel probability measures on Σ. The map $M\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ induces a map $M_*\colon \mathcal{M}_\sigma(\Sigma) \to \mathbb{R}$ given by

$$M_*(\mu) = \lim_{n\to\infty} \frac{1}{n} \int \log \|M(y)M(\sigma y)\dots M(\sigma^{n-1}y)\| d\mu(y), \quad \mu \in \mathcal{M}_\sigma(\Sigma).$$

The limit exists by an subadditive argument. In 1960, Furstenberg and Kesten [21] considered the products of random matrices and proved that for each ergodic measure μ on Σ,

$$\lambda_M(x) = M_*(\mu), \qquad \mu \ a.s.\ x \in \Sigma.$$

The above fact follows also by Kingman's Subadditive Ergodic Theorem (see [37]).

In this paper, we investigate the sizes of the sets with given Lyapunov exponents:

$$E_M(\alpha) = \{x \in \Sigma\colon \lambda_M(x) = \alpha\} \qquad (\alpha \in L_M).$$

Recall that Σ is a metric space where a metric is defined by $d(x, y) = m^{-n}$ for $x = (x_j)_{j\ge1}$ and $y = (y_j)_{\ge1}$ where n is the largest one such that $x_j = y_j$ $(1 \le j \le n)$. Different notions of dimensions are then defined on Σ. We shall talk about the Hausdorff dimension $\dim_H$, the packing dimension $\dim_P$ and the upper box dimension $\overline{\dim}_B$ (see [11, 28] for a general account of dimensions). The sizes of the sets in question will be described by their dimensions.

In the special case $d = 1$, M is just a real-valued continuous function; we would rather write Φ instead of M in this case. The first historical example of this type is due to Besicovitch [4] and Eggleston [10], they proved that for $0 \le \alpha \le 1$, the set

$$\left\{ x = (x_n) \in \{1, 2\}^{\mathbb{N}}\colon \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (x_j - 1) = \alpha \right\}$$

has Hausdorff dimension $-\big[\alpha \log_2 \alpha + (1 - \alpha) \log_2(1 - \alpha)\big]$. In this case the corresponding function Φ is given by $\Phi(x) = 1$ if $x_1 = 1$, and $\Phi(x) = e$ if $x_1 = 2$. A slightly more elaborate example was given by Billingsley in [5]. Some further consideration of the multifractal formalism for Hölder continuous Φ was given in [12, 14, 33, 38]. The case that Φ is only assumed to be continuous, was considered by Fan, Feng and Wu [13], Feng, Lau and Wu [17] and Olivier [29].

In the case $d \geq 2$, M is a matrix-valued continuous function. As we know, there are few results about this topic. In [27], Ledrappier and Porzio considered a special kind of product of matrices of order two, and obtained a local result of multifractal spectrum by using some classical random matrix products theory and perturbative theory; Porzio [35] strengthened that result somewhat by a study of Ruelle-Perron-Frobenius operator associated with random matrix products.

The main result of the present paper is the following theorem.

Theorem 1.1. *Suppose* $M\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ *is a continuous function taking values in the set of* $d \times d$ *positive matrices. For any* $\alpha \in L_M$, *we have the following formula*

$$\begin{aligned} & \dim_H E_M(\alpha) = \dim_P E_M(\alpha) \\ (1.2) \qquad & = \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{-\alpha q + P_M(q)\} \\ (1.3) \qquad & = \frac{1}{\log m} \sup\{h(\mu)\colon \mu \in \mathcal{M}_\sigma(\Sigma),\ M_*(\mu) = \alpha\}. \end{aligned}$$

Moreover, $\dim_H E_M(\alpha)$ *is a concave and continuous function of* α *on* L_M.

We remark that under this setting, the pressure function $P_M(q)$ of q may be not differentiable. Under a stronger condition that M is Hölder continuous, the formula (1.2) has been proved by Feng and Lau [16], and in that case $P_M(q)$ is a differentiable function of q over $\mathbb{R}$.

What we state in Theorem 1.1 is a kind of multifractal analysis. But it is a little different from the multifractal analysis of measures to which the term " multifractal " is often attached. Let us mention [1, 2, 7, 9, 8, 14, 20, 22, 23, 26, 30, 32, 34] (it is far from exhaustive). Another kind of multifractal analysis was engaged in [25] (see more references herein) where functions rather than measures are studied.

Now we state some ideas in the proof of Theorem 1.1. First we consider a special case that the map $M(x)$ depends only upon finitely many coordinates of x. In this case, we prove that the corresponding product of matrices is associated with a measure ν on Σ satisfying the so-called **quasi-Bernoulli property**: there is a constant $C \geq 1$ such that

$$\frac{1}{C}\nu([I])\nu([J]) \leq \nu([IJ]) \leq C\nu([I])\nu([J]), \qquad \forall\ I,\ J \in \bigcup_{n \geq 1} \Sigma_n.$$

By using some multifractal results on quasi-Bernoulli measures obtained by Brown, Michon & Peyriere [7] and Heurteaux [23], we can prove the desired results for matrix products. To consider the general case, we first prove a formal formula for $\dim_H E_M(\alpha)$. More precisely, for any $\alpha \in L_M$, $n \geq 1$ and $\epsilon > 0$, we define

$$f(\alpha; n, \epsilon) = \#F(\alpha; n, \epsilon)$$

with

$$F(\alpha; n, \epsilon) = \left\{\omega \in \Sigma_n \colon \left|\frac{1}{n}\log\|M(x)\dots M(\sigma^{n-1}x)\| - \alpha\right| < \epsilon \text{ for some } x \in [\omega]\right\}.$$

We prove (Proposition 3.2, Proposition 3.3)

(1.4)

$$\dim_H E_M(\alpha) = \lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log f(\alpha; n, \epsilon)}{\log m^n} = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log f(\alpha; n, \epsilon)}{\log m^n}.$$

Using the above formula, we can prove the general results by approximating M by a sequence of maps $\{M_k\}$ such that M_k depends only upon the first k coordinates.

We organize the materials in the paper as follows. In Section 2, we give some properties of the set L_M and the pressure function $P_M(q)$. In Section 3, we prove (1.4) by using a dimensional result for the homogeneous Moran sets. In Section 4, we consider the case that M depends upon finitely many coordinates. In Section 5, we complete the proof of Theorem 1.1. In Section 6, we give several remarks.

2. Lyapunov exponents and the pressure function

Let $M\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ be a continuous map. In this section, we will consider the set L_M of possible Lyapunov exponents and some relations between L_M and the pressure function $P_M(q)$. We also give some elementary results about convex functions and invariant measures on Σ. For convenience, we write $\pi_n M(x)$ for the product $M(x)M(\sigma x)\dots M(\sigma^{n-1}x)$ throughout this paper.

Let us start from a simple lemma.

