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Multifractal analysis of weak Gibbs measures and phase transition—application to some Bernoulli convolutions

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Abstract. For a given expanding d-fold covering transformation of the one-dimensional torus, the notion of weak Gibbs measure is defined by a natural generalization of the classical Gibbs property. For these measures, we prove that the singularity spectrum and the L^q-spectrum form a Legendre transform pair. The main difficulty comes from the possible existence of first-order phase transition points, that is, points where the L^q-spectrum is not differentiable. We give examples of weak Gibbs measure with phase transition, including the so-called Erdös measure.

0. *Introduction*

The one-dimensional torus $\mathbf{S}^1 := \mathbb{R}/\mathbb{Z}$ is endowed with the natural metric and $\dim_H M$ denotes the Hausdorff dimension of any $M \subset \mathbf{S}^1$ (by convention, $\dim_H \emptyset = -\infty$). Let $B_r(x)$ be the closed ball of radius $r > 0$ centered at $x \in \mathbf{S}^1$; the *local dimension* of a Borel probability measure η at x is by definition

$$\text{DIM}_\eta(x) := \lim_{r\to 0} \frac{\log \eta(B_r(x))}{\log r}, \tag{1}$$

provided that the limit exists. The *level set* $E(\alpha)$ ($\alpha \in \mathbb{R}$) associated to η is the set of points $x \in \mathbf{S}^1$ such that $\text{DIM}_\eta(x)$ exists and is equal to α. The map $\alpha \mapsto \dim_H E(\alpha)$ is called the *singularity spectrum* of η. Heuristic arguments using techniques of statistical mechanics (see [**16**] for example) show that the singularity spectrum should be finite on a compact interval denoted $\text{DOM}(\eta)$ and is expected to be the Legendre transform conjugate of the *L^q-spectrum* τ associated to η (see Definition 1.3); that is, for all $\alpha \in \text{DOM}(\eta)$,

$$\dim_H E(\alpha) = \inf\{\alpha q - \tau(q);\, q \in \mathbb{R}\} =: \tau^*(\alpha). \tag{2}$$

The *multifractal analysis* of a probability measure is concerned with rigorous arguments ensuring that the Legendre transform formula (2) holds; the first multifractal formalisms were established for *Gibbs* [**7, 36, 39**], *quasi-Bernoulli* [**4, 17**] and *self-similar measures* [**6, 25, 33**]: we refer to [**34**] for a general overview of the subject and complete references. In this paper, we focus our attention on the notion of *weak Gibbs measure* defined in [**46**] by a natural generalization of the classical Gibbs property (see Definition 1.1). Of special interest are the g-measures, as considered in [**22**], which turn out to be weak Gibbs, as well as a large class of the *conformal measures* studied in [**19**]. It follows from the variational characterization of the g-measure proved in [**26**] that a non-ergodic g-measure cannot be Gibbs. The existence of non-ergodic g-measures established in [**3**] shows that a weak Gibbs measure need not be Gibbs.

Let T be an expanding $\mathtt{d}$-fold covering transformation of $\mathbf{S}^1$ (with $\mathtt{d} \geq 2$) so that there exists a *Markov partition* $\mathcal{M}_1$ of T, by $\mathtt{d}$ (semi-open) intervals; if $\mathcal{M}_n$ ($n \geq 1$) denotes the partition of $\mathbf{S}^1$ by the *n-step basic intervals* then $\mathcal{M} := \bigcup_n \mathcal{M}_n$ is called the *Markovian net* of T. From now on, we assume that η is a weak Gibbs measure of a potential ϕ defined on the symbolic space which codes the Markovian dynamics of T. Our approach to the multifractal analysis of η is classical; we consider the level sets $E(\alpha|\mathcal{M})$ associated to the Markovian local dimension $\mathrm{DIM}_\eta(\cdot|\mathcal{M})$ and we give (Theorem A) an intermediate Legendre transform formula, say, for any α in the interior of $\mathrm{DOM}(\eta)$,

$$\dim_H E(\alpha|\mathcal{M}) = \tau_\phi^*(\alpha), \tag{3}$$

where the concave map τ_ϕ is implicitly defined by a pressure equation. An important and non-trivial step is to prove (Theorem B) that $\dim_H E(\alpha|\mathcal{M}) = \dim_H E(\alpha)$, when η is a weak Gibbs measure. The last step is achieved by proving (Theorem C) that τ_ϕ coincides with the L^q-spectrum τ and we conclude (Theorem A′) that the Legendre transform formula (2) holds, when η is weak Gibbs.

These generalizations are relevant essentially because the L^q-spectrum τ need not be real-analytic/differentiable when η is weak Gibbs. For the thermodynamic formalism on lattices, a system is said to exhibit a phase transition when a thermodynamic function displays a defect of analyticity at some critical value (see [**40**, ch. 5]). We shall say that the real number q_c is a *phase transition point* (respectively a *first-order phase transition point*) if τ is not real-analytic (respectively not differentiable) at q_c: this will make sense, for we shall prove (Theorem C) that τ coincides with a thermodynamic function determined by a pressure equation. Let us denote by $\tau'(q_c^+)$ (respectively $\tau'(q_c^-)$) the right (respectively left) derivative of τ at q_c; if $\tau'(q_c^+) < \tau'(q_c^-)$ then the mass distribution principle does not apply to give the desired lower bound of $\dim_H E(\alpha|\mathcal{M})$ when $\tau'(q_c^+) < \alpha < \tau'(q_c^-)$. Our argument depends on the tangency property of the topological pressure, which yields a thermodynamic characterization of $\tau'(q^+)$ and $\tau'(q^-)$ for any $q \in \mathbb{R}$ (Lemma 3.2) and on a formula which gives the *Billingsley dimension* of the *generic points* of a (not necessarily ergodic) shift-invariant measure.

In §1 we describe the framework of the expanding $\mathtt{d}$-fold covering transformation of the one-dimensional torus and the thermodynamic formalism of the equilibrium state. Then we give a definition of the weak Gibbs measure (Definition 1.1) making possible a rigorous statement of Theorems A, B, C and A′; the proofs of these theorems are given in §3.

Section 2 is devoted to an illustration of the previous results through the analysis of two examples of *Bernoulli convolutions*. We first study (§2.1) the so-called (2, 3)-*Bernoulli convolution*: this measure is proved to be weak Gibbs (Theorem 2.4) and we show that its L^q-spectrum displays a phase transition point (Theorem 2.5). The case of the (2, 3)-Bernoulli convolution is closely related to our main application, concerned with the multifractal analysis of the *Erdös measure*. The Bernoulli convolution ν_β $(1 < \beta < 2)$ defined in §2.2 is a non-atomic probability measure supported by the unit interval which is either continuous or purely singular (see [**21**]). Erdös proved in [**10**] that ν_β is purely singular when β is a Pisot number (i.e. an algebraic integer whose conjugates have modulus less than 1). When $\beta = (1 + \sqrt{5})/2$ the measure $\nu := \nu_\beta$ is called [**42**] the Erdös measure. The multifractal analysis of ν has been partially studied in [**13, 24, 27**]; we prove (Theorem 2.9) that ν is a weak Gibbs measure (but not Gibbs) with respect to a suitable 3-fold covering transformation (the potential of ν is defined by means of continued fractions), so that the full multifractal formalism is completely established (our contribution is concerned with the decreasing part of the singularity spectrum of ν). In [**13**], the first author completes a result in [**24**] by giving an explicit formula for the L^q-spectrum of the Erdös measure, proving in addition that there exists a negative q_c such that: (i) $\tau(q) = q \log 2/\log \beta$ for any $q \leq q_c$; (ii) τ is infinitely differentiable at any $q > q_c$; and (iii) τ is not differentiable at q_c. The weak Gibbs property of ν makes possible (Theorem C) the use of the thermodynamic formalism and one may interpret q_c as a first-order phase transition. The variational principle allows an alternative approach to (i) (Theorem 2.10), which is actually enough to ensure that τ is not real-analytic at $q_c < 0$; the fact that τ is not differentiable at q_c is more difficult to establish: Appendix A is devoted to a self-contained proof of this result based on the original argument in [**13**]. We point out that the multifractal formalism is valid for ν_{β_n}, when β_n is the Pisot number such that $\beta_n^n = \beta_n^{n-1} + \cdots + \beta_n + 1$ $(n \geq 3)$ [**13, 32**], the corresponding L^q-spectrum being differentiable on the whole real line [**13**]. Even if some partial results can be achieved (see e.g. [**14**]), the general case of a Pisot number seems to remain a difficult problem.

1. *Multifractal formalism of weak Gibbs measures*

1.1. *General framework.* Let T be an expanding continuous transformation of $\mathbf{S}^1$ such that $T^{-1}\{x\}$ is of cardinality $\mathtt{d} > 1$ for any $x \in \mathbf{S}^1$. Assuming that $T(0) = 0$, there exist $\mathtt{d}$ points $x_{\mathtt{d}} = 0 = x_0, x_1, \ldots, x_{\mathtt{d}-1}$ such that T is a one-to-one, onto mapping from $[i] := [x_i, x_{i+1}[$ to $\mathbf{S}^1$. The bilateral restrictions $T_i :]x_i, x_{i+1}[\to \mathbf{S}^1 \backslash \{0\}$ are diffeomorphisms with Hölder-continuous derivative (the transformation T may not be differentiable at the points x_i): we say that T is a regular $\mathtt{d}$-fold covering transformation ($\mathtt{d}$-f.c.t.) of $\mathbf{S}^1$. Let $\Omega_{\mathtt{d}}$ be the one-sided direct product of an infinite number of copies of the alphabet $\mathcal{A}_{\mathtt{d}} := \{0, \ldots, \mathtt{d} - 1\}$, i.e. $\Omega_{\mathtt{d}} := \prod_0^\infty \mathcal{A}_{\mathtt{d}}$; we assume that $\Omega_{\mathtt{d}}$ is endowed with the product topology and $\sigma : \Omega_{\mathtt{d}} \to \Omega_{\mathtt{d}}$ denotes the (one-sided) shift transformation. Each $x \in \mathbf{S}^1$ is associated to a unique point $\chi(x) := (\omega_k) \in \Omega_{\mathtt{d}}$ such that $T^k(x) \in [\omega_k]$, for any integer $k \geq 0$, and $\chi : \mathbf{S}^1 \to \Omega_{\mathtt{d}}$ is a one-to-one map such that the following diagram commutes

$$\begin{array}{ccc} \mathbf{S}^1 & \xrightarrow{T} & \mathbf{S}^1 \\ \chi \big\downarrow & & \big\downarrow \chi \\ \Omega_{\mathrm{d}} & \xrightarrow{\sigma} & \Omega_{\mathrm{d}} \end{array}$$

(we usually call χ the coding map of T). For any $x \in \mathbf{S}^1$ such that $\omega = \chi(x)$, we denote by $I_n(x) = [\omega_0 \cdots \omega_{n-1}]$ the *n-step basic interval* about x, that is, the set of points $y \in \mathbf{S}^1$ such that $T^k(y) \in [\omega_k]$, for $k = 0, \dots, n-1$. Let $[\![w]\!]$ be the set of the $\xi \in \Omega_{\mathrm{d}}$ such that $\xi_0 \cdots \xi_{n-1} = w$, which is usually called the n-step cylinder set about w (clearly, $\chi^{-1}([\![w]\!]) = [w]$). If $\mathcal{M}_0 := \{\mathbf{S}^1\}$ and if $\mathcal{M}_n$ $(n > 0)$ is the collection of the n-step basic intervals then $\mathcal{M} := \{\mathcal{M}\}_{n=0}^{\infty}$ is by definition the Markovian net of T. The Hölder-continuous function $\Psi_0 : \Omega_{\mathrm{d}} \to \mathbb{R}$ such that $\Psi_0(\omega) = -\log|T'(x)|$ when $\omega = \chi(x)$ (and extended to Ω_{d} by continuity) is called the *volume-derivative potential* of T with respect to $\mathcal{M}$. By a classical application of the Mean Value Theorem, there exists a constant $K > 1$ such that, for any $\omega \in \chi(\mathbf{S}^1)$ and any integer $n > 0$,

$$\frac{1}{K} \leq \frac{|[\omega_1 \cdots \omega_n]|}{\exp(S_n \Psi_0(\omega))} \leq K \tag{4}$$

($|J|$ stands for the length of any interval $J \subset \mathbf{S}^1$ and $S_n \Psi_0(\omega) := \sum_{k=0}^{n-1} \Psi_0(\sigma^k \omega)$).

Definition 1.1. The measure η defined on $\mathbf{S}^1$ is said to be a weak Gibbs measure of the potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$, if there exists a sub-exponential sequence of real numbers $K(n) > 1$ (i.e. $\lim_n (1/n) \log K(n) = 0$) such that, for any $n > 0$, and any $\omega \in \chi(\mathbf{S}^1)$,

$$\frac{1}{K(n)} \leq \frac{\eta[\omega_0 \cdots \omega_{n-1}]}{\exp(S_n \phi(\omega))} \leq K(n); \tag{5}$$

without loss of generality, we assume that $K(n)$ increases with n.

If $K(n)$ is constant, one recovers the classical notion of Gibbs measures [**2**]; accordingly, by (4), the Lebesgue measure is a Gibbs measure of the volume-derivative potential Ψ_0.

Suppose that the probability measure η is fully supported by $\mathbf{S}^1$; for $\omega \in \chi(\mathbf{S}^1)$, we set $\phi_1(\omega) := \log \eta[\omega_0]$ and for any $n > 1$,

$$\phi_n(\omega) := \log \frac{\eta[\omega_0 \cdots \omega_{n-1}]}{\eta[\omega_1 \cdots \omega_{n-1}]}. \tag{6}$$

Using the density of $\chi(\mathbf{S}^1)$ in Ω_{d}, one extends ϕ_n to a continuous map defined on the whole of Ω_{d} and called the *n-step potential* of η. The following lemma provides a useful way to prove that η is weak Gibbs (the proof is left to the reader).

LEMMA 1.2. *Let ϕ_n be the n-step potential of a fully supported probability measure η; if ϕ_n converges uniformly to a potential ϕ then η is a weak Gibbs measure of ϕ.*

Even if a weak Gibbs measure need not be invariant under the dynamics of T, the notion is closely related to the theory of equilibrium states. The topological pressure of a potential

$\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$ (simply assumed to be continuous) is, by definition,

$$P(\phi) := \lim_n \frac{1}{n} \log \sum_{w \in \mathcal{A}_{\mathrm{d}}^n} \exp(S_n\phi^*[\![w]\!]), \tag{7}$$

where $S_n\phi^*[\![w]\!] := \max\{S_n\phi(\xi);\ \xi \in [\![w]\!]\}$ (a sub-additive argument ensures that the limit in (7) does exist). The variational principle of Walters [**45**] asserts that, for any σ-invariant probability measure η of metric entropy $\mathrm{h}_\sigma(\eta)$, one has $\mathrm{h}_\sigma(\eta) + \eta(\phi) \le P(\phi)$, equality being obtained when η is an equilibrium state of ϕ; the set of equilibrium states of ϕ is a non-empty weak-$*$ compact convex set (in fact a Choquet simplex), whose extreme points are σ-ergodic measures. We shall use the basic properties of the topological pressure listed in [**45**, Theorem 9.7].

It is worth noting that the weak Gibbs property is satisfied by a g-measure in the sense of Keane [**22**] as well as the so-called conformal measures. More precisely, given a potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$, the probability measure η defined on $\mathbf{S}^1$ is said to be $e^{-\phi\circ\chi}$-conformal, if for any Borel set A, with $A \subset [i]$ for some $i \in \mathcal{A}_{\mathrm{d}}$, one has

$$\eta \circ T(A) = \int_A e^{-\phi\circ\chi(x)}\, d\eta(x). \tag{8}$$

Under the condition that η is fully supported by $\mathbf{S}^1$, it is easily seen that (8) implies the uniform convergence of the n-step potentials of η, ensuring by Lemma 1.2 that η is weak Gibbs. However, the converse is not true in general, even if there exists a partial reciprocal; to see this, we notice that if in addition to being of full support and $e^{-\phi\circ\chi}$-conformal, one also assumes that η is T-invariant, then, according to the previous remark, η is a weak Gibbs measure of ϕ, but it is also necessary that ϕ is normalized in the sense that $\sum_{\sigma\xi=\omega} e^{\phi(\xi)} \equiv 1$ (the n-step potentials are trivially normalized and, using the uniform convergence, one deduces that ϕ is also normalized). Now, if one assumes that η is a T-invariant weak Gibbs measure of the normalized potential ϕ, then it is easily seen that η is an equilibrium state of ϕ and by the variational principle in [**26**, Théorème 1], one deduces that η is $e^{-\phi\circ\chi}$-conformal. When the potential ϕ is normalized one usually writes $g = e^{\phi\circ\chi}$ so that the T-invariant $e^{-\phi\circ\chi}$-conformal measures are exactly the g-measures.

1.2. *Statement of the multifractal theorems.* The classical starting point of the multifractal analysis is to make possible an application of the Shannon–McMillan Theorem, by the introduction of the *Markovian local dimension*

$$\mathrm{DIM}_\eta(x|\mathcal{M}) := \lim_n \frac{\log \eta(I_n(x))}{\log |I_n(x)|}, \tag{9}$$

provided that the limit exists; then the α-level set corresponding to this local dimension is by definition

$$E(\alpha|\mathcal{M}) := \{x \in \mathbf{S}_1;\ \mathrm{DIM}_\eta(x|\mathcal{M}) = \alpha\}. \tag{10}$$

From now on, suppose that the probability measure η defined on $\mathbf{S}^1$ is a weak Gibbs measure of the negative potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$. For any $(q,t) \in \mathbb{R}\times\mathbb{R}$ we consider the *partition function*

$$\mathbb{Z}_n(q,t) := \sum_{w\in\mathcal{A}_{\mathrm{d}}^n} \frac{\eta[w]^q}{|[w]|^t}. \tag{11}$$

Using the Gibbs property (4) of the Lebesgue measure and the weak Gibbs property (5) of η, it follows from the definition given in (7) that

$$P(q\phi - t\Psi_0) = \lim_n \frac{1}{n} \log \mathbb{Z}_n(q,t) =: \mathbb{P}_\phi(q,t). \tag{12}$$

The convex property of P implies that $\mathbb{P}_\phi$ is a convex map on $\mathbb{R} \times \mathbb{R}$; for any $q, q' \in \mathbb{R}$ with $q' > q$, one has

$$q'\phi - t\Psi_0 \leq (q' - q)\sup(\phi) + (q\phi - t\Psi_0),$$

where $\sup(\phi) := \sup\{\phi(\omega); \omega \in \Omega_{\mathrm{d}}\}$; hence

$$P(q'\phi - t\Psi_0) \leq (q' - q)\sup(\phi) + P(q\phi - t\Psi_0),$$

and thus

$$\frac{\mathbb{P}_\phi(q,t) - \mathbb{P}_\phi(q',t)}{q - q'} \leq \sup(\phi). \tag{13}$$

Since ϕ is supposed continuous and negative, $\sup(\phi) < 0$, and one deduces from (13) that the convex map $q \mapsto \mathbb{P}_\phi(q,t)$ decreases from $+\infty$ to $-\infty$. Using the assumption that $\sup(\Psi_0) < 0$, the same argument shows that, for any $q \in \mathbb{R}$ fixed, the convex map $t \mapsto \mathbb{P}_\phi(q,t)$ increases from $-\infty$ to $+\infty$. Therefore, there exists an increasing concave map $\tau_\phi : \mathbb{R} \to \mathbb{R}$ defined by the implicit equation $\mathbb{P}_\phi(q, \tau_\phi(q)) = 0$. The underlying thermodynamic formalism related to the convex map $\mathbb{P}_\phi$ leads to a characterization of the points where τ_ϕ is not differentiable (i.e. the first-order phase transition points): actually, according to Lemma 3.2 proved in §3.1,

$$\tau'_\phi(q^-) = \sup_\mu \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} \quad \text{and} \quad \tau'_\phi(q^+) = \inf_\mu \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\},$$

where the infimum and the supremum are respectively taken over the probability measures μ, equilibrium states of the potential $q\phi - \tau_\phi(q)\Psi_0$. Lemma 3.2 also provides a description of the behavior of τ_ϕ about $+\infty$ and $-\infty$: more precisely we shall prove that

$$\inf_\mu \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} = \lim_{q \to +\infty} \frac{\tau_\phi(q)}{q} =: \underline{\alpha} \quad \text{and} \quad \sup_\mu \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} = \lim_{q \to -\infty} \frac{\tau_\phi(q)}{q} =: \overline{\alpha},$$

where the infimum and the supremum are respectively taken over the probability measures μ which are σ-invariant on Ω_{d}.

It is now possible to state the multifractal formalism of a weak Gibbs measure with respect to the Markovian local dimension.

THEOREM A. *Let $\mathcal{M}$ be the Markovian net of a* d*-f.c.t. T and η be a weak Gibbs measure of a negative potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$. The set of all points α with $E(\alpha\,|\mathcal{M}) \neq \emptyset$ is the interval $[\underline{\alpha}, \overline{\alpha}]$ and $\dim_H E(\alpha\,|\mathcal{M}) = \tau^*_\phi(\alpha)$, for any $\underline{\alpha} < \alpha < \overline{\alpha}$.*

When T is differentiable on the whole torus and η is a Gibbs measure associated to a potential which is continuous with respect to the natural topology on $\mathbf{S}^1$, it is well known [**36**] that $E(\alpha) = E(\alpha|\mathcal{M})$. We emphasize that, in our framework, the transformation T (respectively the potential of the weak Gibbs measure) may not be differentiable (respectively continuous) for the natural topology on $\mathbf{S}^1$: in that case one may have $E(\alpha) \neq E(\alpha|\mathcal{M})$. The following theorem is a crucial point of our multifractal analysis.

THEOREM B. *Let $\mathcal{M}$ be the Markovian net of a* $\mathtt{d}$*-f.c.t. T and η be a weak Gibbs measure. Then,* $\dim_H E(\alpha|\mathcal{M}) = \dim_H E(\alpha)$*, for any $\alpha \in \mathbb{R}$.*

Definition 1.3. The L^q-spectrum of η is the concave map $\tau : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\tau(q) := \liminf_{r \to 0} \frac{\log \inf \big\{ \sum_i \eta(B_i)^q ; \{B_i\}_i \big\}}{\log r},$$

where $\{B_i\}_i$ runs over the family of covers of $\mathbf{S}^1$ by closed balls of radius r.

THEOREM C. *Let T be a* $\mathtt{d}$*-f.c.t. and η be a weak Gibbs measure of a negative potential $\phi : \Omega_{\mathtt{d}} \to \mathbb{R}$. Then, $\tau_\phi(q) = \tau(q)$, for any $q \in \mathbb{R}$.*

Finally, we can state the strong version of Theorem A.

THEOREM A$'$. *Let T be a* $\mathtt{d}$*-f.c.t. and η be a weak Gibbs measure of a negative potential $\phi : \Omega_{\mathtt{d}} \to \mathbb{R}$. The domain of the finite values of* $\dim_H E(\alpha)$ *is* $\mathrm{DOM}(\eta) = [\underline{\alpha}, \overline{\alpha}]$ *and* $\dim_H E(\alpha) = \tau^*(\alpha)$*, for any $\underline{\alpha} < \alpha < \overline{\alpha}$.*

2. *Bernoulli convolutions*

For practical reasons, we shall need basic notions about the set of words on an alphabet. Given $\mathcal{A} := \{0, \dots, s-1\}$ $(s \geq 2)$ a finite alphabet, each element in $\mathcal{A}^n$ $(n \geq 1)$ is denoted by a string of n letters/digits in $\mathcal{A}$ that we call a word; by convention $\mathcal{A}^0$ is reduced to the empty word $\emptyset$. By definition, $\mathcal{A}^*$ is the set of words on $\mathcal{A}$, that is, $\mathcal{A}^* := \bigcup_{n=0}^{\infty} \mathcal{A}^n$. We denote by wm the concatenation of the two words w and m so that $\mathcal{A}^*$, endowed with the concatenation, is a monoid with unit element $\emptyset$. Whenever $x_0, \dots, x_{s-1}$ are s elements of a monoid $(X, \star)$ with identity element $\mathtt{e}$, we denote $x_\emptyset := \mathtt{e}$ and $x_w := x_{\xi_0} \star \cdots \star x_{\xi_{n-1}}$, for any word $w = \xi_0 \cdots \xi_{n-1} \in \mathcal{A}^*$.

2.1. *The (2,3)-Bernoulli convolution.*

Let $\mathtt{b}$ and $\mathtt{d}$ be two integers with $2 \leq \mathtt{b} < \mathtt{d}$ ($\mathtt{b}$ is for *basis* and $\mathtt{d}$ is for *digit*). The (uniform) $(\mathtt{b}, \mathtt{d})$-*Bernoulli convolution* μ is defined as the probability distribution of the random variable $X : \Omega_{\mathtt{d}} \to \mathbb{R}$ such that

$$X(\omega) = \frac{1}{\mathtt{b}} \frac{\mathtt{b}-1}{\mathtt{d}-1} \sum_{n=0}^{\infty} \frac{\omega_n}{\mathtt{b}^n},$$

for any $\omega = (\omega_i)_{i=0}^{\infty} \in \Omega_{\mathtt{d}}$, when $\Omega_{\mathtt{d}}$ is endowed with the equidistributed Bernoulli probability. The measure μ is non-atomic, is supported by the whole unit interval $I = [0, 1]$, and is either absolutely continuous or purely singular (see [**21**, Theorem 35]). In this section we focus our attention on the $(2, 3)$-Bernoulli convolution μ, when Ω_3 is endowed with the equidistributed Bernoulli measure λ_3 (by [**8**, Propositions 5.2 and 5.3], μ is known to be purely singular). The measure μ turns out to be self-similar; to see this, notice that for $M \subset I$ one has $X(\omega) \in M$ if and only if $\omega_0/4 + X(\sigma\omega)/2 \in M$, that is, $S_{\omega_0} \circ X \circ \sigma(\omega) \in M$, where $S_{\omega_0}(x) := x/2 + \omega_0/4$; since λ_3 is a Bernoulli measure, for $\varepsilon = 0, 1$ or 2 one obtains

$$\begin{aligned}\lambda_3([\![\varepsilon]\!] \cap \{S_\varepsilon \circ X \circ \sigma(\omega) \in M\}) &= \lambda_3[\![\varepsilon]\!]\lambda_3\{S_\varepsilon \circ X \circ \sigma(\omega) \in M\} \\ &= \tfrac{1}{3}\lambda_3\{S_\varepsilon \circ X(\omega) \in M\} = \tfrac{1}{3}\mu(S_\varepsilon^{-1}(M)).\end{aligned}$$

Finally, one deduces that μ satisfies the following self-similarity equation:

$$\mu = \tfrac{1}{3}\mu \circ S_0^{-1} + \tfrac{1}{3}\mu \circ S_1^{-1} + \tfrac{1}{3}\mu \circ S_2^{-1}. \tag{14}$$

Each dyadic sub-interval of I is coded by a word $w \in \{0,2\}^*$ and in what follows we denote $[w] := S_w(I)$. From the self-similarity property of the measure μ given in (14) and the fact that $S_1^{-1}S_{00}(M) = \{0\}$ and $S_2^{-1}S_{00}(M) = \emptyset$, for any $M \subset I$, one obtains

$$\begin{aligned}\mu[00w] &= \tfrac{1}{3}\mu(S_0^{-1}[00w]) + \tfrac{1}{3}\mu(S_1^{-1}[00w]) + \tfrac{1}{3}\mu(S_2^{-1}[00w])\\ &= \tfrac{1}{3}\mu(S_0^{-1}S_0[0w]) + \tfrac{1}{3}\mu(S_1^{-1}S_{00}[w]) + \tfrac{1}{3}\mu(S_2^{-1}S_{00}[w]) = \tfrac{1}{3}\mu([0w]).\end{aligned}$$

Likewise, using the identity $S_{12} = S_{20}$ and the fact that $S_0^{-1}S_{20}(M) \subset \{1\}$, for any $M \subset I$,

$$\begin{aligned}\mu[20w] &= \tfrac{1}{3}\mu(S_0^{-1}S_{20}[w]) + \tfrac{1}{3}\mu(S_1^{-1}S_{20}[w]) + \tfrac{1}{3}\mu(S_2^{-1}S_{20}[w])\\ &= \tfrac{1}{3}\mu(S_1^{-1}S_{12}[w]) + \tfrac{1}{3}\mu[0w]\\ &= \tfrac{1}{3}\mu([2w]) + \tfrac{1}{3}\mu([0w]),\end{aligned}$$

and the following matricial identity holds:

$$\begin{pmatrix}\mu[00w]\\ \mu[20w]\end{pmatrix} = \tfrac{1}{3}P_0\begin{pmatrix}\mu[0w]\\ \mu[2w]\end{pmatrix}, \quad \text{where } P_0 := \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}. \tag{15}$$

Since $S_{10} = S_{02}$, one gets in the same way that

$$\begin{pmatrix}\mu[02w]\\ \mu[22w]\end{pmatrix} = \tfrac{1}{3}P_2\begin{pmatrix}\mu[0w]\\ \mu[2w]\end{pmatrix}, \quad \text{where } P_2 := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \tag{16}$$

A simple induction using (15), (16) and the fact that $\mu[0] = \mu[2] = 1/2$ yields, for any $\omega_0\cdots\omega_{n-1} \in \{0,2\}^n$,

$$\mu[\omega_0\cdots\omega_{n-1}] = \frac{1}{2}\frac{1}{3^{n-1}}\,{}^tV_{\omega_0}P_{\omega_0\cdots\omega_{n-1}}V, \tag{17}$$

where

$$V_0 := \begin{pmatrix}1\\0\end{pmatrix}, \quad V_2 := \begin{pmatrix}0\\1\end{pmatrix}, \quad V := \begin{pmatrix}1\\1\end{pmatrix}.$$

In order to fit our framework, the measure μ is identified to a measure on the torus $\mathbf{S}^1$. Let $T_2 : \mathbf{S}^1 \to \mathbf{S}^1$ be the multiplication by 2 (mod 1); it is a 2-f.c.t. coded by the full shift $\sigma : \Sigma \to \Sigma$ where $\Sigma := \Pi_0^\infty\{0,2\}$ and the volume-derivative potential is $\Psi_0 : \Sigma \to \mathbb{R}$ such that $\Psi_0 \equiv -\log 2$. A dyadic interval $[\omega_0\cdots\omega_{n-1}]$ becomes an n-step basic interval, i.e. $[0] = [0, 1/2[$, $[2] = [1/2, 1[$ and $x \in [\omega_0\cdots\omega_{n-1}]$ if and only if $T_2^k(x) \in [\omega_k]$ whenever $0 \le k \le n-1$.