Lemma 2.1. *There exists a constant $C > 0$ (depending on M) such that for any $x \in \Sigma$ and $n, m \in \mathbb{N}$,*

$$C\|\pi_n M(x)\|\,\|\pi_m M(\sigma^n x)\| \le \|\pi_{n+m}M(x)\| \le \|\pi_n M(x)\|\,\|\pi_m M(\sigma^n x)\|.$$

Proof. The second inequality is obvious. We only need to prove the first one. Since M is continuous, there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} \ge dC, \qquad \forall\ x \in \Sigma,$$

which implies that $M(x) \ge CEM(x)$ (here and afterwards we write $A \ge B$ for two matrices A, B if $A_{i,j} \ge B_{i,j}$ for each index (i,j)), here $E = (E_{i,j})_{1\le i,j\le d}$ is the matrix whose entries are all equal to 1. Let $\mathbf{1}$ be the d-dimensional column vector each coordinate of which is 1. Then

$$\begin{aligned}\|\pi_{n+m}M(x)\| &\ge \|(\pi_n M(x))CE(\pi_m M(\sigma^n x))\|\\ &= C\|(\pi_n M(x))\mathbf{1}^\tau\mathbf{1}(\pi_m M(\sigma^n x))\|\\ &= C\|\pi_n M(x)\|\cdot\|\pi_m M(\sigma^n x)\|.\end{aligned}$$

□

Proposition 2.2. *Set*

$$\alpha_M = \lim_{n\to\infty} \frac{1}{n} \inf_{x\in\Sigma} \log \|\pi_n M(x)\|, \tag{2.1}$$

$$\beta_M = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in\Sigma} \log \|\pi_n M(x)\|. \tag{2.2}$$

Then $L_M = [\alpha_M, \beta_M]$.

Proof. We first show that the limits in (2.1) and (2.2) exist. To see this, write

$$a_n = \inf_{x\in\Sigma} \log \|\pi_n M(x)\|, \qquad b_n = \sup_{x\in\Sigma} \log \|\pi_n M(x)\|. \tag{2.3}$$

By Lemma 2.1, we have

$$a_{n+m} \geq \log C + a_n + a_m, \qquad b_{n+m} \leq b_n + b_m, \qquad \forall\, n, m \geq 1,$$

where C is the constant in Lemma 2.1. This declares that the sequences $\{\log C + a_n\}$ and $\{b_n\}$ are superadditive and subadditive respectively, from which the existence of the limits follows.

By the definition of upper Lyapunov exponents, we have $L_M \subset [\alpha_M, \beta_M]$ immediately. Hence, to prove the proposition, it suffices to prove that for any $t \in [\alpha_M, \beta_M]$, there exists $y \in \Sigma$ such that $\lambda_M(y) = t$.

Now fix a real number $t \in [\alpha_M, \beta_M]$. Then there is a number $p \in [0, 1]$ such that $t = p\alpha_M + (1-p)\beta_M$. For convenience, we define a sequence of real numbers $\{r_n\}$ by $r_{2n} = \alpha_M$ and $r_{2n-1} = \beta_M$ for $n \geq 1$. By the continuity of M and the definitions of α_M and β_M, there exist a sequence of words $\{\omega_n\}$ ($\omega_n \in \Sigma_n$) and a sequence of positive numbers $\{\epsilon_n\}$ which tend to 0 such that

$$\left| \frac{1}{n} \log \|\pi_n M(x)\| - r_n \right| < \epsilon_n, \qquad \forall x \in [\omega_n]. \tag{2.4}$$

Now construct a sequence of positive integers $\{N_n\}$ by

$$N_n = \begin{cases} [\![pn + \log n]\!], & \text{if } n \text{ is odd}, \\ [\![(1-p)n + \log n]\!], & \text{otherwise}, \end{cases}$$

where $[\![x]\!]$ denotes the integral part of x. It can be checked directly that

$$\lim_{n\to\infty} N_n = \infty, \quad \lim_{n\to\infty} \frac{nN_n}{\sum_{i=1}^n iN_i} = 0, \quad \lim_{n\to\infty} \frac{\sum_{i=1}^n (2i-1)N_{2i-1}}{\sum_{j=1}^{2n} jN_j} = p.$$

Now define

$$y = \underbrace{\omega_1 \dots \omega_1}_{N_1} \underbrace{\omega_2 \dots \omega_2}_{N_2} \dots \underbrace{\omega_n \dots \omega_n}_{N_n} \dots.$$

In the following we show that $\lambda(y) = t$. In fact, for each integer $k > N_1$, there is an integer $n > 0$ such that

$$\sum_{i=1}^n iN_i \leq k < \sum_{i=1}^{n+1} iN_i.$$

By Lemma 2.1 and (2.4), we have

$$\begin{aligned}\|\pi_k M(y)\| &\leq \|\pi_{N_1+\ldots+nN_n-1}M(y)\|\,\|\pi_{k-N_1-\ldots-nN_n}M(\sigma^{N_1+\ldots+nN_n}y)\| \\ &\leq \exp\Big(\sum_{i=1}^{n} iN_i(r_i+\epsilon_i)\Big)\cdot\exp\Big((k-(N_1+\ldots+nN_n))b_1\Big),\end{aligned}$$

which implies that

$$\frac{1}{k}\log\|\pi_k M(y)\| \leq \frac{\sum_{i=1}^{n} iN_i(r_i+\epsilon_i)}{k} + \frac{k-(N_1+\ldots+nN_n)}{k}\cdot b_1,$$

where b_1 is defined by (2.3). Letting k tend to the infinity we have

$$\limsup_{k\to\infty}\frac{1}{k}\log\|\pi_k M(y)\| \leq t.$$

Now by Lemma 2.1, we have also that

$$\begin{aligned}\|\pi_k M(y)\| &\geq C\|\pi_{N_1+\ldots+nN_n-1}M(y)\|\exp\Big((k-(N_1+\ldots+nN_n))a_1\Big) \\ &\geq C^{N_1+N_2+\ldots+N_{n+1}}\exp\Big(\sum_{i=1}^{n} iN_i(r_i-\epsilon_i)\Big) \\ &\quad\cdot\exp\Big((k-(N_1+\ldots+nN_n))a_1\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{k}\log\|\pi_k M(y)\| \quad\geq\quad & \frac{\sum_{i=1}^{n} iN_i(r_i-\epsilon_i)}{k} + \frac{N_1+\ldots+N_{n+1}}{k}\log C \\ & + \frac{k-(N_1+\ldots+nN_n)}{k}\cdot a_1.\end{aligned}$$

By taking the limit we have

$$\liminf_{k\to\infty}\frac{1}{k}\log\|\pi_k M(y)\| \geq t.$$

This finishes the proof. $\square$

The following proposition gives some relations between L_M and the pressure function $P_M(q)$.

Proposition 2.3. *$P_M(q)$ is a convex function of q on $\mathbb{R}$. Furthermore, let α_M and β_M be defined as in Proposition 2.2, then we have*

$$\lim_{q\to-\infty}\frac{P_M(q)}{q} = \alpha_M, \qquad \lim_{q\to+\infty}\frac{P_M(q)}{q} = \beta_M.$$

Proof. The convexity of $P_M(q)$ follows by a standard argument.

Let the sequences $\{a_n\}, \{b_n\}$ be defined as in (2.3). Then for each $n \geq 1$,

$$\begin{cases}\exp(b_n q) \leq \sum_{\omega\in\Sigma_n}\sup_{x\in[\omega]}\|\pi_n M(x)\|^q \leq m^n\exp(b_n q), & \forall\ q\geq 0\\ \exp(a_n q) \leq \sum_{\omega\in\Sigma_n}\sup_{x\in[\omega]}\|\pi_n M(x)\|^q \leq m^n\exp(a_n q), & \forall\ q<0\end{cases}$$

which implies that

$$\begin{cases} q\beta_M \leq P_M(q) \leq \log m + q\beta_M, & \forall\ q \geq 0 \\ q\alpha_M \leq P_M(q) \leq \log m + q\alpha_M, & \forall\ q < 0 \end{cases} \tag{2.5}$$

By taking the limit we obtain the desired result. □

Proposition 2.4. *Suppose that $N\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ is a continuous map, and there is a real number $\delta > 0$ such that*

$$(1+\delta)^{-1} M(x) \leq N(x) \leq (1+\delta) M(x), \qquad \forall\ x \in \Sigma.$$

Let L_N denote the set of all possible upper Lyapunov exponents of N, and $P_N(q)$ denote the pressure function of N. Then

$$L_N \supset [\alpha_M + \log(1+\delta),\ \beta_M - \log(1+\delta)].$$

Moreover, we have

$$|P_N(q) - P_M(q)| \leq |q \log(1+\delta)|.$$

Proof. It follows immediately from Proposition 2.2 and the definitions of L_N and $P_N(q)$. □

Proposition 2.5. *Let f be a convex real-valued function on $\mathbb{R}$. Denote*

$$a = \lim_{x\to-\infty} \frac{f(x)}{x}, \qquad b = \lim_{x\to\infty} \frac{f(x)}{x}. \tag{2.6}$$

(i) *Suppose that $\{f_n\}$ is a sequence of differentiable convex functions converging to f pointwisely. Then for any $c \in (a, b)$, there exist $N > 0$ and a uniformly bounded sequence of real numbers $\{x_n\}_{n\geq N}$ such that $f_n'(x_n) = c$.*

(ii) *Assume $-\infty < a < b < \infty$. Then we have*

$$\overline{\lim_{z\uparrow b}} \inf_{x\in\mathbb{R}} \{-zx + f(x)\} \geq \inf_{x\in\mathbb{R}} \{-bx + f(x)\},$$

and

$$\overline{\lim_{z\downarrow a}} \inf_{x\in\mathbb{R}} \{-zx + f(x)\} \geq \inf_{x\in\mathbb{R}} \{-ax + f(x)\}.$$

Proof. Since f is convex, $\dfrac{f(x)-f(0)}{x}$ is an increasing function of x. Thus the limits in (2.6) exist. Take $\epsilon > 0$ with $a + \epsilon < c < b - \epsilon$. Pick $t > 0$ large enough so that

$$\frac{f(t)-f(0)}{t} \geq c + \epsilon, \qquad \frac{f(-t)-f(0)}{-t} \leq c - \epsilon.$$

Since the sequence $\{f_n\}$ converges to f pointwisely, there exists $N > 0$ such that for each $n \geq N$,

$$\frac{f_n(t)-f_n(0)}{t} \geq c + \epsilon/2, \qquad \frac{f_n(-t)-f_n(0)}{-t} \leq c - \epsilon/2.$$

Note that each f_n is continuously differentiable since it is differentiable convex (see [36, Theorem 25.3]). By using the Mean Value Theorem and the Intermediate Value Theorem, we see that for each $n \geq N$, there exists $x_n \in (-t,t)$ such that $f_n'(x_n) = c$. This concludes statement (i).

To prove statement (ii), denote $f^*(z) = \inf_{x\in\mathbb{R}}\{-zx + f(x)\}$. It can be checked directly that f^* is a concave function on $[a,b]$, and thus it is lower semi-continuous on $[a,b]$ (see [36, Theorem 10.2]), which concludes statement (ii). $\square$

The following proposition is needed in the proof of (1.3).

Proposition 2.6. *For any $\mu \in \mathcal{M}_\sigma(\Sigma)$, there is a sequence of ergodic measures $\{\mu_k\}_{k\geq 1} \subset \mathcal{M}_\sigma(\Sigma)$ such that*

$$\mu = w^*\text{-}\lim_{k\to\infty} \mu_k, \qquad h(\mu) = \lim_{k\to\infty} h(\mu_k).$$

Proof. First we assume that μ is fully supported on Σ. For each integer $n \geq 2$, let μ_n be the unique equilibrium state (see [6]) of the potential $\phi_n\colon \Sigma \to \mathbb{R}$ defined by

$$\phi_n(x) = \log \mu([x_1 \ldots x_n]) - \log \mu([x_1 \ldots x_{n-1}]), \qquad \forall\ x = (x_i).$$

One may check that μ_n has the following property: for any integer $\ell > 0$ and $i_1 \ldots i_\ell \in \Sigma_\ell$,

$$\mu_n([i_1 \ldots i_\ell]) = \begin{cases} \mu([i_1 \ldots i_\ell]), & \text{if } \ell \leq n, \\ \mu([i_1 \ldots i_n]) \prod_{j=2}^{\ell-n+1} \dfrac{\mu([i_j \ldots i_{j+n-1}])}{\mu([i_j \ldots i_{j+n-2}])}, & \text{otherwise.} \end{cases}$$

This means that μ_n converges to μ in the weak-star topology. By the upper-semi continuity of the entropy of μ, we have

$$h(\mu) \geq \limsup_{n\to\infty} h(\mu_n). \tag{2.7}$$

Furthermore, by using the Variational Principle for equilibrium states (see [37]), we obtain

$$\int \phi_n d\mu + h(\mu) \leq \int \phi_n d\mu_n + h(\mu_n),$$

which yields $h(\mu) \leq h(\mu_n)$. This together with (2.7) yields $h(\mu) = \lim_{n\to\infty} h(\mu_n)$.

Now assume that μ is not fully supported. Denote by ν a fully supported invariant measure on Σ. Then we can approximate μ by a sequence of fully supported invariant measures $\{\frac{n-1}{n}\mu + \frac{1}{n}\nu\}$. We can see that these measures converge to μ in the weak-star topology, and their entropies converge to $h(\mu)$ (since $h(\frac{n-1}{n}\mu + \frac{1}{n}\nu) = \frac{n-1}{n}h(\mu) + \frac{1}{n}h(\nu)$). Combining this with the results in the last paragraph, we can obtain the desired result. $\square$

3. Homogeneous Moran sets and A formal formula of $\dim_H E_M(\alpha)$

In this section, we first recall the definition and some dimensional results of homogeneous Moran sets; then by using these results and some furthermore constructions we give a formal formula of $\dim_H E_M(\alpha)$. The main results in this section are Proposition 3.2 and Proposition 3.3, in their proof we adopt some ideas from the proof of [12, Theorem 4].

It is helpful to think of Σ as the interval $[0,1]$ and cylinders as subintervals. Let $\{n_k\}_{k\geq 1}$ be a sequence of positive integers and $\{c_k\}_{k\geq 1}$ be a sequence of positive numbers satisfying $n_k \geq 2$, $0 < c_k < 1$, $n_1 c_1 \leq \delta$ and $n_k c_k \leq 1$ $(k \geq 2)$, where δ is some positive number. Let

$$D = \bigcup_{k\geq 0} D_k$$

with $D_0 = \{\emptyset\}$ and $D_k = \{(i_1, \dots, i_k);\quad 1 \leq i_j \leq n_j,\quad 1 \leq j \leq k\}$. Suppose that J is an interval of length δ. A collection $\mathcal{F} = \{J_\sigma\colon\ \sigma \in D\}$ of subintervals of J is said to have a **homogeneous Moran structure** if it satisfies

(1) $J_\emptyset = J$;
(2) For any $k \geq 0$ and $\sigma \in D_k$, $J_{\sigma i}$ $(i = 1, \dots, n_{k+1})$ are disjoint subintervals of J_σ such that

$$\frac{|J_{\sigma i}|}{|J_\sigma|} = c_{k+1}, \qquad \forall\ 1 \leq i \leq n_{k+1},$$

where $|A|$ denotes the length of A.