Denote by $\overline{0} := (\omega_i = 0)_{i=0}^\infty$; then, for any potential $\psi : \Sigma \to \mathbb{R}$,

$$\exp(S_n\psi(2\overline{0})) = \frac{\exp(\psi(2\overline{0}))}{\exp(\psi(\overline{0}))}\{\exp(\psi(\overline{0}))\}^n.$$

A direct computation using (17) gives $\mu[20^{n-1}] = n/(2\cdot 3^{n-1})$, for any $n > 0$; if μ is a Gibbs measure of ψ, then there exists a constant $K > 1$ such that, for any $n \ge 1$,

$$\frac{1}{K} \le \frac{1}{n}\{3\exp(\psi(\overline{0}))\}^n \le K;$$

this is impossible and one concludes that μ is not a Gibbs measure. Our aim is to prove that μ satisfies the weak Gibbs property with respect to some potential to be identified. We consider the probability measure μ' defined on $\mathbf{S}^1$ by setting, for any word $\omega_0 \cdots \omega_{n-1} \in \{0, 2\}^n$,

$$\mu'[\omega_0 \cdots \omega_{n-1}] = \frac{1}{2}\frac{1}{3^n}\, {}^tVP_{\omega_0\cdots\omega_{n-1}}V. \tag{18}$$

It is clear that μ' is T_2-invariant and the next proposition shows how it is related to μ.

PROPOSITION 2.1. *For any $\omega \in \Sigma$, and any integer $n \geq 1$,*

$$\frac{3}{n+2} \leq \frac{\mu[\omega_0 \cdots \omega_{n-1}]}{\mu'[\omega_0 \cdots \omega_{n-1}]} \leq 3.$$

Proof. Let $w \in \{0, 2\}^n$; it is easily seen that $\mu[w]/\mu'[w] \leq 3$ and thus it remains to prove that $\mu[w]/\mu'[w] \geq 3/(n+2)$. Assume that $w = 0^{a_1}2^{a_2} \cdots \epsilon^{a_k}$, with $\epsilon \in \{0, 2\}$, $k \geq 2$ and $a_1, \ldots, a_k > 0$ (the cases $a_1 = 0$ or n are similar); if we denote

$$P_2^{a_2-1} \cdots P_\epsilon^{a_k} V =: \begin{pmatrix} p \\ q \end{pmatrix},$$

then

$$\frac{\mu[w]}{\mu'[w]} = \frac{3(1\ 1)\begin{pmatrix} p \\ q \end{pmatrix}}{(1\ 1)P_0^{a_1}\, P_2\begin{pmatrix} p \\ q \end{pmatrix}} = \frac{3(p+q)}{(1+a_1)(p+q)+q} \geq \frac{3}{a_1+2}.$$

Since $n = a_1 + \cdots + a_k$, it follows that $a_1 \leq n$ and thus $\mu[w]/\mu'[w] \geq 3/(n+2)$. □

In order to define the potential associated to μ' in Theorem 2.2 below (as well as for the Erdös measure in §2.2), we introduce some notations and ideas.

Given a sequence $a_0, a_1, \ldots$ of integers with $a_0 \geq 0$ and $a_i > 0$ for $i > 0$, we denote

$$[a_0; a_1, \ldots, a_k] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}} =: \frac{p_k}{q_k},$$

where the irreducible fraction p_k/q_k is the kth convergent of the continued fraction $[a_0; a_1, \ldots] = \lim_k [a_0; a_1, \ldots, a_k]$. The integers p_k and q_k satisfy a well-known linear recurrence, say

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix} = \begin{pmatrix} a_k p_{k-1} + p_{k-2} \\ a_k q_{k-1} + q_{k-2} \end{pmatrix} = Q_{a_0} \cdots Q_{a_k} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{where } Q_{a_i} := \begin{pmatrix} a_i & 1 \\ 1 & 0 \end{pmatrix}; \tag{19}$$

we refer to [**23**] for a general presentation of continued fraction theory.

For $\epsilon \in \{0, 2\}$ and a any positive integer, we denote $\langle \epsilon | a \rangle = \epsilon^a$; given any sequence $a_1, a_2, \ldots, a_k$ ($k \geq 2$) of positive integers, we define $\langle \epsilon | a_1, a_2, \ldots, a_k \rangle$ by means of the induction formula $\langle\epsilon|a_1, a_2, \ldots, a_k\rangle = \epsilon^{a_1}\langle 2-\epsilon|a_2, \ldots, a_k\rangle$. Let $\overline{0} := (\omega_i = 0)_{i=0}^{\infty}$, $\overline{2} := (\omega_i = 2)_{i=0}^{\infty}$ and $\theta : \Omega_3 \to \mathbf{N} \cup \{\infty\}$ such that

$$\theta(\omega) = \begin{cases} \min\{k \geq 0; \sigma^k\omega \in [\![1]\!] \cup \{\overline{0}, \overline{2}\}\} & \text{if } \exists k \geq 0, \sigma^k\omega \in [\![1]\!] \cup \{\overline{0}, \overline{2}\}; \\ \infty & \text{if } \forall k \geq 0, \sigma^k\omega \notin [\![1]\!] \cup \{\overline{0}, \overline{2}\}; \end{cases}$$

hence the map $\theta : \Omega_3 \to \mathbf{N} \cup \{\infty\}$ is the hitting time of $[\![1]\!] \cup \{\overline{0}, \overline{2}\}$. Moreover, if $\theta(\omega) = 0$ then we set $\mathbf{n}(\omega) = 0$, and if $0 < \theta(\omega) < \infty$, there exists a unique finite sequence of $\mathbf{n}(\omega)$ positive integers $a_1^\omega, \ldots, a_{\mathbf{n}(\omega)}^\omega$ with $a_1^\omega + \cdots + a_{\mathbf{n}(\omega)}^\omega = \theta(\omega)$ and such that, $\omega_0 \cdots \omega_{\theta(\omega)-1} = \langle\omega_0|a_1^\omega, \ldots, a_{\mathbf{n}(\omega)}^\omega\rangle$; when $\theta(\omega) = \infty$ we set $\mathbf{n}(\omega) := \infty$ and there exists a unique (infinite) sequence of integers $a_1^\omega, a_2^\omega, \ldots$ with $a_i^\omega > 0$ and such that $\omega_0 \cdots \omega_{n-1} = \langle\omega_0|a_1^\omega, \ldots, a_k^\omega, \alpha\rangle$, where $0 \leq \alpha < a_{k+1}^\omega$ and $a_1^\omega + \cdots + a_k^\omega + \alpha = n$.

THEOREM 2.2. *The T_2-invariant probability measure μ' defined on $\mathbf{S}^1$ by (18) is a g-measure of the normalized potential $\varphi : \Sigma \to \mathbb{R}$ such that*

$$\varphi(\omega) = \begin{cases} \log(1/3), & \text{if } \mathbf{n}(\omega) = \theta(\omega) = 0, \\ \log([1; a_1^\omega, \ldots, a_{\mathbf{n}(\omega)}^\omega]/3), & \text{if } 0 < \mathbf{n}(\omega), \theta(\omega) < \infty, \\ \log([1; a_1^\omega, \ldots]/3), & \text{if } \mathbf{n}(\omega) = \theta(\omega) = \infty. \end{cases}$$

The proof of Theorem 2.2 relies on the following lemma, which, according to (19), makes the link between the matrix product formula in (18) defining μ' and the continued fractions involved in the definition of the potential φ associated to μ' in Theorem 2.2.

LEMMA 2.3. *For $w = \langle\varepsilon|a_1, \ldots, a_k\rangle$, one has ${}^tVP_wV = {}^tVQ_{a_1} \cdots Q_{a_k}V$.*

Proof. Notice that Q_0Q_0 is the identity matrix and that $P_2^aQ_0 = Q_0P_0^a = Q_a$, for each integer $a \geq 0$; given any finite sequence of integers $a_1, \ldots, a_k$ with $a_2 \cdots a_{k-1} > 0$, one has

$$\begin{aligned} {}^tVP_2^{a_1}P_0^{a_2} \cdots P_2^{a_k}V &= {}^tV(P_2^{a_1}Q_0)(Q_0P_0^{a_2}) \cdots (P_2^{a_k}Q_0)Q_0V \\ &= {}^tVQ_{a_1}Q_{a_2} \cdots Q_{a_k}V. \end{aligned}$$

□

Proof of Theorem 2.2. Assume that $\theta(\omega) = \infty$ (the case $\theta(\omega) < \infty$ being similar). For $n \geq 1$ one writes $\omega_0 \cdots \omega_{n-1} = \langle\omega_0|a_1^\omega, \ldots, a_k^\omega, \alpha\rangle$, with $n = a_1^\omega + \cdots + a_k^\omega + \alpha$ and $0 \leq \alpha < a_{k+1}^\omega$; one considers $x := [1; a_1^\omega, \ldots]$ and for any $k \geq 1$,

$$\frac{p_k}{q_k} = [1; a_1^\omega, \ldots, a_k^\omega] \quad \text{and} \quad \frac{p'_k}{q'_k} := [1; a_1^\omega, \ldots, a_k^\omega, \alpha + 1].$$

By the definition of the n-step potential ϕ_n of μ', it follows from Lemma 2.3 that

$$\begin{aligned} 3\exp(\phi_n(\omega)) &= \frac{{}^tVQ_{a_1^\omega}Q_{a_2^\omega} \cdots Q_{a_k^\omega}Q_\alpha V}{{}^tVQ_{a_1^\omega-1}Q_{a_2^\omega} \cdots Q_{a_k^\omega}Q_\alpha V} \\ &= \frac{(1 \quad 0)Q_1Q_{a_1^\omega} \cdots Q_{a_k^\omega}Q_{\alpha+1}\,{}^t(1 \quad 0)}{(0 \quad 1)Q_1Q_{a_1^\omega} \cdots Q_{a_k^\omega}Q_{\alpha+1}\,{}^t(1 \quad 0)} = \frac{(1 \quad 0)\,{}^t(p'_k \quad q'_k)}{(0 \quad 1)\,{}^t(p'_k \quad q'_k)} = \frac{p'_k}{q'_k}. \end{aligned}$$

Using classical inequalities of the theory of continued fractions,

$$\left|x - \frac{p'_k}{q'_k}\right| \le \left|x - \frac{p_k}{q_k}\right| + \left|\frac{p_k}{q_k} - \frac{p'_k}{q'_k}\right| \le \frac{1}{q_k q_{k+1}} + \frac{1}{q_k q'_k} \le \frac{1}{q_{k+1}} + \frac{1}{q'_k}.$$

Since $n = a_1^\omega + \cdots + a_k^\omega + \alpha$, one gets $q_{k+1} \ge q'_k > n$, which gives the following upper bound:

$$|\exp(\varphi(\omega)) - \exp(\phi_n(\omega))| \le \frac{2}{3n}.$$

The desired result is obtained by an application of Lemma 1.2. □

From Proposition 2.1 and Theorem 2.2, one deduces the following.

THEOREM 2.4. *The* (2, 3)*-Bernoulli convolution* μ *is a weak Gibbs measure of* φ.

Since μ is a weak Gibbs measure of the potential φ, it satisfies the multifractal formalism as stated in Theorem A′; moreover, by Theorem C, its L^q-spectrum τ coincides with the concave function τ_φ, a solution of the implicit equation $\mathbb{P}_\varphi(q, \tau_\varphi(q)) = 0$. The following theorem proves that τ is not real-analytic at a critical point $q_c < 0$, meaning that q_c is a phase transition point.

THEOREM 2.5. *The* L^q*-spectrum* τ *of the* (2, 3)*-Bernoulli convolution* μ *is a concave function for which there exists* $q_c < 0$ *such that* $\tau(q) = q \log 3/\log 2$ *if and only if* $q \le q_c$.

The proof of Theorem 2.5 depends on the following lemma.

LEMMA 2.6. $\lim_{q\to-\infty} \sum_{k=1}^{\infty} \sum_{a_1,\dots,a_k>0} ({}^tVQ_{a_1} \cdots Q_{a_k} V)^q = 0.$

Proof. We first prove that if $a_1, \dots, a_k$ are positive integers then

$${}^tVQ_{a_1} \cdots Q_{a_k} V \ge \left(\frac{a_1 + \beta^2}{\beta^2}\right) \cdots \left(\frac{a_k + \beta^2}{\beta^2}\right), \tag{20}$$

where $\beta = (1 + \sqrt{5})/2$. It is readily checked that ${}^tVQ_a V \ge (a + \beta^2)/\beta^2$ when $a > 0$. Given k positive integers $a_1, \dots, a_k$, one has, for any integer $a > 0$,

$${}^tVQ_{a_1} \cdots Q_{a_k} Q_a V \ge {}^tVQ_{a_1} \cdots Q_{a_{k-1}} Q_{a_k - 1} \begin{pmatrix} a+1 \\ a+1 \end{pmatrix}.$$

If (20) is valid for rank 1 up to rank $k \ge 1$, then (even when $a_k - 1 = 0$),

$$\begin{aligned} {}^tVQ_{a_1} \cdots Q_{a_k} Q_a V &\ge \left(\frac{a_1 + \beta^2}{\beta^2}\right) \cdots \left(\frac{a_{k-1} + \beta^2}{\beta^2}\right) \left(\frac{a_k - 1 + \beta^2}{\beta^2}\right) (a+1) \\ &\ge \left(\frac{a_1 + \beta^2}{\beta^2}\right) \cdots \left(\frac{a_{k-1} + \beta^2}{\beta^2}\right) \left(\frac{a_k + \beta^2}{\beta^2}\right) \left(\frac{a + \beta^2}{\beta^2}\right), \end{aligned}$$

and (20) follows by induction. Using (20), one gets, for any $q < 0$,

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{a_1,\dots,a_k>0} ({}^tVQ_{a_1} \cdots Q_{a_k} V)^q &\le \sum_{k=1}^{\infty} \sum_{a_1,\dots,a_k>0} \left\{\left(\frac{a_1 + \beta^2}{\beta^2}\right) \cdots \left(\frac{a_k + \beta^2}{\beta^2}\right)\right\}^q \\ &\le \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \left(\frac{n + \beta^2}{\beta^2}\right)^q\right)^k \end{aligned}$$

and the desired result holds. □

Proof of Theorem 2.5. The Bernoulli convolution μ being a weak Gibbs measure of φ, one deduces from Theorem C that $\tau(q) = \tau_\varphi(q)$: this means that, for any $q \in \mathbb{R}$,

$$\tau(q) = \max\{t; \mathbb{P}_\varphi(q,t) \geq 0\} = \min\{t; \mathbb{P}_\varphi(q,t) \leq 0\}. \tag{21}$$

Let $\alpha_0 := \log 3/\log 2$; for $\overline{0} = (\omega_i = 0)_{i=0}^\infty$ one has $\Psi_0(\overline{0}) = -\log 2$ and $\varphi(\overline{0}) = -\log 3$, so that $q\varphi(\overline{0}) - \alpha_0 q \Psi_0(\overline{0}) = 0$, for any $q \in \mathbb{R}$: it follows from the variational principle that

$$0 = \mathrm{h}_\sigma(\delta_{\overline{0}}) + \int \{q\varphi(\xi) - \alpha_0 q \Psi_0(\xi)\}\, d\delta_{\overline{0}}(\xi) \leq P(q\varphi - \alpha_0 q \Psi_0) =: \mathbb{P}_\varphi(q, \alpha_0 q),$$

which by (21) implies that $\tau(q) \leq \alpha_0 q$. We now prove that $\tau(q) \geq \alpha_0 q$ when q is sufficiently close to $-\infty$. Since μ' is also a weak Gibbs measure of φ, one can write

$$\mathbb{P}_\varphi(q,t) = \lim_n \frac{1}{n} \log \mathbb{Z}_n(q,t), \quad \text{where } \mathbb{Z}_n(q,t) := \sum_{w\in\{0,2\}^n} \mu'[w]^q / |[w]|^t.$$