If $\mathcal{F}$ is such a collection, $E := \bigcap_{k\geq 1} \bigcup_{\sigma\in D_k} J_\sigma$ is called a **homogeneous Moran set** determined by $\mathcal{F}$. One may refer to [19, 18] for more information about homogeneous Moran sets. For the purpose of the present paper, we only need the following simplified version of a result contained in [19], whose simpler proof was given in [12, Proposition 3].

Proposition 3.1. *For the homogeneous Moran set defined above, we have*

$$\dim_H E \geq \liminf_{n\to\infty} \frac{\log n_1 n_2 \dots n_k}{-\log c_1 c_2 \dots c_{k+1} n_{k+1}}.$$

For $x = (x_i) \in \Sigma$, denote $I_n(x) = \{y = (y_i) \in \Sigma\colon x_i = y_i, 1 \leq i \leq n\}$. We call $I_n(x)$ the *n-cylinder* about x. Write $M(x) = \Big(M_{i,j}(x)\Big)_{1\leq i,j\leq d}$. For each $n \in \mathbb{N}$, define

$$\delta_n(M) = \sup_{y\in\Sigma} \left\{ \max_{1\leq i,j\leq d} \frac{M_{i,j}(x)}{M_{i,j}(y)},\ x \in I_n(y) \right\}.$$

Since $M\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ is continuous, we have $\lim_{n\to\infty} \delta_n(M) = 1$.

For any $\alpha \in L_M$, $n \geq 1$ and $\epsilon > 0$, we define

$$\begin{aligned} &F(\alpha; n, \epsilon) \\ = \quad &\left\{ \omega \in \Sigma_n\colon \left| \frac{1}{n} \log \|\pi_n M(x)\| - \alpha \right| < \epsilon \text{ for some } x \in [\omega] \right\} \end{aligned}$$

and $f(\alpha;n,\epsilon)=\#F(\alpha;n,\epsilon)$.

Proposition 3.2. *For $\alpha\in L_M$, we have*

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{\log f(\alpha;n,\epsilon)}{\log m^n}=\lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{\log f(\alpha;n,\epsilon)}{\log m^n}\quad (=:\Lambda_M(\alpha)).$$

The function $\Lambda_M\colon L_M\to[0,1]$ is concave and continuous.

Proof. We first show that $\log f(\alpha;n,\epsilon)$, as a sequence of n, has a kind of subadditivity. More precisely, for any $\epsilon>0$, there is an N such that

$$[f(\alpha;n,\epsilon)]^p\le f(\alpha;np,2\epsilon)\qquad(\forall n\ge N,\forall p\ge 1).$$

In fact, suppose $\{\omega_1,\dots,\omega_p\}\subset F(\alpha;n,\epsilon)$. Let $\omega=\omega_1\dots\omega_p$. Let $x_k\in[\omega_k]$ $(1\le k\le p)$ be a point such that

$$\left|\frac{1}{n}\log\|M(x_k)\dots M(\sigma^{n-1}x_k)\|-\alpha\right|<\epsilon.$$

Let x be a point in $[\omega]$. Note that for any $1\le j\le p$,

$$\begin{aligned}\frac{\pi_n M(x_j)}{\delta_1(M)\dots\delta_n(M)} &\le \pi_n M(\sigma^{(j-1)n}x)\\ &\le \delta_1(M)\dots\delta_n(M)\pi_n M(x_j)\end{aligned}$$

We have

$$\left|\frac{1}{n}\log\|\pi_n M(\sigma^{(j-1)n}x)\|-\alpha\right|<\epsilon+\frac{1}{n}\log(\delta_1(M)\dots\delta_n(M))$$

for all $1\le j\le p$. It follows that

$$\left|\frac{1}{np}\log\|\pi_{pn}M(x)\|-\alpha\right|<\epsilon+\frac{1}{n}\log(\delta_1(M)\dots\delta_n(M))+\frac{\log C}{n},$$

where C is the constant in Lemma 2.1. Since $\lim_{n\to\infty}\delta_n(M)=1$, there exists N such that $\frac{1}{n}\log(\delta_1(M)\dots\delta_n(M))+\frac{\log C}{n}<\epsilon$ for $n\ge N$. It follows that

$$\left|\frac{1}{np}\log\|\pi_{np}M(x)\|-\alpha\right|<2\epsilon$$

for $n\ge N$ and for all $p\ge 1$. Then $[\omega]$, which contains x, is in $F(\alpha;np,2\epsilon)$. Notice that different choices $\{\omega_1,\dots,\omega_p\}$ give rise to different ω's. Thus we get the desired subadditivity. By using this subadditivity, it is easy to get

$$\limsup_{n\to\infty}\frac{\log f(\alpha;n,\epsilon)}{\log m^n}\le\liminf_{n\to\infty}\frac{\log f(\alpha;n,2\epsilon)}{\log m^n}$$

from which the equality of the two limits follows.

It is evident that $0\le\Lambda_M(\alpha)\le 1$. Let $\alpha,\beta\in L_M$. Let p,q be two positive integers. By subadditivity, for large n we have

$$[f(\alpha;n,\epsilon)]^p[f(\beta;n,\epsilon)]^q\le f(\alpha;np,2\epsilon)f(\beta;nq,2\epsilon).$$

Let $u \in F(\alpha; np, 2\epsilon)$ and $v \in F(\beta; nq, 2\epsilon)$. Take a point $x \in [uv]$. As above, we can get

$$\begin{aligned} & |\log \|\pi_{np+nq} M(x)\| - np\alpha - nq\beta| \\ \leq \quad & 2\epsilon n(p+q) + \log(\delta_1(M)\dots\delta_{np}(M)) \\ & \quad + \log(\delta_1(M)\dots\delta_{nq}(M)) + \log C. \end{aligned}$$

It follows that if n is sufficiently large, $uv \in F(\frac{p\alpha+q\beta}{p+q}; n(p+q), 3\epsilon)$. Consequently, for large n we have

$$f(\alpha; np, 2\epsilon) f(\beta; nq, 2\epsilon) \leq f(\frac{p\alpha + q\beta}{p+q}; n(p+q), 3\epsilon)$$

By the equality of the two limits that we have already proved, we can get

$$\frac{p}{p+q}\Lambda_M(\alpha) + \frac{q}{p+q}\Lambda_M(\beta) \leq \Lambda_M(\frac{p}{p+q}\alpha + \frac{q}{p+q}\beta).$$

This gives the rational concavity of the (bounded) function Λ_M. However, the concavity of Λ_M on the interval L_M is a consequence of its rational concavity and its upper semi-continuity that we prove below.

Given $\alpha \in L_M$. For any $\eta > 0$, there is $\epsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} < \Lambda_M(\alpha) + \eta.$$

As above, it can be proved that for $\beta \in L_M$ with $|\beta - \alpha| < \frac{\epsilon}{3}$ we have

$$F(\beta; n, \epsilon/3) \subset F(\alpha; n, \epsilon)$$

when n is sufficiently large. It follows that $f(\beta; n, \epsilon/3) \leq f(\alpha; n, \epsilon)$. Therefore

$$\begin{aligned} \Lambda_M(\beta) \quad & \leq \quad \liminf_{n\to\infty} \frac{\log f(\beta; n, \epsilon/3)}{\log m^n} \leq \liminf_{n\to\infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} \\ & \leq \quad \Lambda_M(\alpha) + \eta. \end{aligned}$$

This establishes the upper semi-continuity of Λ_M at α.

The continuity of Λ_M on the interval L_M follows from its concavity and its upper semi-continuity. □

Proposition 3.3. *For $\alpha \in L_M$, we have*

$$\dim_H E_M(\alpha) = \dim_P E_M(\alpha) = \Lambda_M(\alpha).$$

PROOF. *Step 1. For $\alpha \in L_M$, we have* $\dim_P E_M(\alpha) \leq \Lambda_M(\alpha)$.

Let

$$G(\alpha; k, \epsilon) = \bigcap_{n=k}^{\infty} \left\{ x \in \Sigma \colon \left| \frac{1}{n} \|\pi_n M(x)\| - \alpha \right| < \epsilon \right\}.$$

It is clear that for any $\epsilon > 0$,

$$E_M(\alpha) \subset \bigcup_{k=1}^{\infty} G(\alpha; k, \epsilon).$$

Notice that if $n \geq k$, $G(\alpha; k, \epsilon)$ is covered by the union of all cylinders $[\omega]$ with $\omega \in F(\alpha; n, \epsilon)$ whose total number is $f(\alpha; n, \epsilon)$. Therefore we have the following estimate

$$\overline{\dim}_B G(\alpha; k, \epsilon) \leq \limsup_{n\to\infty} \frac{\log f(\alpha; n, \epsilon)}{\log m^n} \quad (\forall \epsilon > 0, \forall k \geq 1).$$

On the other hand, by using the σ-stability of the packing dimension, we have

$$\begin{aligned} \dim_P E_M(\alpha) &\leq \dim_P \left(\bigcup_{k=1}^{\infty} G(\alpha; k, \epsilon) \right) \leq \sup_k \dim_P G(\alpha; k, \epsilon) \\ &\leq \sup_k \overline{\dim}_B G(\alpha; k, \epsilon). \end{aligned}$$

This, together with the last proposition, leads to the desired result.

Step 2. For $\alpha \in L_M$, we have $\dim_H E_M(\alpha) \geq \Lambda_M(\alpha)$.

Given $\delta > 0$. By the last proposition, there are $\ell_j \uparrow \infty$ and $\epsilon_j \downarrow 0$ such that

$$f(\alpha; \ell_j, \epsilon_j) > m^{\ell_j(\Lambda_M(\alpha) - \frac{\delta}{2})}.$$

Write simply $F_{\ell_j} = F(\alpha; \ell_j, \epsilon_j)$ and $f_{\ell_j} = f(\alpha; \ell_j, \epsilon_j)$. Define a new sequence $\{\ell_j^*\}$ in the following manner

$$\underbrace{\ell_1, \ldots, \ell_1}_{N_1}; \underbrace{\ell_2, \ldots, \ell_2}_{N_2}; \ldots; \underbrace{\ell_j, \ldots, \ell_j}_{N_j}; \ldots$$

where N_j is defined recursively by

$$N_j = 2^{\ell_{j+1} + N_{j-1}} \quad (j \geq 2); \qquad N_1 = 1.$$

Denote $n_j = f_{\ell_j^*}$ and $c_j = m^{-\ell_j^*}$. Define

$$\Theta^* = \prod_{j=1}^{\infty} F_{\ell_j^*}.$$

Observe that Θ^* is a homogeneous Moran set in Σ. More precisely Θ^* is constructed as follows. At level 0, we have only the initial cylinder Σ. In step j, cut a cylinder of level $j-1$ into $m^{\ell_j^*}$ cylinders and pick up n_j ones. By Proposition 3.1, we have

$$\begin{aligned} \dim_H \Theta^* &\geq \liminf_{k\to\infty} \frac{\log(n_1 \ldots n_k)}{-\log(c_1 \ldots c_k c_{k+1} n_{k+1})} \\ &\geq \liminf_{k\to\infty} \frac{\log(f_{\ell_1^*} \ldots f_{\ell_k^*})}{\log(2^{\ell_1^* + \cdots + \ell_k^* + \ell_{k+1}^*})} \\ &= \liminf_{k\to\infty} \frac{\log(f_{\ell_1^*} \ldots f_{\ell_k^*})}{\log(2^{\ell_1^* + \cdots + \ell_k^*})} \\ &\geq \Lambda_M(\alpha) - \delta. \end{aligned}$$

However by a direct check, Θ^* is a set in $E_M(\alpha)$. Hence $\dim_H E_M(\alpha) \geq \Lambda_M(\alpha) - \delta$. And thus $\dim_H E_M(\alpha) \geq \Lambda_M(\alpha)$ since δ can be picked small arbitrary. $\square$

4. The case that M depends upon finitely many coordinates

In this section, we always assume that M depends upon finitely many coordinates. That is, there exists an integer $k \geq 1$ such that $M(x)$ depends upon the first k coordinates of x for all $x = (x_i) \in \Sigma$. For simplicity, we write $M(x) = M(x_1 \dots x_k)$. We will prove the following proposition by using some multifractal results about quasi-Bernoulli measures.

Proposition 4.1. *Suppose that the map $M\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ depends only upon the first k coordinates. Then $P_M(q)$ is a differentiable function of q on $\mathbb{R}$. Moreover, if $\alpha = P_M'(t)$ for some $t \in \mathbb{R}$, then*

(i) $\dim_H E_M(\alpha) = \dfrac{1}{\log m} \inf_{q\in\mathbb{R}} \{-\alpha q + P_M(q)\} = \dfrac{1}{\log m} \left(-\alpha t + P_M(t)\right).$

(ii) *There exists an ergodic measure μ_t on Σ such that*

$$M_*(\mu_t) = \alpha \quad \textit{and} \quad \dim_H \mu_t = \frac{h(\mu_t)}{\log m} = \frac{1}{\log m} \left(-\alpha t + P_M(t)\right).$$

Before giving the proof of the above proposition, we recall some multifractal results about quasi-Bernoulli measures. Let ν be a Borel probability measure on Σ. We recall that ν is **quasi-Bernoulli** if there exists a constant $C > 1$ such that

$$\frac{1}{C}\nu([I])\nu([J]) \leq \nu([IJ]) \leq C\nu([I])\nu([J]), \qquad \forall\ I,\ J \in \bigcup_{n\geq 1} \Sigma_n. \tag{4.1}$$

Let μ be a Borel probability measure on Σ. For any $q \in \mathbb{R}$, the L^q-spectrum of μ is defined by

$$\tau_\mu(q) = \liminf_{n\to\infty} \frac{1}{n} \log \sum_I \mu([I])^q,$$

where the summation is taken over all $I \in \Sigma_n$ with $\mu([I]) > 0$.

Brown, Michon & Peyriere [7] and Heurteaux [23] have considered the multifractal properties of quasi-Bernoulli measures. They proved

Proposition 4.2. *Suppose that ν is a quasi-Bernoulli measure. Then the L^q-spectrum $\tau_\nu(q)$ is differentiable for $q \in \mathbb{R}$. Moreover, if $\alpha = \tau_\nu'(t)$ for some $t \in \mathbb{R}$, then*

(i)

$$\begin{aligned} \dim_H \left\{ x \in \Sigma\colon \lim_{r\to\infty} \frac{\log \nu(B_r(x))}{\log r} = \alpha \right\} &= \inf_{q\in\mathbb{R}} \{\alpha q - \tau_\nu(q)\} \\ &= \alpha t - \tau_\nu(t); \end{aligned}$$

(ii) *there exists an ergodic measure* μ_t *on* Σ *such that*

$$\mu_t\left\{x\in\Sigma\colon \lim_{r\to\infty}\frac{\log\nu(B_r(x))}{\log r}=\alpha\right\}=1$$

and $\dim_H\mu_t=\frac{h(\mu_t)}{\log m}=\alpha t-\tau_\nu(t)$.