By Lemma 2.3, one has, for any $q < 0$,

$$\begin{aligned}\mathbb{Z}_n(q, \alpha_0 q) &:= \sum_{w\in\{0,2\}^n} (\mu'[w]/|[w]|^{\alpha_0})^q \\ &= \frac{1}{2^q} \sum_{w\in\{0,2\}^n} ({}^tVP_w V)^q = \frac{2}{2^q} \sum_{k=1}^n \sum_{a_1+\cdots+a_k=n} ({}^tVQ_{a_1}\cdots Q_{a_k} V)^q\end{aligned}$$

so that

$$\mathbb{Z}_n(q, \alpha_0 q) \leq \frac{2}{2^q} \sum_{k=1}^\infty \sum_{a_1,\ldots,a_k>0} ({}^tVQ_{a_1}\cdots Q_{a_k} V)^q.$$

Hence, by Lemma 2.6, there exists $q_0 < 0$ such that $\mathbb{Z}_n(q, \tilde{\alpha} q) \leq 1/2^q$ for any $q < q_0$ and each $n \geq 1$. Therefore, $\mathbb{P}_\varphi(q, \alpha_0 q) \leq 0$ and (21) gives $\tau(q) \geq \alpha_0 q$ when $q \leq q_0$; since τ is a concave map with $\tau(0) = -1$, there exists $q_c < 0$ such that $\tau(q) = \alpha_0 q$ if and only if $q \leq q_c$. □

2.2. *The Erdös measure.* Suppose that $1 < \beta < 2$ and let $\alpha := 1/(\beta - 1)$. The Bernoulli convolution ν_β can be defined as the probability distribution of the random variable $Y_\beta : \Omega_2 \to \mathbb{R}$ such that $Y_\beta(\omega) = (1/\alpha)\sum_{k=0}^\infty \omega_k/\beta^{k+1}$, where Ω_2 is endowed with the equidistributed Bernoulli measure. As in the case of a (b, d)-Bernoulli convolution, ν_β satisfies a self-similarity equation, say

$$\nu_\beta = \tfrac{1}{2}\nu_\beta \circ S_0^{-1} + \tfrac{1}{2}\nu_\beta \circ S_1^{-1}, \tag{22}$$

with $S_\varepsilon(x) = (x + \varepsilon/\alpha)/\beta$ and $\varepsilon \in \{0, 1\}$. From now on, we assume that $\beta = (1+\sqrt{5})/2$, that is $\nu = \nu_\beta$ is the Erdös measure. The algebraic equation $\beta^2 = \beta + 1$ satisfied by β implies that $1/\beta = \beta - 1 =: \rho$, so that $S_0(x) = \rho x$ and $S_1(x) = \rho x + \rho^2$. It is easily seen that the intervals $S_{00}[0, 1[$, $S_{100}[0, 1[= S_{011}[0, 1[$ and $S_{11}[0, 1[$ form a partition of $[0, 1[$; this means that $R_0 := S_{00}$, $R_1 := S_{100} = S_{011}$ and $R_2 := S_{11}$ define a non-overlapping system of affine contractions on $[0, 1[$ and for any word w on the alphabet $\{0, 1, 2\}$, we denote $[w] = R_w(I)$. By symmetry $\nu[0] = \nu[2]$ and according to the self-similarity property (22) satisfied by ν, one gets $\nu[1] = (\nu[0] + \nu[2])/2$, so that

$$\nu[0] = \nu[1] = \nu[2] = \tfrac{1}{3}. \tag{23}$$

Likewise, for any word w on the alphabet $\{0, 1, 2\}$,

$$\begin{aligned}\nu[1w] &= \tfrac{1}{2}\nu(S_0^{-1}[1w]) + \tfrac{1}{2}\nu(S_1^{-1}[1w]) \\ &= \tfrac{1}{2}\nu(S_0^{-1}S_{011}[w]) + \tfrac{1}{2}\nu(S_1^{-1}S_{100}[w]) \\ &= \tfrac{1}{2}\nu(R_2[w]) + \tfrac{1}{2}\nu(R_0[w]) = \tfrac{1}{2}\nu[2w] + \tfrac{1}{2}\nu[0w],\end{aligned}$$

which we can write in the following matricial way:

$$\begin{pmatrix}\nu[1w]\\ \nu[1w]\end{pmatrix} = P_1 \begin{pmatrix}\nu[0w]\\ \nu[2w]\end{pmatrix} \quad \text{where } P_1 := \begin{pmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{pmatrix}. \tag{24}$$

As initially noticed by Strichartz *et al.* [**43**], the identity $S_{100} = S_{011}$ plays a crucial role in the overlapping situation involved in the self-similarity of ν. It follows from the same kind of elementary computation which yields (24) that

$$\begin{pmatrix}\nu[01w]\\ \nu[21w]\end{pmatrix} = \frac{1}{4}P_1 \begin{pmatrix}\nu[0w]\\ \nu[2w]\end{pmatrix}; \tag{25}$$

$$\begin{pmatrix}\nu[00w]\\ \nu[20w]\end{pmatrix} = \frac{1}{4}P_0 \begin{pmatrix}\nu[0w]\\ \nu[2w]\end{pmatrix} \quad \text{where } P_0 := \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}; \tag{26}$$

$$\begin{pmatrix}\nu[02w]\\ \nu[22w]\end{pmatrix} = \frac{1}{4}P_2 \begin{pmatrix}\nu[0w]\\ \nu[2w]\end{pmatrix} \quad \text{where } P_2 := \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}. \tag{27}$$

By induction using (23), (24), (25), (26) and (27), one gets for any $\omega_0 \cdots \omega_{n-1} \in \{0, 1, 2\}^n$,

$$\nu[\omega_0 \cdots \omega_{n-1}] = \frac{1}{3}\frac{1}{4^{n-1}}\, {}^t V_{\omega_0} P_{\omega_0} \cdots P_{\omega_{n-1}} V, \tag{28}$$

where

$$V_0 := \begin{pmatrix}1\\ 0\end{pmatrix}, \quad V_2 := \begin{pmatrix}0\\ 1\end{pmatrix}, \quad V_1 = \begin{pmatrix}1/2\\ 1/2\end{pmatrix}.$$

Similarly to the $(2, 3)$-Bernoulli convolution, the Erdös measure ν is considered as a probability measure on $\mathbf{S}^1$; moreover, the transformations R_0, R_1 and R_2 are identified to the local inverses of a 3-f.c.t. of $\mathbf{S}^1$ denoted T which is coded by the full shift $\sigma : \Omega_3 \to \Omega_3$ and associated to the volume-derivative potential $\Psi_0 : \Omega_3 \to \mathbb{R}$, such that

$$\Psi_0(\omega) = \log \rho^2 \mathbf{1}_{[\![0]\!]}(\omega) + \log \rho^3 \mathbf{1}_{[\![1]\!]}(\omega) + \log \rho^2 \mathbf{1}_{[\![2]\!]}(\omega).$$

We now consider the intervals $[w]$ ($w \in \{0, 1, 2\}^*$) as subsets of $\mathbf{S}^1$ in such a way that $[0] := [0, \rho^2[$, $[1] := [\rho^2, \rho[$, $[2] := [\rho, 1[$ (defining the one-step basic intervals of T) and for each word $\omega_0 \cdots \omega_{n-1} \in \{0, 1, 2\}^n$ ($n \geq 1$)

$$x \in [\omega_0 \cdots \omega_{n-1}] \iff T^k(x) \in [\omega_k] \quad \forall k = 0, \ldots, n-1,$$

(defining the n-step basic intervals of T).

By (28) one gets that $\nu[10^n] = (2n+4)/(3\cdot 4^{n+1})$ and thus ν is not a Gibbs measure with respect to T. Following the same idea as in the case of the $(2,3)$-Bernoulli convolution, we introduce the T-invariant probability measure ν', such that

$$\nu'[\omega_0\cdots\omega_{n-1}] = \frac{1}{2}\frac{1}{4^n}\,{}^tVP_{\omega_0}\cdots P_{\omega_{n-1}}V. \tag{29}$$

The same argument leading to Proposition 2.1 gives the following.

PROPOSITION 2.7. *For any $\omega\in\Omega_3$, and any integer $n\ge 1$,*

$$\frac{2}{n+2}\le\frac{\nu[\omega_0\cdots\omega_{n-1}]}{\nu'[\omega_0\cdots\omega_{n-1}]}\le 4.$$

In order to establish the following theorem, we use formula (29) defining ν' together with Lemma 2.3 and we apply the argument of the uniform convergence of the n-step potentials (Lemma 1.2) in a similar way leading to Theorem 2.2.

THEOREM 2.8. *The T-invariant probability measure ν' defined on $\mathbf{S}^1$ by (29) is a g-measure of the normalized potential $\phi:\Omega_3\to\mathbb{R}$ such that*

$$\phi(\omega)=\begin{cases}\log(1/4), & \text{if } \mathbf{n}(\omega)=\theta(\omega)=0,\\ \log([1;a_1^\omega,\ldots,a_{\mathbf{n}(\omega)}^\omega+1]/4), & \text{if } 0<\mathbf{n}(\omega),\theta(\omega)<\infty,\\ \log([1;a_1^\omega,\ldots]/4), & \text{if } \mathbf{n}(\omega)=\theta(\omega)=\infty.\end{cases}$$

As a corollary of Proposition 2.7 and Theorem 2.8, one has the following.

THEOREM 2.9. *The Erdös measure ν is a weak Gibbs measure of ϕ.*

The case of the Erdös measure ν is similar to the one of the $(2,3)$-Bernoulli convolution studied in §1.1; since ν is a weak Gibbs measure of the potential ϕ, it satisfies the multifractal formalism as stated in Theorem A′ and, by Theorem C, its L^q-spectrum τ coincides with the concave function τ_ϕ, a solution of the implicit equation $\mathbb{P}_\phi(q,\tau_\phi(q))=0$. Theorem 2.10 below is the analog of Theorem 2.5: it proves that τ is not real-analytic at a critical point $q_c<0$, meaning that q_c is a phase transition point.

THEOREM 2.10. *The L^q-spectrum τ of the Erdös measure is a concave function for which there exists a real number $q_c<0$ such that $\tau(q)=q\log 2/\log\beta$ if and only if $q\le q_c$.*

Proof. The Erdös measure ν being a weak Gibbs measure of ϕ, one deduces from Theorem C that $\tau(q)=\tau_\phi(q)$: this means that, for any $q\in\mathbb{R}$,

$$\tau(q)=\max\{t;\mathbb{P}_\phi(q,t)\ge 0\}=\min\{t;\mathbb{P}_\phi(q,t)\le 0\}. \tag{30}$$

For $\alpha_0:=\log 2/\log\beta$ one can check that $q\phi(\overline{0})-\alpha_0 q\Psi_0(\overline{0})=0$, for any $q\in\mathbb{R}$: it follows from the variational principle that

$$0=\mathrm{h}_\sigma(\delta_{\overline{0}})+\int\{q\phi(\xi)-\alpha_0 q\Psi_0(\xi)\}\,d\delta_{\overline{0}}(\xi)\le P(q\phi-\alpha_0 q\Psi_0)=:\mathbb{P}_\phi(q,\alpha_0 q),$$

which implies that $\tau(q)\le\alpha_0 q$. We now prove that $\tau(q)\ge\alpha_0 q$ when q is sufficiently close to $-\infty$. Using the fact that ν' is also a weak Gibbs measure of ϕ, it is easily seen that

$$\mathbb{P}_\phi(q,t)=\lim_n\frac{1}{n}\log\tilde{\mathbb{Z}}_n(q,t)\quad\text{where }\tilde{\mathbb{Z}}_n(q,t):=\sum_{w\in\{0,1,2\}^n}\frac{\nu'[w1]^q}{|[w1]|^t}. \tag{31}$$

Notice that any word $w \in \{0, 1, 2\}^n$ is associated to a unique sequence of (possibly empty) words $m_1, \ldots, m_k$ ($1 \leq k \leq n$) on the alphabet $\{0, 2\}$ such that $w1 = m_1 1 \cdots m_k 1$; since $\nu'[w1] = \nu'[m_1 1] \cdots \nu'[m_k 1]$ and $|[w1]| = |[m_1 1]| \cdots |[m_k 1]|$, one has

$$\tilde{\mathbb{Z}}_n(q, t) \leq \sum_{k=1}^{\infty} \sum_{\substack{m_1, \ldots, m_k \\ m_i \in \{0,2\}^*}} \left(\frac{\nu'[m_1 1]^q}{|[m_1 1]|^t} \right) \cdots \left(\frac{\nu'[m_k 1]^q}{|[m_k 1]|^t} \right) \leq \sum_{k=1}^{\infty} \left(\sum_{m \in \{0,2\}^*} \frac{\nu}{[m1]^q} |[m1]|^t \right)^k$$

(recall that $\{0, 2\}^* := \{\emptyset\} \cup \bigcup_{n=1}^{\infty} \{0, 2\}^n$). However, according to Lemma 2.3, one gets

$$\sum_{m \in \{0,2\}^*} \frac{\nu'[m1]^q}{|[m1]|^{\alpha_0 q}} = 2^q + 2 \sum_{k=1}^{\infty} \sum_{a_1 \cdots a_k > 0} ({}^t V Q_{a_1} \cdots Q_{a_k} V)^q,$$

which implies that $\tilde{\mathbb{Z}}_n(q, \alpha_0 q)$ is bounded for $q < q_0$, with a small enough q_0 given by Lemma 2.6: by (31) one has $\mathbb{P}_\phi(q, \alpha_0 q) \leq 0$ and thus $\tau(q) \geq \alpha_0 q$ when $q \leq q_0$; since τ is concave with $\tau(0) = -1$, there exists $q_c < 0$ such that $\tau(q) = \alpha_0 q$ if and only if $q \leq q_c$. □

Remark 2.11. (1) Starting from Definition 1.3 of τ, it is proved in [**13**] that τ is not differentiable at q_c; according to our terminology and the fact that $\tau = \tau_\phi$, this means that q_c is a critical value of a first-order phase transition: we include in Appendix A a proof of this result, based on the approach developed in [**13**].

(2) The phase transitions occurring in the multifractal formalism of the (2, 3)-Bernoulli convolution and the Erdös measure are to be related to the problem of phase transition on one-dimensional lattice systems and one-sided full-shift respectively studied in [**9, 15**] and [**18**] (see also [**19**] and [**30, 31**] for the relationship with the multifractal formalism).

(3) We point out the similarity between the potentials associated with the (2, 3)-Bernoulli convolution (Theorem 2.2), the Erdös measure (Theorem 2.8) and the potentials considered in [**41**].

(4) The L^q-spectra (for $q > 0$) and the sets of possible local dimensions for a special class of self-similar measures with overlaps were studied in [**12, 20**]; the Hausdorff dimension of the corresponding self-similar sets was determined in [**38**].