We remark that statement (ii) is only implicit in [23].

The following lemma plays a crucial role in the proof of Proposition 4.1.

Lemma 4.3. *There exist a Borel probability measure* μ *on* Σ *and two positive constants* ρ, C *such that for any* $n\geq 1$ *and* $i_1\ldots i_{n+k-1}\in\Sigma_{n+k-1}$,

$$\begin{aligned}&C^{-1}\rho^n\|M(i_1\ldots i_k)M(i_2\ldots i_{k+1})\ldots M(i_n\ldots i_{n+k-1})\|\\ \leq\ &\mu([i_1\ldots i_{n+k-1}])\\ \leq\ &C\rho^n\|M(i_1\ldots i_k)M(i_2\ldots i_{k+1})\ldots M(i_n\ldots i_{n+k-1})\|.\end{aligned}$$

Proof. At first we declare that, there exist positive numbers ρ_1, ρ_2 and d-dimensional column vectors $\mathbf{u}(i_1\ldots i_k)$, $\mathbf{v}(i_1\ldots i_k)$ $(i_1\ldots i_k\in\Sigma_n)$ with positive entries such that for any $i_1\ldots i_k\in\Sigma_k$,

$$\mathbf{u}(i_1\ldots i_k)^\tau=\frac{1}{\rho_1}\sum_i\mathbf{u}(ii_1\ldots i_{k-1})^\tau M(ii_1\ldots i_{k-1}),\tag{4.2}$$

$$\mathbf{v}(i_1\ldots i_k)=\frac{1}{\rho_2}\sum_i M(i_2\ldots i_ki)\mathbf{v}(i_2\ldots i_ki).\tag{4.3}$$

To see it, without loss of generality we assume $m=2$ and $k=2$. We construct a new $4d\times 4d$ matrix H by

$$H=\begin{bmatrix}M(11)&\mathbf{0}&M(21)&\mathbf{0}\\ M(11)&\mathbf{0}&M(21)&\mathbf{0}\\ \mathbf{0}&M(12)&\mathbf{0}&M(22)\\ \mathbf{0}&M(12)&\mathbf{0}&M(22)\end{bmatrix}.$$

Since $M(ij)$ $(ij\in\Sigma_2)$ are positive matrices, H are primitive (one checks that H^2 is positive). Thus by the Perron-Frobenius theorem (see [24]), there exist a positive number ρ_1 and a $4d$-dimensional positive column vector $\mathbf{s}$ such that $\mathbf{s}^\tau=\dfrac{1}{\rho_1}\mathbf{s}^\tau H$. Write $\mathbf{s}^\tau$ as the form

$$\mathbf{s}^\tau=(\mathbf{u}(11)^\tau,\mathbf{u}(12)^\tau,\mathbf{u}(21)^\tau,\mathbf{u}(22)^\tau),$$

where $\mathbf{u}(ij)$ are d-dimensional column vectors. Then it is clear that the vectors $\mathbf{u}(ij)$ satisfy (4.2). The proof of (4.3) follows by a similar discussion.

Define two functions η_1 and η_2 on $\bigcup_{n\geq k}\Sigma_n$ by

$$\begin{aligned}\eta_1(i_1i_2\ldots i_{n+k-1})\ =\ &\rho_1^{-n}\mathbf{u}(i_1\ldots i_k)^\tau M(i_1\ldots i_k)M(i_2\ldots i_{k+1})\\ &\ldots M(i_n\ldots i_{n+k-1})\mathbf{v}(i_n\ldots i_{n+k-1})\end{aligned}$$

and

$$\begin{aligned}\eta_2(i_1 i_2 \dots i_{n+k-1}) &= \rho_2^{-n} \mathbf{u}(i_1 \dots i_k)^\tau M(i_1 \dots i_k) M(i_2 \dots i_{k+1}) \\ &\quad \dots M(i_n \dots i_{n+k-1}) \mathbf{v}(i_n \dots i_{n+k-1}).\end{aligned}$$

By (4.2) and (4.3) we have

$$\left\{ \begin{array}{ll} \sum_i \eta_1(i i_1 i_2 \dots i_{n+k-1}) & = \eta_1(i_1 i_2 \dots i_{n+k-1}), \\ \sum_i \eta_2(i_1 i_2 \dots i_{n+k-1} i) & = \eta_2(i_1 i_2 \dots i_{n+k-1}), \end{array} \right. \tag{4.4}$$

which implies that for each $n \geq k$,

$$\sum_{\omega \in \Sigma_n} \eta_1(\omega) = \sum_{\omega' \in \Sigma_k} \eta_1(\omega'), \qquad \sum_{\omega \in \Sigma_n} \eta_2(\omega) = \sum_{\omega' \in \Sigma_k} \eta_2(\omega').$$

We deduce from the above equalities that $\rho_1 = \rho_2$ since

$$(\rho_1/\rho_2)^n = \sum_{\omega \in \Sigma_n} \eta_1(\omega) / \sum_{\omega \in \Sigma_n} \eta_2(\omega) = \sum_{\omega \in \Sigma_k} \eta_1(\omega) / \sum_{\omega \in \Sigma_k} \eta_2(\omega).$$

And thus $\eta_1 = \eta_2$. Define η on $\bigcup_{n \geq k} \Sigma_n$ by

$$\eta(\omega) = \eta_1(\omega) / \sum_{\omega' \in \Sigma_k} \eta_1(\omega'), \qquad \forall\, \omega \in \bigcup_{n \geq k} \Sigma_n.$$

By the Kolmogrov consistence theorem, there is a unique invariant Borel probability measure μ on Σ such that $\mu([\omega]) = \eta(\omega)$ for any $\omega \in \bigcup_{n \geq k} \Sigma_n$. This completes the proof. □

Proof of Proposition 4.1. Let μ be the measure as in Lemma 4.3 and ρ the corresponding constant. By Lemma 4.3 and Lemma 2.1, μ is a quasi-Bernoulli measure. Moreover,

$$\tau_\mu(q) = \frac{q \log \rho - P_M(q)}{\log m} \qquad (\forall\, q \in \mathbb{R})$$

and

$$E_M(\alpha) = \left\{ x \in \Sigma\colon \lim_{r \to \infty} \frac{\log \mu(B_r(x))}{\log r} = \frac{\log \rho - \alpha}{\log m} \right\} \qquad (\forall\, \alpha \in L_M).$$

Using Proposition 4.2, we obtain the desired result. □

5. **The Proof of Theorem 1.1**

We divide the proof into 4 small steps:

Step 1. $\dim_P E_M(\alpha) \leq \frac{1}{\log m}(-\alpha q + P_M(q)) \quad (\alpha \in L_M,\ q \in \mathbb{R})$.

For any $\alpha \in L_M$, $\epsilon > 0$ and $n \in \mathbb{N}$, let $f(\alpha; n, \epsilon)$ be defined as in Section 3. Then

$$\sum_{\omega \in \Sigma_n} \sup_{x \in [\omega]} \|\pi_n M(x)\|^q \geq \left\{ \begin{array}{ll} f(\alpha; n, \epsilon) \exp(nq(\alpha - \epsilon)), & \text{if } q \geq 0 \\ f(\alpha; n, \epsilon) \exp(nq(\alpha + \epsilon)), & \text{if } q < 0 \end{array} \right.$$

which implies that for any $q \in \mathbb{R}$,

$$P_M(q) \geq q\alpha + \lim_{\epsilon\to\infty} \liminf_{n\to\infty} \frac{\log f(\alpha;n,\epsilon)}{n}.$$

Combining it with Propositions 3.2 and 3.3, we obtain

$$\dim_P E_M(\alpha) \leq \frac{1}{\log m}(-q\alpha + P_M(q)).$$

Step 2. We prove the following inequality:

$$\dim_H E_M(\alpha) \geq \frac{1}{\log m} \inf_{q\in\mathbb{R}}\{-\alpha q + P_M(q)\} \qquad (\alpha \in L_M). \tag{5.1}$$

At first we consider a trivial case: $\alpha_M = \beta_M$ (α_M and β_M are defined as in Proposition 2.2). In this case, we have $\lambda_M(x) = \alpha_M$ for all $x \in \Sigma$. By (2.5), we have

$$\dim_H E_M(\alpha_M) = \dim_H \Sigma = 1 \geq \frac{1}{\log m} \inf_{q\in\mathbb{R}}\{-\alpha_M q + P_M(q)\}.$$

From now on we assume that $\alpha_M \neq \beta_M$.

First we consider $\alpha \in (\alpha_M, \beta_M)$. For each $k \in \mathbb{N}$, we define a map $M_k\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ such that M_k depends upon the first k coordinates of x and $M_k(x) = M(y)$ for some $y \in I_n(x)$. It is clear that M_k is continuous. Moreover there is a sequence of real numbers $\{\delta_k\} \downarrow 0$ such that

$$(1+\delta_k)^{-1}M(x) \leq M_k(x) \leq (1+\delta_k)M(x), \qquad \forall\ x \in \Sigma. \tag{5.2}$$

Pick $\epsilon > 0$ with $\epsilon < \frac{1}{2}\min\{\alpha - \alpha_M, \beta_M - \alpha\}$. For each $k, n \in \mathbb{N}$, define

$$\begin{aligned} &F_k(\alpha; n, \epsilon/2) \\ = &\left\{\omega \in \Sigma_n\colon \left|\frac{1}{n}\log\|\pi_n M_k(x)\| - \alpha\right| < \frac{\epsilon}{2} \text{ for some } x \in [\omega]\right\} \end{aligned}$$

and

$$f_k(\alpha; n, \epsilon/2) = \#F_k(\alpha; n, \epsilon/2).$$

Take a large integer k_0 such that $\log(1+\delta_k) \leq \epsilon/2$ for any $k \geq k_0$. Then by (5.2) we have $F_k(\alpha; n, \epsilon/2) \subset F(\alpha; n, \epsilon)$ and hence

$$f_k(\alpha; n, \epsilon/2) \leq f(\alpha; n, \epsilon) \qquad (k \geq k_0). \tag{5.3}$$

By (5.2) and Proposition 2.4, $P_{M_k}(q)$ converges to $P_M(q)$ uniformly on compact sets. And thus by Proposition 2.5, there exists $k_1 > k_0$ and a bounded sequence of real numbers $\{q_k\}_{k\geq k_1}$ such that $\alpha = P'_{M_k}(q_k)$. By Proposition 3.2, Proposition 3.3 and Proposition 4.1,

$$\begin{aligned} \limsup_{n\to\infty} \frac{\log f_k(\alpha; n, \epsilon/2)}{n} &\geq \log m \cdot \dim_H E_{M_k}(\alpha) \\ &= \inf_{q\in\mathbb{R}}\{-\alpha q + P_{M_k}(q)\} \\ &= -\alpha q_k + P_{M_k}(q_k). \end{aligned} \tag{5.4}$$

Since the sequence $\{q_k\}$ is bounded, there is a subsequence $\{q_{k_i}\}$ which converges to a finite point q_∞. It follows from Proposition 2.4 that

$$\begin{aligned} & |P_{M_{k_i}}(q_{k_i}) - P_M(q_\infty)| \\ \leq\ & |P_{M_{k_i}}(q_{k_i}) - P_M(q_{k_i})| + |P_M(q_{k_i}) - P_M(q_\infty)| \\ \leq\ & |q_{k_i}| \cdot \log(1+\delta_{k_i}) + |P_M(q_{k_i}) - P_M(q_\infty)|. \end{aligned}$$

By the continuity of $P_M(q)$, we have $\lim_{i\to\infty} P_{M_{k_i}}(q_{k_i}) = P_M(q_\infty)$. Thus by (5.3) and (5.4) we have

$$\limsup_{n\to\infty} \frac{\log f(\alpha; n, \epsilon)}{n} \geq -\alpha q_\infty + P_M(q_\infty) \geq \inf_{q\in\mathbb{R}}\{-\alpha q + P_M(q)\}.$$

Since ϵ can be picked arbitrary small, by Proposition 3.2 and 3.3, we obtain (5.1) for $\alpha \in (\alpha_M, \beta_M)$.

Now we consider the case $\alpha = \alpha_M$ or $\alpha = \beta_M$. By Proposition 3.2 and 3.3, we have

$$\dim_H E_M(\alpha_M) = \lim_{z\downarrow\alpha_M} \dim_H E_M(z)$$

and

$$\dim_H E_M(\beta_M) = \lim_{z\uparrow\beta_M} \dim_H E_M(z).$$

Thus

$$\dim_H E_M(\alpha_M) \geq \frac{1}{\log m} \lim_{z\downarrow\alpha_M} \inf_{q\in\mathbb{R}}\{-zq + P_M(q)\}$$

and

$$\dim_H E_M(\beta_M) \geq \frac{1}{\log m} \lim_{z\uparrow\beta_M} \inf_{q\in\mathbb{R}}\{-zq + P_M(q)\}.$$

By Proposition 2.5, we have

$$\dim_H E_M(\alpha_M) \geq \frac{1}{\log m} \inf_{q\in\mathbb{R}}\{-\alpha_M q + P_M(q)\}$$

and

$$\dim_H E_M(\beta_M) \geq \frac{1}{\log m} \inf_{q\in\mathbb{R}}\{-\beta_M q + P_M(q)\},$$

which finishes the proof of (5.1).