3. *Proof of the multifractal theorems*

3.1. *Proof of Theorem A: lower bound.*

For any probability measure μ, σ-invariant on Ω_{d}, we denote by $G_\sigma(\mu)$ the set of μ-generic points of Ω_{d}, meaning that $\omega \in G_\sigma(\mu)$ if and only if $S_n f(\omega)/n$ tends to $\mu(f)$, for any real-valued continuous function f defined on Ω_{d}. Since the full-shift Ω_{d} satisfies the specification property, the set $G_\sigma(\mu)$ is never empty (especially when μ is not ergodic).

PROPOSITION 3.1. *Let T be a d-f.c.t. of $\mathbf{S}^1$ and $\chi : \mathbf{S}^1 \to \Omega_{\mathrm{d}}$ the associated coding map; for any σ-invariant measure μ (not necessarily ergodic) on Ω_{d}, one has*

$$\dim_H \chi^{-1}(G_\sigma(\mu)) = -\frac{\mathrm{h}_\sigma(\mu)}{\mu(\Psi_0)}.$$

Hint. For any Borel set $M \subset \Omega_{\mathrm{d}}$, we denote by $\lambda(M)$ the Lebesgue measure of $\chi^{-1}(M)$; the probability λ is non-atomic and fully supported by Ω_{d}: by setting for any $\omega, \xi \in \Omega_{\mathrm{d}}$,

$$\mathbf{d}_\lambda(\omega, \xi) = \begin{cases} 1, & \text{if } \omega_0 \neq \xi_0, \\ \inf\{\lambda[\![\omega_0 \cdots \omega_{n-1}]\!]; \xi \in [\![\omega_0 \cdots \omega_{n-1}]\!]\}, & \text{if } \omega_0 = \xi_0, \end{cases}$$

one defines a metric $\mathbf{d}_\lambda$ compatible with the product topology on Ω_{d}. Let $\dim_\lambda$ be the λ-Billingsley dimension [**1**], that is, the Hausdorff dimension on the metric space $(\Omega_{\mathrm{d}}, \mathbf{d}_\lambda)$. Given μ a probability measure σ-invariant on Ω_{d}, we claim that

$$\dim_\lambda G_\sigma(\mu) = -\frac{\mathrm{h}_\sigma(\mu)}{\mu(\Psi_0)}. \tag{32}$$

When μ is ergodic, this formula can be deduced by a classical argument using the Shannon–McMillan Theorem and the fact that $\mu(G_\sigma(\mu)) = 1$. However, (32) is still valid when μ is σ-invariant without being ergodic: this follows from [**5**, Theorems 7.1 and 7.2] and [**29**, Théorème 2], where the fact that λ is a Gibbs measure is needed.

Actually, it is easily seen from (4) that λ is a Gibbs measure of Ψ_0 in the sense that there exists a constant $K > 1$ such that, for any $\omega \in \Omega_{\mathrm{d}}$ and any integer $n \geq 1$,

$$\frac{1}{K} \leq \frac{\lambda[\![\omega_0 \cdots \omega_{n-1}]\!]}{\exp(S_n \Psi_0(\omega))} \leq K;$$

then we can apply [**37**, Théorème 1.2.2], ensuring that $\dim_H \chi^{-1}(M) = \dim_\lambda M$, for any $M \subset \Omega_{\mathrm{d}}$: according to (32), one concludes that Proposition 3.1 holds.

We now prove a lemma that gives the characterization of the points where τ_ϕ is not differentiable (i.e. the first-order phase transition points); it is essentially a corollary of the tangency property of the topological pressure [**45**, Theorem 9.15] saying that μ is an equilibrium state of the potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$ if and only if $P(\phi + \psi) - P(\phi) \geq \mu(\psi)$, for any real-valued continuous function ψ defined on Ω_{d}.

LEMMA 3.2. *Given a negative potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$ and τ_ϕ the concave map such that $\mathbb{P}_\phi(q, \tau_\phi(q)) = 0$, the following two propositions hold:*

(i) *if $\mathcal{I}_q$ is the simplex of the equilibrium states of $q\phi - \tau_\phi(q)\Psi_0$ then*

$$\tau'_\phi(q^-) = \sup_{\mu \in \mathcal{I}_q} \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} \quad \textit{and} \quad \tau'_\phi(q^+) = \inf_{\mu \in \mathcal{I}_q} \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\};$$

(ii) *if $\mathcal{I}_\sigma(\Omega_{\mathrm{d}})$ is the simplex of the σ-invariant probability measures on Ω_{d} then*

$$\inf_{\mu \in \mathcal{I}_\sigma(\Omega_{\mathrm{d}})} \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} = \lim_{q \to +\infty} \frac{\tau_\phi(q)}{q} =: \underline{\alpha}$$

and

$$\sup_{\mu \in \mathcal{I}_\sigma(\Omega_{\mathrm{d}})} \left\{ \frac{\mu(\phi)}{\mu(\Psi_0)} \right\} = \lim_{q \to -\infty} \frac{\tau_\phi(q)}{q} =: \overline{\alpha}.$$

Proof. (i) Let $q > 0$; from the definition of the concave map $\tau_\phi : \mathbb{R} \to \mathbb{R}$ one has $P(q\phi - \tau_\phi(q)\Psi_0) = 0$; similarly, for any $h > 0$

$$0 = P((q+h)\phi - \tau_\phi(q+h)\Psi_0) = P(q\phi - \tau_\phi(q)\Psi_0 + h(\phi - (\tau'_\phi(q^+) + \varepsilon_h)\Psi_0)),$$

where we write $\tau_\phi(q+h) = \tau_\phi(q) + h\tau'_\phi(q^+) + h\varepsilon_h$ with ε_h tending to 0 when h tends to 0. In conclusion, one has the equation

$$P(q\phi - \tau_\phi(q)\Psi_0 + h(\phi - (\tau'_\phi(q^+) + \varepsilon_h)\Psi_0)) - P(q\phi - \tau_\phi(q)\Psi_0) = 0;$$

which, by the tangency property of the pressure, ensures that

$$\mu(\phi) - (\tau'_\phi(q^+) + \varepsilon_h)\mu(\Psi_0) \leq 0,$$

for every $\mu \in \mathcal{I}_q$, that is, $\tau'_\phi(q^+) \leq \mu(\phi)/\mu(\Psi_0) - \varepsilon_h$. When h tends to 0, one gets $\tau'_\phi(q^+) \leq \mu(\phi)/\mu(\Psi_0)$ and thus

$$\tau'_\phi(q^+) \leq \inf\left\{\frac{\mu(\phi)}{\mu(\Psi_0)}; \mu \in \mathcal{I}_q\right\}.$$

By the same argument one gets

$$\sup\left\{\frac{\mu(\phi)}{\mu(\Psi_0)}; \mu \in \mathcal{I}_q\right\} \leq \tau'_\phi(q^-).$$

It is clear that if τ_ϕ is differentiable at q then $\tau'_\phi(q) = \mu(\phi)/\mu(\Psi_0)$, for every $\mu \in \mathcal{I}_q$. Since τ_ϕ is not differentiable on an (at most) countable subset of $\mathbb{R}$, there exists a sequence of real numbers $q_n > q$ which tend to q and such that τ_ϕ is differentiable at q_n for any n. Let $\mu_n \in \mathcal{I}_{q_n}$ and assume (by compactness) that μ_n tends to a σ-invariant probability measure μ in the weak-$*$ sense; since the potentials $q_n\phi - \tau_\phi(q_n)\Psi_0$ converge uniformly to $q\phi - \tau_\phi(q)\Psi_0$, it follows from the variational principle and the upper semi-continuity of the entropy that $\mu \in \mathcal{I}_q$. Using the fact that τ_ϕ is concave,

$$\tau'_\phi(q^+) = \lim_n \tau'_\phi(q_n) = \lim_n \frac{\mu_n(\phi)}{\mu_n(\Psi_0)} = \frac{\mu(\phi)}{\mu(\Psi_0)},$$

and then $\tau'_\phi(q^+) = \inf\{\mu(\phi)/\mu(\Psi_0); \mu \in \mathcal{I}_q\}$. The same argument applies to $\tau'_\phi(q^-)$.

(ii) We prove the assertion for $\underline{\alpha}$ (the same argument applies to $\overline{\alpha}$). By the concavity of τ_ϕ it is clear that $\underline{\alpha} \leq \lim_{q\to+\infty} \tau_\phi(q^+)$; this fact together with part (i) yields

$$\inf\left\{\frac{\mu(\phi)}{\mu(\Psi_0)}; \mu \in \mathcal{I}_\sigma(\Omega_{\mathrm{d}})\right\} \leq \underline{\alpha}.$$

Moreover, if $\mu \in \mathcal{I}_\sigma(\Omega_{\mathrm{d}})$ then, by the variational principle, $\mathrm{h}_\sigma(\mu) + \mu(q\phi - \tau_\phi(q)\Psi_0) \leq \mathbb{P}_\phi(q, \tau_\phi(q)) = 0$, for any $q > 0$. Since Ψ_0 is negative and uniformly bounded away from 0,

$$\frac{\tau_\phi(q)}{q} \leq \frac{\mathrm{h}_\sigma(\mu)}{q\mu(\phi\Psi_0)} + \frac{\mu(\phi)}{\mu(\Psi_0)} \leq \frac{\mu(\phi)}{\mu(\Psi_0)}.$$

Therefore $\underline{\alpha} = \lim_{q\to+\infty} \tau_\phi(q)/q \leq \mu(\phi)/\mu(\Psi_0)$, that is,

$$\underline{\alpha} \leq \inf\left\{\frac{\mu(\phi)}{\mu(\Psi_0)}; \mu \in \mathcal{I}_\sigma(\Omega_{\mathrm{d}})\right\}. \qquad \square$$

LEMMA 3.3. *For any $\underline{\alpha} < \alpha < \overline{\alpha}$, let $q_\alpha \in \mathbb{R}$ such that $\tau'(q_\alpha^+) \leq \alpha \leq \tau(q_\alpha^-)$; then there exists $\mu \in \mathcal{I}_{q_\alpha}$ such that $\mu(\phi)/\mu(\Psi_0) = \alpha$ and for such a measure one has*

$$-\frac{\mathrm{h}_\sigma(\mu)}{\mu(\Psi_0)} = \sup\{q \in \mathbb{R}; \alpha q - \tau_\alpha(q)\} =: \tau_\phi^*(\alpha).$$

Proof. We consider the map $\Xi : \mu \mapsto \mu(\phi)/\mu(\Psi_0)$ which is continuous on the convex set of probability measures on Ω_{d}, endowed with the weak-$*$ topology. Given $\underline{\alpha} < \alpha < \overline{\alpha}$, the set $\Xi(\mathcal{I}_{q_\alpha})$ is a non-empty closed interval of $\mathbb{R}$, for $\mathcal{I}_{q_\alpha}$ is a non-empty closed convex subset of the σ-invariant measure. By Lemma 3.2, $[\tau'(q_\alpha^+), \tau(q_\alpha^-)] = \Xi(\mathcal{I}_{q_\alpha})$ and thus there exists $\mu \in \mathcal{I}_{q_\alpha}$ (not necessarily ergodic) such that $\Xi(\mu) = \mu(\phi)/\mu(\Psi_0) = \alpha$. We now apply the variational principle: on the one hand, for every $q \in \mathbb{R}$,

$$\mathrm{h}_\sigma(\mu) + \mu(q\phi - \tau(q)\Psi_0) \leq P(q\phi - \tau(q)\Psi_0) = 0;$$

on the other hand, since $\mu \in \mathcal{I}_{q_\alpha}$,

$$\mathrm{h}_\sigma(\mu) + \mu(q_\alpha\phi - \tau(q_\alpha)\Psi_0) = P(q_\alpha\phi - \tau(q_\alpha)\Psi_0) = 0.$$

Therefore, $-\mathrm{h}_\sigma(\mu)/\mu(\Psi_0) = \alpha q_\alpha - \tau(q_\alpha)$ and $-\mathrm{h}_\sigma(\mu)/\mu(\Psi_0) \leq \alpha q - \tau(q)$, for any $q \in \mathbb{R}$, which means, by definition, that $\tau^*(\alpha) = -\mathrm{h}_\sigma(\mu)/\mu(\Psi_0)$. □

We are now in a position to prove the lower bound $\tau_\phi^*(\alpha) \leq \dim_H E(\alpha|M)$, for $\underline{\alpha} < \alpha < \overline{\alpha}$. Given $x \in \mathbf{S}^1$ with $\chi(x) = \omega$, the Gibbs property of the Lebesgue measure (with respect to the volume-derivative potential Ψ_0) ensures the existence of $C > 1$ such that, for any integer n,

$$S_n\Psi_0(\omega) - \log C \leq \log |I_n(x)| \leq S_n\Psi_0(\omega) + \log C.$$

Likewise, by the weak Gibbs properties of η, there exists a sub-exponential sequence of real numbers $K(n) > 1$ such that

$$S_n\phi(\omega) - \log K(n) \leq \log \eta(I_n(x)) \leq S_n\phi(\omega) + \log K(n).$$

The potential ϕ being negative, $\phi(\xi) < \varepsilon < 0$ uniformly on Ω_{d} and $S_n\phi(\omega)/n < \varepsilon$ for any n; moreover, since $(1/n)\log K(n)$ tends to 0 when n goes to infinity, one deduces that $S_n\phi(\omega) + \log K(n) < 0$, when n is sufficiently large. For the same reason that Ψ_0 is negative, one also has $S_n\Psi_0(\omega) + \log C < 0$, when n is sufficiently large, and thus

$$\frac{S_n\phi(\omega) + \log K(n)}{S_n\Psi_0(\omega) - \log C} \leq \frac{\log \eta(I_n(x))}{\log |I_n(x))|} \leq \frac{S_n\phi(\omega) - \log K(n)}{S_n\Psi_0(\omega) + \log C}. \tag{33}$$

Now assume that $\omega = \chi(x) \in G_\sigma(\mu)$, where $\mu \in \mathcal{I}_{q_\alpha}$ (given by Lemma 3.3), satisfies $\mu(\phi)/\mu(\Psi_0) = \alpha$; by the definition of $G_\sigma(\mu)$, one has $\lim_n S_n\phi(\omega)/n = \mu(\phi) < 0$ and $\lim_n S_n\Psi_0(\omega)/n = \mu(\Psi_0) < 0$, so that

$$\lim_n \frac{\log \eta(I_n(x))}{\log |I_n(x)|} = \frac{\mu(\phi)}{\mu(\Psi_0)} = \alpha;$$

the point x being arbitrarily taken in $\chi^{-1}(G_\sigma(\mu))$, one deduces that

$$\chi^{-1}(G_\sigma(\mu)) \subset E(\alpha|\mathcal{M}).$$

Using successively Lemma 3.2 and Proposition 3.1, one concludes

$$\tau_\phi^*(\alpha) = -\frac{\mathrm{h}_\sigma(\mu)}{\mu(\Psi_0)} = \dim_H \chi^{-1}(G_\sigma(\mu)) \leq \dim_H E(\alpha|\mathcal{M}).$$

The required lower bound is proved.