Step 3. $\quad \dim E_M(\alpha) \geq \dfrac{1}{\log m} \max\limits_{\mu}\{h(u)\colon M_*(\mu) = \alpha\} \qquad (\forall \alpha \in L_M).$

To see it, if $\mu \in \mathcal{M}_\sigma(\Sigma)$ satisfies $M_*(\mu) = \alpha$, then by Proposition 2.6, there exists a sequence of ergodic measures μ_k on Σ converging to μ in the weak-star topology, satisfying $\lim_{k\to\infty} h(\mu_k) = h(\mu)$. Let $\alpha_k = M_*(\mu_k)$. Then by (2.1), $\lim_{k\to\infty} \alpha_k = \alpha$. By Furstenberg and Kesten's Theorem [21], $\mu_k(E_M(\alpha_k)) = 1$. By the Shannon-McMillan-Breiman theorem (see

[37]), $\dim_H \mu_k = \dfrac{h(\mu_k)}{\log m}$. Hence we have $\dim_H E_M(\alpha_k) \geq \dfrac{h(\mu_k)}{\log m}$. Thus, by Proposition 3.2 and 3.3,

$$\dim_H E_M(\alpha) = \lim_{k\to\infty} \dim_H E_M(\alpha_k) \geq \lim_{k\to\infty} \frac{h(\mu_k)}{\log m} = \frac{h(\mu)}{\log m}.$$

Step 4. $\displaystyle \dim E_M(\alpha) \leq \frac{1}{\log m} \max_{\mu}\{h(u)\colon\ M_*(\mu) = \alpha\} \qquad (\forall \alpha \in L_M).$

For the trivial case $\alpha_M = \beta_M$, take μ to be the Parry measure on Σ (i.e. $\mu([I]) = m^{-n}$ for each $I \in \Sigma_n$). Then one can check directly that $M_*(\mu) = \alpha_M$ and

$$\dim_H E_M(\alpha_M) \leq \dim_H \Sigma = 1 = \frac{h(\mu)}{\log m}.$$

In what follows we assume that $\alpha_M < \beta_M$. First we consider $\alpha \in (\alpha_M, \beta_M)$. We define the maps $M_k\colon \Sigma \to L^+(\mathbb{R}^d, \mathbb{R}^d)$ for $k \in \mathbb{N}$ the same as in Step 2. As we have mentioned, there exists $k_1 > k_0$ and a bounded sequence of real numbers $\{q_k\}_{k\geq k_1}$ such that $\alpha = P'_{M_k}(q_k)$. By Proposition 4.1, there exists a sequence of ergodic measures ν_k on Σ such that

$$(M_k)_*(\nu_k) = \alpha \quad \text{and} \quad h(\nu_k) = -\alpha q_k + P_{M_k}(q_k). \tag{5.5}$$

Since the sequence $\{q_k\}$ is bounded, there is a subsequence $\{q_{k_i}\}$ which converges to a finite point q_∞; in the mean time ν_{k_i} converges to an invariant measure ν in the weak-star topology. By (2.1) and (5.2), we see that $M_*(\nu) = \lim_{i\to\infty} M_*(\nu_{k_i}) = \lim_{i\to\infty}(M_{k_i})_*(\nu_{k_i}) = \alpha$. By the upper semi-continuity of the entropy of invariant measures on Σ and the result proved in Step 1, we have

$$\begin{aligned} h(\nu) &\geq \limsup_{i\to\infty} h(\nu_{k_i}) \\ &= \limsup_{i\to\infty}(-\alpha q_{k_i} + P_{M_{k_i}}(q_{k_i})) = -\alpha q_\infty + P_M(q_\infty) \\ &\geq \log m \cdot \dim_H E_M(\alpha). \end{aligned}$$

Now assume $\alpha = \alpha_M$ or β_M. Pick $\alpha_n \in (\alpha_M, \beta_M)$ such that

$$\lim_{n\to\infty} \alpha_n = \alpha.$$

Choose $\nu_n \in \mathcal{M}_\sigma(\Sigma)$ such that

$$M_*(\nu_n) = \alpha_n \quad \text{and} \quad h(\nu_n)/\log m \geq \dim_H E_M(\alpha_n).$$

Let ν be a cluster point of $\{\nu_n\}$ in the weak-star topology. Then by (5.2)

$$M_*(\nu) = \lim_{n\to\infty} M_*(\nu_n) = \lim_{n\to\infty} \alpha_n = \alpha.$$

By Proposition 3.2 and 3.3, and the upper semi-continuity of the entropy of invariant measures on Σ,

$$\dim_H E_M(\alpha) = \lim_{n\to\infty} \dim_H E_M(\alpha_n) \le \lim_{n\to\infty} \frac{h(\nu_n)}{\log m} \le \frac{h(\nu)}{\log m},$$

which completes the proof. $\square$

6. Final remarks

In this section we give several remarks.

First Theorem 1.1 can be extended from the full shift space (Σ, σ) to a subshift space (Σ_A, σ) where A is a $m \times m$ 0-1 primitive matrix. To attain this, one needs to modify our proof slightly.

The reader may care about how to deal with the points x at which $\lambda_M(x)$ does not exist. Actually we can define $\overline{\lambda}_M(x)$ and $\underline{\lambda}_M(x)$ by taking limsup and liminf in (1.1), respectively. By Proposition 2.2, the ranges of $\overline{\lambda}_M(x)$ and $\underline{\lambda}_M(x)$ are both equal to L_M.

We remark that for any $\alpha \in L_M$,

$$\begin{aligned} \dim_H\{x \in \Sigma\colon \overline{\lambda}_M(x) = \alpha\} &= \dim_H\{x \in \Sigma\colon \underline{\lambda}_M(x) = \alpha\} \\ &= \Lambda_M(\alpha) \\ &= \dim_H\{x \in \Sigma\colon \lambda_M(x) = \alpha\}. \end{aligned}$$

It is obvious that $\dim_H\{x \in \Sigma\colon \overline{\lambda}_M(x) = \alpha\} \ge \Lambda_M(\alpha)$ and $\dim_H\{x \in \Sigma\colon \underline{\lambda}_M(x) = \alpha\} \ge \Lambda_M(\alpha)$. Now we prove the "$\le$". Assume that $\Lambda_M(\alpha) < t$. By Proposition 3.2, there exist $\epsilon > 0$, $\delta > 0$ and $N_0 \in \mathbb{N}$ such that

$$f(\alpha; n, \epsilon) < m^{n(t-\delta)}, \qquad \forall\, n \ge N_0.$$

Note that for any $\ell > N_0$, $\{x \in \Sigma\colon \overline{\lambda}_M(x) = \alpha\}$ and $\{x \in \Sigma\colon \underline{\lambda}_M(x) = \alpha\}$ are subsets of

$$\bigcap_{k=\ell}^{\infty} \bigcup_{n\ge k} F(\alpha; n, \epsilon).$$

Therefore, for any $\ell > N_0$, the collection

$$\mathcal{G}_\ell = \{[\omega]\colon \omega \in F(\alpha; n, \epsilon) \text{ for some } n \ge \ell\}$$

is a cover of the sets $\{x \in \Sigma\colon \overline{\lambda}_M(x) = \alpha\}$ and $\{x \in \Sigma\colon \underline{\lambda}_M(x) = \alpha\}$. Since

$$\begin{aligned} \sum_{[\omega]\in\mathcal{G}_\ell} (\mathrm{diam}[\omega])^t &= \sum_{n=\ell}^{\infty} \sum_{[\omega]\in F(\alpha;n,\epsilon)} (\mathrm{diam}[\omega])^t \\ &\le \sum_{n=\ell}^{\infty} m^{n(t-\delta)} m^{-nt} < \frac{1}{1-m^{-\delta}} \end{aligned}$$

for each $\ell > N_0$, we have $\dim_H\{x \in \Sigma\colon \overline{\lambda}_M(x) = \alpha\} \le t$ and $\dim_H\{x \in \Sigma\colon \underline{\lambda}_M(x) = \alpha\} \le t$. This finishes the proof.

Using a method similar to that in [13] or [17], one can prove that if $\alpha_M < \beta_M$, then

$$\dim_H\{x \in \Sigma\colon \underline{\lambda}_M(x) < \overline{\lambda}_M(x)\} = \dim_H \Sigma.$$

For related results in the scalar function case, see e.g. [3, 13, 17, 31].

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084 and Department of Mathematics, The Chinese University of Hong Kong, Hong Kong

E-mail address: dfeng@math.tsinghua.edu.cn