3.2. *Proof of Theorem A: upper bound.* The following proof is essentially the argument given by Brown *et al.* [**4**] that we adapt to our framework. Notice that the hypothesis that η is a weak Gibbs measure of the potential ϕ is implicitly needed to ensure that the concave map τ_ϕ is well defined. We use the notions of ε-packing, box dimension (denoted $\dim_B$) and packing dimension (denoted $\dim_P$), presented in Appendix B. We shall prove, in Theorem 3.4 below, a stronger result than the upper-bound $\dim_H E(\alpha|\mathcal{M}) \le \tau_\phi^*(\alpha)$ involved in Theorem A; actually in place of the level sets $E(\alpha|\mathcal{M})$, we shall deal with what we call the *'fat level sets'*, defined by setting, for any $\alpha \in \mathbb{R}$,

$$F_+(\alpha|\mathcal{M}) := \left\{ x \in \mathbf{S}^1 : \limsup_{n\to\infty} \frac{\log \eta(I_n(x))}{\log |I_n(x)|} \le \alpha \right\}$$

and

$$F_-(\alpha|\mathcal{M}) := \left\{ x \in \mathbf{S}^1 : \liminf_{n\to\infty} \frac{\log \eta(I_n(x))}{\log |I_n(x)|} \ge \alpha \right\}.$$

THEOREM 3.4. *Let $\mathcal{M}$ be the Markov net of a regular* d*-f.c.t. T and η be a weak Gibbs measure of a negative potential $\phi : \Omega_{\mathrm{d}} \to \mathbb{R}$. Then, the following propositions hold:*
(i) *if $\underline{\alpha} < \alpha \le \tau'_\phi(0^-)$ then $\dim_P F_+(\alpha\,|\mathcal{M}) \le \tau_\phi^*(\alpha)$;*
(ii) *if $\tau'_\phi(0^+) \le \alpha < \overline{\alpha}$ then $\dim_P F_-(\alpha\,|\mathcal{M}) \le \tau_\phi^*(\alpha)$.*

Proof. We prove part (i) while part (ii) can be handled in a similar way. See Figure 1 for the graph of $\tau_\phi(q)$. To begin with, notice that, for $\tau'_\phi(0^+) \le \alpha \le \tau'_\phi(0^-)$, one has $\tau_\phi^*(\alpha) = -\tau_\phi(0) = 1$, so that the upper bound $\dim_P F_+(\alpha\,|\mathcal{M}) \le \tau_\phi^*(\alpha)$ is trivial in that case. We now consider that $\underline{\alpha} < \alpha < \tau'_\phi(0^+)$; given β such that $\alpha < \beta < \tau'_\phi(0^+)$, we define

$$F_+^n(\beta) := \{x \in \mathbf{S}^1;\ |I_n(x)|^\beta \le \eta(I_n(x))\},$$

for any $n \ge 0$, so that

$$F_+(\alpha|\mathcal{M}) \subset \bigcup_{m=0}^{\infty} \bigcap_{n=m}^{\infty} F_+^n(\beta). \tag{34}$$

Let $m \ge 0$ and $\varepsilon > 0$ be arbitrarily chosen; we consider $\mathcal{J} = \bigcup_{n=m}^{\infty} \mathcal{J}_n$, where $\mathcal{J}_n \subset \mathcal{A}_{\mathrm{d}}^n$ and such that $\{[w];\ w \in \mathcal{J}\}$ is an ε-packing of $\bigcap_{n=m}^{\infty} F_+^n(\beta)$. By definition of an ε-packing, it is clear that, for any word $w \in \mathcal{J}$, there exists $x \in \bigcap_{n=m}^{\infty} F_+^n(\beta)$ such that $[w] = I_n(x)$ for some $n \ge m$, so that $|[w]|^\beta \le \eta[w]$. Accordingly, for any $\delta \ge 0$ and $q > 0$, one can write

$$\begin{aligned}\sum_{w\in\mathcal{J}} |[w]|^\delta &= \sum_{n=m}^{\infty} \sum_{w\in\mathcal{J}_n} |[w]|^{\beta q}/|[w]|^{\beta q-\delta} \\ &\le \sum_{n=m}^{\infty} \sum_{w\in\mathcal{J}_n} \eta[w]^q/|[w]|^{\beta q-\delta} \le \sum_{n=1}^{\infty} \mathbb{Z}_n(q, \beta q - \delta),\end{aligned}$$

where $\mathbb{Z}_n(\cdot,\cdot)$ is as in (11). Since $\beta < \tau'_\phi(0^+)$, there exists $q_0 > 0$ with $\beta q_0 - \tau_\phi(q_0) = \tau_\phi^*(\beta)$. When $\delta > \tau_\phi^*(\beta)$ one has $\beta q_0 - \delta < \tau_\phi(q_0)$, which, by definition of τ_ϕ, implies that

$$P_\phi(\beta q, \beta q_0 - \delta) = P(q_0\phi - (\beta q_0 - \delta)\Psi_0) < 0.$$

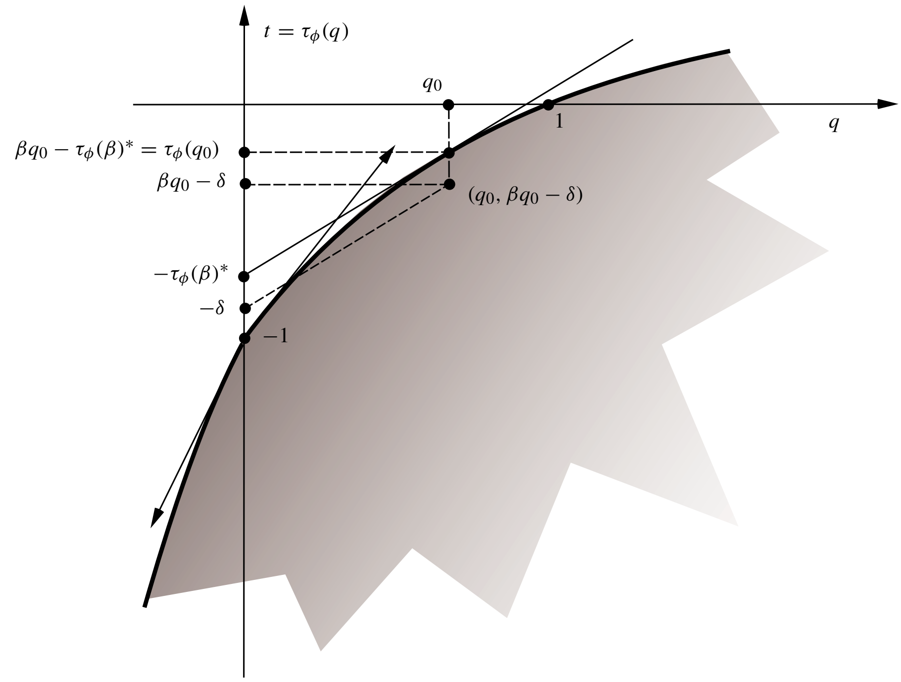

FIGURE 1. We represent here the graph of the concave map $q \mapsto \tau_\phi(q)$ (with a discontinuity of the derivative at $q = 0$) and the gray area is the open domain of the $(q, t) \in \mathbb{R} \times \mathbb{R}$ for which $P_\phi(q, t) < 0$; when $\underline{\alpha} < \beta < \tau_\phi(0^+)$, the straight line with equation $t = \beta q - \tau_\phi^*(\beta)$ is tangent to the graph of the map τ_ϕ at the point with abscissa $q_0 > 0$, and $P_\phi(q_0, \beta q_0 - \delta) < 0$ when $\delta > \tau_\phi^*(\beta)$.

Hence, there exists $\rho < 0$ and a rank $n_0 \geq 0$ such that $\mathbb{Z}_n(q_0, \beta q_0 - \delta) \leq \exp(n\rho)$ for any $n \geq n_0$; it follows that $\sum_{w \in \mathcal{J}} \eta[w]^\delta < \infty$, for any $\delta > \tau_\phi^*(\beta)$. This proves that $\dim_B \bigcap_{n=m}^\infty F_+^n(\beta) \leq \tau_\phi^*(\beta)$, for any $m \geq 0$, and from (34) one concludes that $\dim_P F_+(\alpha|\mathcal{M}) \leq \tau_\phi^*(\beta)$. Since β is an arbitrary real number such that $\alpha < \beta < \tau_\phi'(0^+)$, the continuity of τ_ϕ^* at α yields the desired upper bound stated in (i); the proof of the theorem is complete. □

3.3. *Proof of Theorem B: upper bound.* There exist conditions on the d-f.c.t. T ensuring that $E(\alpha) = E(\alpha|\mathcal{M})$: for instance, this occurs when T is differentiable at the point $\mathbf{0} := 0$ (mod 1). In our setting, the inclusion $E(\alpha) \subset E(\alpha|\mathcal{M})$ does not hold in general; however, we shall establish the upper bound $\dim_H E(\alpha) \leq \tau_\phi(\alpha)^* = \dim_H E(\alpha \mid \mathcal{M})$ (see part (iii) of Proposition 3.5 below), by means of the upper bounds in Theorem 3.4 involving the fat level sets $F_-(\alpha|\mathcal{M})$ and $F_+(\alpha|\mathcal{M})$.

PROPOSITION 3.5. *Let $\mathcal{M}$ be the Markov net of a regular d-f.c.t. T; if η is a Borel probability measure then the following two properties hold:*
(i) $E(\alpha) \subset F_-(\alpha|\mathcal{M})$*;*
(ii) $\dim_H E(\alpha) \leq \dim_H F_+(\alpha\,|\mathcal{M})$*;*
if in addition η is a weak Gibbs measure, then one has:
(iii) $\dim_H E(\alpha) \leq \tau_\phi(\alpha)^*$.

In order to prove Proposition 3.5 we use ideas taken from the approach of Hofbauer in [**19**]. Let $\mathrm{dist}(\cdot,\cdot)$ be the usual distance on $\mathbf{S}^1$; for any $x \in \mathbf{S}^1$ and $0 < d < 1/2$, we define

$$\mathbf{N}_d(x) := \{n \in \mathbf{N};\, \mathrm{dist}(T^n(x), \mathbf{0}) > d\} \quad \text{and} \quad \partial\mathcal{M}_d := \{x \in \mathbf{S}^1;\, \#\mathbf{N}_d(x) < \infty\}$$

($\#A$ stands for the cardinality of the countable set A.)

LEMMA 3.6. [**19**, Lemma 13] *Suppose that T is a regular* d*-f.c.t. of the torus $\mathbf{S}^1$; then*

$$\lim_{d\to 0} \dim_H \partial\mathcal{M}_d = 0.$$

We define the n-step variation of the volume-derivative potential Ψ_0, by setting

$$\begin{cases} \mathrm{V}_0(\Psi_0) := \max\{\Psi_0(\omega) - \Psi_0(\xi);\, \omega, \xi \in \Omega_{\mathrm{d}}\}, \\ \mathrm{V}_n(\Psi_0) := \max\{\Psi_0(\xi) - \Psi_0(\omega);\, \xi_0 \cdots \xi_{n-1} = \omega_0 \cdots \omega_{n-1}\} \quad \text{for } n \geq 1, \end{cases} \tag{35}$$

and we consider in addition

$$\Lambda_n(\Psi_0) := \mathrm{V}_1(\Psi_0) + \cdots + \mathrm{V}_n(\Psi_0). \tag{36}$$

Notice that, since Ψ_0 is continuous on Ω_{d}, the variation $\mathrm{V}_n(\Psi_0)$ tends to 0 when n goes to infinity and, by a classical lemma on Cesàro averages, $\Lambda_n(\Psi_0)/n$ tends to 0 as well. We denote by $d_n(x)$ the distance of x to the boundary of the basic interval $I_n(x)$; the following lemma shows that the ratio $|I_n(x)|/d_n(x)$ is a sub-exponential sequence, when $x \in \mathbf{S}^1 \backslash \partial\mathcal{M}_d$, for some $0 < d < 1/2$.

LEMMA 3.7. *Given any $x \in \mathbf{S}^1 \backslash \partial\mathcal{M}_d$, there exists a rank n_0 (depending on x) such that, for any $n \geq n_0$,*

$$0 < \log\left(\frac{|I_n(x)|}{d_n(x)}\right) \leq \Lambda_n(\Psi_0) + \log d.$$

Proof. By an application of the Mean Value Theorem, it is clear that

$$\log|I_n(x)| \leq \sup_{y \in I_n(x)} \{S_n\Psi_0 \circ \chi(y)\};$$

using the fact that $x \notin \partial\mathcal{M}_d$, another application of the Mean Value Theorem yields

$$\log d_n(x) \geq \inf_{y \in I_n(x)} \{S_n\Psi_0 \circ \chi(y)\} + \log d,$$

and one concludes using the definition of $\Lambda_n(\Psi_0)$. □

Proof of Proposition 3.5. (iii) When η is a weak Gibbs measure, the upper bound $\dim_H E(\alpha) \leq \tau_\phi(\alpha)^*$ is a trivial consequence of Theorem 3.4 together with parts (i) and (ii), which we now establish.

(i) Fix $x \in \mathbf{S}_1$ and set $\varepsilon_n := |I_n(x)|$, for any $n \geq 0$. Then, for any rank $n \geq 0$,

$$\frac{\log \eta(B_{\varepsilon_n}(x))}{\log \varepsilon_n} \leq \frac{\log \eta(I_n(x))}{\log|I_n(x)|},$$

ensuring that $E(\alpha) \subset F_-(\alpha|\mathcal{M})$.

(ii) Since T is regular and uniformly expanding,

$$0 < \underline{\gamma} := \inf_{\omega \in \Omega_{\mathtt{d}}} \{\exp(\Psi_0(\omega))\} \leq \sup_{\omega \in \Omega_{\mathtt{d}}} \{\exp(\Psi_0(\omega))\} =: \overline{\gamma} < 1$$

and, by a classical application of the Mean Value Theorem, $\underline{\gamma}^n \leq |I_n(x)| \leq \overline{\gamma}^n$, for any $x \in \mathbf{S}^1$ and any integer $n \geq 0$. Let $x \in \mathbf{S}^1 \backslash \partial \mathcal{M}_d$ and denote by $d_n(x)$ the distance of x to the boundary of the basic interval $I_n(x)$; it follows from the definition of $\partial \mathcal{M}_d$ that $B_{d_n(x)}(x) \subset I_n(x)$ and thus

$$\frac{\log \eta(B_{d_n(x)}(x))}{\log d_n(x)} \geq \frac{\log \eta(I_n(x))}{\log d_n(x)} = \frac{\log \eta(I_n(x))}{\log |I_n(x)|} \left(1 + \frac{\log(d_n(x)/|I_n(x)|)}{\log |I_n(x)|}\right)^{-1}. \quad (37)$$

Since one clearly has $-(1/n) \log |I_n(x)| \geq -\log \overline{\gamma} > 0$, one deduces from Lemma 3.7 that

$$\frac{\log(d_n(x)/|I_n(x)|)}{\log |I_n(x)|} = \frac{(1/n) \log(|I_n(x)|/d_n(x))}{-(1/n) \log |I_n(x)|} \leq -\frac{(\Lambda_n(\Psi_0) + \log d)/n}{\log \overline{\gamma}},$$

which according to (37) yields

$$\frac{\log \eta(B_{d_n(x)}(x))}{\log d_n(x)} \geq \frac{\log \eta(I_n(x))}{\log |I_n(x)|} \left(1 - \frac{(\Lambda_n(\Psi_0) + \log d)/n}{\log \overline{\gamma}}\right)^{-1},$$

and thus $E(\alpha) \cap (\mathbf{S}^1 \backslash \partial \mathcal{M}_d) \subset F_+(\alpha|\mathcal{M})$, for $\Lambda_n(\Psi_0)/n$ tending to 0. Let $\varepsilon > 0$ be arbitrarily given; by Hofbauer's lemma there exists $d > 0$ such that $\dim_H \partial \mathcal{M}_d \leq \varepsilon$, so that

$$\dim_H E(\alpha) = \dim_H E(\alpha) \cap (\mathbf{S}^1 \backslash \partial \mathcal{M}_d) + \dim_H E(\alpha) \cap \partial \mathcal{M}_d \leq \dim_H F_+(\alpha|\mathcal{M}) + \varepsilon.$$

Part (ii) is established since $\varepsilon > 0$ is arbitrarily chosen. □

3.4. *Proof of Theorem B: lower bound.* In order to prove the lower bound $\tau_\phi(\alpha)^* \leq \dim_H E(\alpha)$ we use the underlying Markov structure to construct a *slim level set* $S(\alpha|\mathcal{M})$, a subset of $E(\alpha)$ having the specified Hausdorff dimension, say $\tau_\phi(\alpha)^*$. Usually this is achieved by constructing a *Frostman measure* on the level set $E(\alpha)$; we shall use this approach with the additional difficulty that the measure to consider may not give a positive measure to $E(\alpha)$ (this is related to the fact that α may correspond to a first-order phase transition point).

The proof of the desired lower bound is a consequence of Theorem 3.8 below. Let us split any $\omega \in \Omega_{\mathtt{d}}$ into an infinite sequence of finite words, say $\tilde{\omega}_1, \tilde{\omega}_2, \ldots,$ so that (by concatenation) $\omega = \tilde{\omega}_1 \tilde{\omega}_2 \cdots$ and with the additional condition that the length of each word $\tilde{\omega}_n$ $(n \geq 1)$ is exactly n. Then we define the one-to-one map $\Theta : \Omega_{\mathtt{d}} \to \Omega_{\mathtt{d}}$ by setting

$$\Theta(\omega) := 1\tilde{\omega}_1 1\tilde{\omega}_2 1 \cdots 1\tilde{\omega}_n 1\tilde{\omega}_{n+1} 1 \cdots =: \omega^* = (\omega_i^*)_{i=0}^{\infty}. \quad (38)$$

THEOREM 3.8. *Let μ be a σ-invariant measure on $\Omega_{\mathtt{d}}$ ($\mathtt{d} > 2$)†; then the following hold:*
(i) $G_\sigma^*(\mu) := \Theta(G_\sigma(\mu)) \subset G_\sigma(\mu)$;

† If T is a 2-f.c.t. one can consider $T \circ T$.

(ii) $\dim_\lambda G^*_\sigma(\mu) = \dim_\lambda G_\sigma(\mu)$;

(iii) *for any* $x \in \mathbf{S}^1$ *such that* $\chi(x) \in \Theta(\Omega_{\mathtt{d}})$ *and any probability measure* η *on* $\mathbf{S}^1$,

$$\mathrm{DIM}_\eta(x\,|\mathcal{M}) = \alpha \iff \mathrm{DIM}_\eta(x) = \alpha.$$

We claim that, for each $\underline{\alpha} < \alpha < \overline{\alpha}$, the upper bound $\tau_\phi(\alpha)^* \leq \dim_H E(\alpha)$ is a consequence of Theorem 3.8. To see this, let $q_\alpha \in \mathbb{R}$ be such that $\tau'(q_\alpha^+) \leq \alpha \leq \tau'(q_\alpha^+)$; using Lemma 3.3, we consider an equilibrium state of the potential $q_\alpha\phi - \tau(q_\alpha)\Psi_0$, say μ_α, for which $\mu_\alpha(\phi)/\mu_\alpha(\Psi_0) = \alpha$ so that $\chi^{-1}(G_\sigma(\mu_\alpha)) \subset E(\alpha|\mathcal{M})$ and

$$\dim_\lambda G_\sigma(\mu_\alpha) = -\frac{\mathrm{h}_\sigma(\mu_\alpha)}{\mu_\alpha(\Psi_0)} = \tau_\phi(\alpha)^*.$$

We define the slim level set $S(\alpha|\mathcal{M}) := \chi^{-1}(G^*_\sigma(\mu_\alpha))$; by part (iii) of Theorem 3.8, one has $\mathrm{DIM}_\eta(x|\mathcal{M}) = \mathrm{DIM}_\eta(x)$ for any $x \in S(\alpha|\mathcal{M})$, and since $\chi^{-1}(G_\sigma(\mu_\alpha)) \subset E(\alpha|\mathcal{M})$, one deduces from part (i) of Theorem 3.8 that $S(\alpha|\mathcal{M}) \subset E(\alpha)$; by part (ii) of Theorem 3.8, one concludes that‡

$$\tau_\phi(\alpha)^* = \dim_\lambda G_\sigma(\mu_\alpha) = \dim_\lambda G^*_\sigma(\mu_\alpha) = \dim_H S(\alpha|\mathcal{M}) \leq \dim_H E(\alpha),$$

completing the proof of the desired lower bound.

Therefore it remains to establish Theorem 3.8. To begin with we shall prove the following lemma.

LEMMA 3.9. *Let* T *be a regular* $\mathtt{d}$*-f.c.t. of* $\mathbf{S}^1$ *(*$d > 2$*) and* η *a Borel probability measure. Suppose that* x *is a point of* $\mathbf{S}^1$ *such that* $T^{m_k}(x) \in [1]$ *(*$k \geq 1$*), where* $m_1, m_2, \ldots$ *form a strictly increasing sequence of integers; if* $\lim_k \log|I_{m_k}(x)|/\log|I_{m_{k+1}}(x)| = 1$, *then*

$$\mathrm{DIM}_\eta(x\,|\mathcal{M}) = \alpha \iff \mathrm{DIM}_\eta(x) = \alpha.$$

Proof. Let $x \in \mathbf{S}^1$ such that $T^{m_k}(x) \in [1]$ ($k \geq 1$); for $\omega = \chi(x)$, it is clear that

$$I_{m_k+1}(x) = [\omega_0 \cdots \omega_{m_k-1}1] \subset I_{m_k}(x) = [\omega_0 \cdots \omega_{m_k-1}].$$

The Lebesgue measure being Gibbs with respect to the Markov net associated to T, there exists a constant $0 < c < 1$ such that, for any $i = 0, 1, 2$,

$$|[\omega_0 \cdots \omega_{m_k-1} i]| \geq c|[\omega_0 \cdots \omega_{m_k-1}]| =: r_k,$$

ensuring that $B_{r_k}(x) \subset I_{m_k}(x)$. Moreover, $I_{m_k+p}(x) \subset B_{r_k}(x)$ for some constant integer $p > 0$ and thus the following sequence of inclusions arises:

$$I_{m_k+p}(x) \subset B_{r_k}(x) \subset I_{m_k}(x) \subset B_{r_k/c}(x).$$

If $\mathrm{DIM}_\eta(x|\mathcal{M}) = \alpha$ then it follows that $\log\eta(B_{r_k}(x))/\log r_k$ tends to α, when k goes to ∞; notice that for $r_{k+1} \leq r \leq r_k$

$$\frac{\log r_k}{\log r_{k+1}} \frac{\log\eta(B_{r_k}(x))}{\log r_k} \leq \frac{\log\eta(B_r(x))}{\log r} \leq \frac{\log r_{k+1}}{\log r_k} \frac{\log\eta(B_{r_{k+1}}(x))}{\log r_{k+1}},$$

and from the assumption that $\lim_k \log|I_{m_k}(x)|/\log|I_{m_{k+1}}(x)| = 1$, one gets $\mathrm{DIM}_\eta(x) = \alpha$. Conversely, if $\mathrm{DIM}_\eta(x) = \alpha$, a similar argument proves that $\mathrm{DIM}_\eta(x|\mathcal{M}) = \alpha$. □

‡ Here, we use again the fact [**37**, Théorème 1.2.2] that $\dim_H \chi^{-1}(M) = \dim_\lambda M$, for any $M \subset \Omega_{\mathtt{d}}$.

Proof of Theorem 3.8. For any integer $p \geq 1$, define $\mathtt{n}_p := \sum_{k=0}^{p-1} k = p(p-1)/2$ and $\mathtt{n}'_p := n_p + p = \mathtt{n}_{p+1}$. For $\omega \in \Omega_{\mathtt{d}}$ given, recall that $\Theta(\omega) = \omega^* = (\omega^*_i)_{i=0}^{\infty}$ and notice that, by construction,

$$\sigma^{\mathtt{n}_p}\omega \in [\![\tilde{\omega}_p]\!], \quad \sigma^{\mathtt{n}'_p - 1}\omega^* \in [\![1]\!] \quad \text{and} \quad \sigma^{\mathtt{n}'_p}\omega^* \in [\![\tilde{\omega}_p]\!].$$

(i) Assume that f is a real-valued continuous function defined on $\Omega_{\mathtt{d}}$. For any $\omega \in G_\sigma(\mu)$ and any integers n, p with $p \geq 2$ and $0 \leq n < p$, one has

$$\begin{aligned}
&S_{\mathtt{n}'_p+n} f(\omega^*) - S_{\mathtt{n}'_p+n} f(\omega) \\
&\quad = \sum_{k=1}^{p-1} \{ f(\sigma^{\mathtt{n}'_k - 1}\omega^*) + S_k f(\sigma^{\mathtt{n}'_k}\omega^*) - S_k f(\sigma^{\mathtt{n}_k}\omega) \} \\
&\qquad + \{ f(\sigma^{\mathtt{n}'_p - 1}\omega^*) + S_n f(\sigma^{\mathtt{n}'_p}\omega^*) - S_n f(\sigma^{\mathtt{n}_p}\omega) \} - S_p f(\sigma^{\mathtt{n}_p + n}\omega), \\
&\quad = \sum_{k=1}^{p-1} \{ S_k f(\sigma^{\mathtt{n}'_k}\omega^*) - S_k f(\sigma^{\mathtt{n}_k}\omega) \} + \{ S_n f(\sigma^{\mathtt{n}'_p}\omega^*) - S_n f(\sigma^{\mathtt{n}_p}\omega) \} \\
&\qquad + \sum_{k=1}^{p} f(\sigma^{\mathtt{n}'_k - 1}\omega^*) - S_p f(\sigma^{\mathtt{n}_p + n}\omega),
\end{aligned}$$

which gives, with $\mathrm{V}_k(f)$ defined as in (35) and $\Lambda_n(f)$ as in (36),

$$\begin{aligned}
\frac{1}{n'_p + n} |S_{\mathtt{n}'_p+n} f(\omega^*) - S_{\mathtt{n}'_p+n} f(\omega)| &\leq \frac{2}{p(p+1)} \left\{ \sum_{k=1}^{p} \sum_{j=1}^{k} \mathrm{V}_j(f) + p\mathrm{V}_0(f) \right\} \\
&\leq \frac{2(\Lambda_p(f) + \mathrm{V}_0(f))}{p+1}.
\end{aligned}$$

With $\Lambda'_k(f) := \Lambda_k(f) + \mathrm{V}_0(f)$, a straightforward computation yields, for any $n \geq 1$,

$$\frac{1}{n} |S_n f(\omega^*) - S_n f(\omega)| \leq \frac{2\Lambda'_{\rho_n}(f)}{\rho_n + 1}, \tag{39}$$

where ρ_n is the integral part of $(\sqrt{8n+1} - 1)/2$. The variation $\mathrm{V}_n(f)$ tends to 0 when n goes to infinity, as well as both $\Lambda_n(f)/n$ and $\Lambda'_n(f)/n$; this completes the proof of (i).

(ii) Let $n := \rho_n(\rho_n - 1)/2 + r$, where r is an integer such that $0 \leq r < \rho_n$ and let $n' = n + \rho_n$. Since the Lebesgue measure is a Gibbs measure of the volume-derivative potential Ψ_0, one gets

$$\log |[\omega^*_0 \cdots \omega^*_{n'-1}]| \geq S_{n'} \Psi_0(\omega^*) - \log K$$

for the constant $K > 1$ given in (4). Thereafter, from (39) one deduces that

$$\begin{aligned}
\log |[\omega^*_0 \cdots \omega^*_{n'-1}]| &\geq S_n \Psi_0(\omega) + S_{\rho_n} \Psi_0(\sigma^n \omega) - 2(n + \rho_n) \frac{\Lambda'_{\rho_n}(\Psi_0)}{\rho_n - 1} - \log K \\
&\geq S_n \Psi_0(\omega) \left\{ 1 - \frac{\rho_n \|\Psi_0\|_\infty}{n \sup(\Psi_0)} - \frac{2(n + \rho_n)}{n \sup(\Psi_0)} \frac{\Lambda'_{\rho_n}(\Psi_0)}{\rho_n - 1} - \frac{\log K}{n \sup(\Psi_0)} \right\},
\end{aligned}$$

with $\sup(\Psi_0) := \sup\{\Psi_0(\xi); \xi \in \Omega_{\mathtt{d}}\} < 0$, that is

$$\log |[\omega^*_0 \cdots \omega^*_{n'-1}]| \geq S_n \Psi_0(\omega)(1 - A_n), \tag{40}$$

where

$$A_n := \frac{\rho_n \|\Psi_0\|_\infty}{n \sup(\Psi_0)} + \frac{2(n+\rho_n)}{n \sup(\Psi_0)} \frac{\Lambda'_{\rho_n}(\Psi_0)}{\rho_n - 1} + \frac{\log K}{n \sup(\Psi_0)}$$

is clearly a quantity that tends to 0 when n goes to infinity. We use again the Gibbs property of the Lebesgue measure, say,

$$\log |[\omega_0 \cdots \omega_{n-1}]| \leq S_n \Psi_0(\omega) + \log K,$$

which yields the following sequence of inequalities:

$$\frac{\log |[\omega_0 \cdots \omega_{n-1}]|}{S_n \Psi_0(\omega)} \geq 1 + \frac{\log K}{S_n \Psi_0(\omega)} \geq 1 + \frac{\log K}{n \sup(\Psi_0)}. \tag{41}$$

Since $\Psi_0 < 0$, it is clear that $B_n := \log K/(n \sup(\Psi_0))$ tends to 0 when n goes to infinity and from (41) (with n large enough),

$$S_n \Psi_0(\omega) \geq \log |[\omega_0 \cdots \omega_{n-1}]|/(1 + B_n). \tag{42}$$

It follows from (40) and (42) that there exists a sequence of real numbers $b_1, b_2, \ldots$ with $\lim_k b_k = 0$ and such that

$$\log |[\omega^*_0 \cdots \omega^*_{n'-1}]| \geq \log |[\omega_0 \cdots \omega_{n-1}]| \left(\frac{1 + A_n}{1 + B_n} \right). \tag{43}$$

In conclusion, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mathbf{d}_\lambda(\omega, \xi) \leq \delta \Rightarrow \mathbf{d}_\lambda(\omega, \xi)^{1+\varepsilon} \leq \mathbf{d}_\lambda(\omega^*, \xi^*), \tag{44}$$

and one deduces that $\dim_\lambda G^*_\sigma(\mu) = \dim_\lambda G_\sigma(\mu)$, for any σ-invariant measure η.

(iii) Let $x \in \mathbf{S}^1$ such that $\chi(x) = \omega \in \Theta(\Omega_{\mathrm{d}})$. For $m_k := \mathrm{n}'_k - 1$ one has $T^{m_k}(x) \in [1]$ (i.e. $\sigma^{m_k}\omega \in [\![1]\!]$) and $\lim_k \log |I_{m_{k+1}}(x)|/\log |I_{m_k}(x)| = 1$. The result follows by an application of Lemma 3.9. □

3.5. *Proof of Theorem C.* Let $\mathcal{F}_0 := \{\mathbf{S}^1\}$ and suppose that $\mathcal{F}_n$ $(n > 0)$ is a partition of $\mathbf{S}^1$ by non-empty intervals. We say that $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is a net if any interval in $\mathcal{F}_n$ is the union of intervals in $\mathcal{F}_{n+1}$. For $x \in \mathbf{S}^1$ we denote by $I_n(x)$ the interval in $\mathcal{F}_n$ such that $x \in I_n(x)$, and we suppose that $\lim_n |I_n(x)| = 0$. Moreover, $\mathcal{F}$ is said to be *regular* if there exists a constant $c > 1$ such that, for any J, J' with $\mathcal{F}_{n+1} \ni J' \subset J \in \mathcal{F}_n$ $(n \geq 0)$, one has $|J|/|J'| \leq c$. Given $0 < r < 1$ and $x \in \mathbf{S}^1$, let $\mathrm{n}_r(x)$ be the rank such that $|I_{\mathrm{n}_r(x)}(x)| < r$ and $|I_{\mathrm{n}_r(x)-1}(x)| \geq r$; furthermore, one defines an equivalence relation on $\mathbf{S}^1$ by setting $x \underset{r}{\sim} y$ if and only if $y \in I_{\mathrm{n}_r(x)}(x)$: the so-called *r-Moran partition* $\mathfrak{F}_r$ is the partition of $\mathbf{S}^1$ generated by this equivalence relation. Notice that, under the condition of $\mathcal{F}$ to be regular, any interval $J \in \mathfrak{F}_r$ has approximately length r, in the sense that

$$r/c \leq |J| < r.$$

We refer to the book of Pesin [**35**] for a systematic presentation of the previous framework. In order to prove Theorem C we first establish the following theorem.

THEOREM 3.10. *Let $\mathcal{F}$ be a regular net of $\mathbf{S}^1$ and η a probability measure on $\mathbf{S}^1$. If η is supposed to be of full support, then, for any $q \in \mathbb{R}$,*

$$\tau(q) := \liminf_{r\to 0} \frac{\log \sum_{J\in\mathfrak{F}_r} \eta(J)^q}{\log r}.$$

Proof. According to Definition 1.3, one has $\tau(q) = \liminf_{r\to 0} \log Z(r)/\log r$, with

$$Z(r) := \inf\left\{ \sum_i \eta(B_i)^q;\ \{B_i\}_i \right\},$$

the infimum being taken over the r-covers of $\mathbf{S}^1$; for any $r > 0$, define

$$\tilde{Z}(r) := \sum_{J\in\mathfrak{F}_r} \eta(J)^q.$$

Let m_r be the integral part of $1/r$ and consider $\mathcal{B}_r := \{B_i\}_{i=0}^{m_r}$ where $B_i := [ir, (i+1)r]$, $(0 \le i < m_r)$ and $B_{m_r} = [1-r, 1]$ ($\mathcal{B}_r$ is an $r/2$-cover of $\mathbf{S}^1$).

First, suppose that $q < 0$; by definition, one has $|J| < r/2$ for any $J \in \mathfrak{F}_{r/2}$ and thus each ball $B_i \in \mathcal{B}_r$ contains at least one interval in $\mathfrak{F}_{r/2}$, say J_i, so that

$$Z(r/2) \le \sum_i \eta(B_i)^q \le 2 \sum_{J\in\mathfrak{F}_{r/2}} \eta(J)^q \le 2\tilde{Z}(r/2)$$

(the factor 2 appears because one may have $J_{m_r-1} = J_{m_r}$). By the regularity of $\mathcal{F}$ one also has $2r \le |J|$, for any $J \in \mathfrak{F}_{2cr}$ (where $c > 1$ stands for the constant involved in the regularity property of $\mathcal{F}$); hence, each $J \in \mathfrak{F}_{2cr}$ contains at least one $B_i \in \mathcal{B}_r$, implying that $\tilde{Z}(2cr) \le 2Z(r/2)$; one deduces that $\tau(q)$ is the lower limit of the ratio $\log \tilde{Z}(r)/\log r$ when r tends to zero.

We now consider the case of $q \ge 0$. By the regularity of $\mathcal{F}$, one has $r \le |J|$ for any $J \in \mathfrak{F}_{cr}$; hence, each $B_i \in \mathcal{B}_r$ intersects no more than two elements of $\mathfrak{F}_{cr}$, one of them, say J_i, satisfying $\eta(B_i) \le 2\eta(J_i)$. One may have $J_i = J_j$ for $i \ne j$, but since each interval in $\mathfrak{F}_{cr}$ intersects at most N elements of $\mathcal{B}_r$, for some N independent of r, one gets

$$\sum_i \eta(B_i)^q \le 2^q N \sum_i \eta(J_i)^q \le 2^q N \sum_{J\in\mathfrak{F}_{cr}} \eta(J)^q, \quad \text{i.e. } \frac{1}{2^q N} Z(r/2) \le \tilde{Z}(cr). \tag{45}$$

Suppose that $\mathcal{C}$ is an arbitrary r-cover of $\mathbf{S}^1$. It follows from Besicovitch's covering Lemma (cf. [**28**, p. 30]) that there exists a sub-cover $\mathcal{C}'$ of $\mathcal{C}$, such that each point $x \in \mathbf{S}^1$ belongs to at most three balls in $\mathcal{C}'$. One can check that, for $0 \le i \le m_r$, the number of $C \in \mathcal{C}'$ which intersects B_i is bounded by 6. Using again the fact that $J \in \mathfrak{F}_{cr}$ intersects at most N elements in $\mathcal{B}_r$, it is clear that J intersects at most $6N$ elements of $\mathcal{C}'$; since any ball in $\mathcal{C}'$ intersects no more than three elements of $\mathfrak{F}_{cr}$,

$$\sum_{J\in\mathfrak{F}_{cr}} \eta(J)^q \le 3(6N)^q \sum_{C\in\mathcal{C}'} \eta(C)^q \le 3(6N)^q \sum_{C\in\mathcal{C}} \eta(C)^q.$$

Taking the infimum over the r-cover $\mathcal{C}$ leads to $\tilde{Z}(cr) \le 2(6N)^q Z(r)$. With (45), one concludes that $\tau(q)$ is the lower limit of the ratio $\log \tilde{Z}(r)/\log r$ when r tends to zero. $\square$

We now turn to the proof of Theorem C itself, which we split into two steps (the first one being inspired by an argument in [**36**]).

First step. To begin with, suppose that η is a Gibbs measure of a Hölder continuous potential ϕ (which, according to our definition, implies that $P(\phi) = 0$). For any $q \in \mathbb{R}$ and any $\omega \in \Omega_{\mathrm{d}}$, the trivial identity

$$\exp(S_n \Phi_q(\omega)) = \frac{\exp(q S_n \phi(\omega))}{\exp(\tau_\phi(q) S_n \Psi_0(\omega))} \tag{46}$$

holds for $\Phi_q := q\phi - \tau_\phi(q)\Psi_0$. Denote by η_q the unique T-ergodic Gibbs measure associated to the Hölder continuous potential Φ_q; since η and the Lebesgue measure are respectively Gibbs measures of ϕ and Ψ_0, the identity (46) gives, for any word w,

$$\eta_q[w]/R' \le \frac{\eta[w]^q}{|[w]|^{\tau_\phi(q)}} \le R' \eta_q[w] \tag{47}$$

for some constant $R' > 1$. The Markovian net $\mathcal{M}$ being regular, it is possible by Theorem 3.10 to consider the L^q-spectrum defined by the mean of the Moran partitions $\mathfrak{M}_r$ $(0 < r < 1)$. By a summation of (47) over the $[w] \in \mathfrak{M}_r$, there exists a constant $R'' > 1$ such that

$$1/R'' \le \tilde{Z}(r, q)/r^{\tau_\phi(q)} \le R'', \quad \text{where } \tilde{Z}(r, q) = \sum_{J \in \mathfrak{M}_r} \eta(J)^q;$$

taking the limit when r tends to 0, one concludes that $\tau(q) = \tau_\phi(q)$, for any $q \in \mathbb{R}$.

Second step. We now turn to the general case. Let us consider a sequence of Hölder continuous potentials ϕ_k which are uniformly convergent to ϕ. Since $P(\phi) = 0$, one has $|P(\phi_k)| \le \|\phi_k - \phi\|_\infty$ and thus $\|(\phi_k - P(\phi_k)) - \phi\|_\infty \le 2\|\phi_k - \phi\|_\infty$; hence one can assume that $P(\phi_k) = 0$. We denote by η_k the unique σ-ergodic Gibbs measure of ϕ_k; by the first step of the proof, $\tau_k(q) = \tau_{\phi_k}(q)$, for any $q \in \mathbb{R}$ (τ_k denotes the L^q-spectrum of η_k). Moreover, it follows from the classical properties of the topological pressure that τ_{ϕ_k} tends to τ_ϕ uniformly on the compact intervals. It remains to prove the pointwise convergence of $\tau_k(q)$ to $\tau(q)$. Given $\delta > 0$, one has $\|\phi_k - \phi\|_\infty \le \delta$ whenever n is large enough; therefore, for any integer $m > 0$ and any word w of length m, one has

$$\eta[w] \le K(m) C_k \exp(2m\delta) \eta_k[w] \tag{48}$$

where $m \mapsto K(m)$ (respectively C_k) is the sub-exponential sequence (respectively constant) which characterizes the weak Gibbs property of η (respectively the Gibbs property of η_k, for $k \ge 1$). Let N_r be the maximal length of the words w such that $[w] \in \mathfrak{M}_r$; for any $k \ge 1$ we consider the partition function $\tilde{Z}_k(r, q) := \sum_{J \in \mathfrak{M}_r} \eta_k(J)^q$, so that from (48)

$$\tilde{Z}(r, q) \le K(N_r) C_k \exp(2N_r\delta) \tilde{Z}_k(r, q),$$

or equivalently

$$\frac{1}{\log r} \log \tilde{Z}(r, q) \ge \frac{1}{\log r} \log \tilde{Z}_k(r, q) + \left(\frac{N_r}{\log r} \frac{\log K(N_r)}{N_r} + \frac{C_k}{\log r} + \frac{2N_r\delta}{\log r} \right).$$

When r tends to 0, one gets $\tau(q) \ge \tau_k(q) - 2a\delta$, where a is a constant such that $N_r/\log(1/r) \le a$ for any $0 < r < 1$. The symmetric argument yields $\tau(q) \le \tau_k(q) + 2a\delta$ and the proof of Theorem C is complete.

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A. *Appendix. First-order phase transition*

It was first proved in [**13**] that the L^q-spectrum τ of the Erdös measure is not differentiable at the critical value q_c defined in Theorem 2.10; the present appendix is devoted to a self-contained proof of this result.

THEOREM A.1. *The L^q-spectrum τ is not differentiable at the critical value q_c.*

Any word $w \in \{0, 1, 2\}^n$ is associated to a unique sequence of (possibly empty) words $m_1, \ldots, m_k$ $(1 \leq k \leq n)$ on the alphabet $\{0, 2\}$ such that $w1 = m_1 1 \cdots m_k 1$; from (29) one has $\nu'[w1] = \nu'[m_1 1] \cdots \nu'[m_k 1]$; since $\|[w1]\| = \|[m_1 1]\| \cdots \|[m_k 1]\|$, one deduces that

$$\frac{\nu'[m_1 1 \cdots m_k 1]^q}{\|[m_1 1 \cdots m_k 1]\|^t} = \prod_{i=1}^{k} ({}^t V P_{m_i} V)^q (\beta^t / 2^q)^{2|m_i|+3}. \tag{49}$$

Recall that

$$\mathbb{P}_\phi(q, t) = \lim_n \frac{1}{n} \log \tilde{\mathbb{Z}}_n(q, t) \quad \text{where } \tilde{\mathbb{Z}}_n(q, t) := \sum_{w \in \{0,1,2\}^n} \frac{\nu'[w1]^q}{\|[w1]\|^t},$$

so that, with $\mathtt{z}_n(q) := \sum_{w \in \{0,2\}^n} ({}^t V P_w V)^q$, one gets†

$$\tilde{\mathbb{Z}}_n(q, t) = \sum_{k=1}^{n} \sum_{\substack{a_1, \ldots, a_k \geq 0 \\ a_1 + \cdots + a_k = n+1-k}} \prod_{i=1}^{k} \mathtt{z}_{a_i}(q) (\beta^t / 2^q)^{2a_i+3}. \tag{50}$$

PROPOSITION A.2.

(i) $q_c < -2.25$ *and*

(ii) $\sum_{n=0}^{\infty} n \mathtt{z}_n(q) < \infty$ *for any* $q < -2.25$.

We shall give the proof of Proposition A.2 at the end of the present appendix.

PROPOSITION A.3. *The following three propositions hold:*

(i) $\mathbb{X}(q) := \beta^{\tau(q)} / 2^q = \sup \left\{ 0 \leq x \leq 1; \sum_{i=0}^{\infty} \mathtt{z}_i(q) x^{2i+3} \leq 1 \right\}$*;*

(ii) $q \leq q_c \iff \sum_{i=0}^{\infty} \mathtt{z}_i(q) \leq 1$*;*

(iii) $q_c \leq q < -2.25 \Rightarrow \sum_{n=0}^{\infty} \mathtt{z}_n(q) \mathbb{X}(q)^{2n+3} = 1$.

Proof. (i) For $x > 0$, define $\mathtt{t}_q(x) := q \log 2 / \log \beta + \log x / \log \beta$ so that, for any $q \in \mathbb{R}$,

$$\mathbb{X}(q) = \max\{x; \mathbb{P}_\phi(q, \mathtt{t}_q(x)) \leq 0\} = \min\{x; \mathbb{P}_\phi(q, \mathtt{t}_q(x)) \geq 0\}.$$

† By definition $\{0, 2\}^0 = \{\emptyset\}$ and $P_\emptyset$ is the identity matrix, so that $\mathtt{z}_0(q) = ({}^t V P_\emptyset V)^q = 2^q$.

For any $(q, x) \in \mathbb{R} \times [0, 1]$ we set $F(q, x) := \sum_n \mathtt{z}_n(q) x^{2n+3}$ ($\in [0, +\infty]$) and we consider, for any $q \in \mathbb{R}$,

$$\mathtt{x}(q) := \sup\{x \geq 0;\, F(q, x) \leq 1\} = \sup\{x \geq 0;\, F(q, x) < 1\}, \tag{51}$$

the second equality being justified by Abel's theorem, for $\mathtt{x}(q)$ is bounded by the radius of convergence of the power series $\sum_i \mathtt{z}_i(q)\, x^{2i+3}$. It follows from (50) that

$$\tilde{\mathbb{Z}}_n(q, \mathtt{t}_q(x)) = \sum_{k=1}^{n} \sum_{\substack{a_1,\ldots,a_k \geq 0 \\ a_1+\cdots+a_k = n+1-k}} \prod_{i=1}^{k} \mathtt{z}_{a_i}(q) x^{2a_i+3} \leq \sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \mathtt{z}_n(q) x^{2n+3} \right)^k,$$

that is,

$$\tilde{\mathbb{Z}}_n(q, \mathtt{t}_q(x)) \leq \sum_{k=0}^{\infty} (F(q, x))^k. \tag{52}$$

Given any $x \geq 0$ with $F(q, x) < 1$, the upper bound in (52) implies that $\tilde{\mathbb{Z}}_n(q, \mathtt{t}_q(x))$ is bounded and thus $\mathbb{P}_\phi(q, \mathtt{t}_q(x)) \leq 0$: this means that $x \leq \mathbb{X}(q)$, and from the second equality in (51), one deduces that $\mathtt{x}(q) \leq \mathbb{X}(q)$.

For the converse inequality, let $n \geq 1$ and $r \geq 1$ be given so that

$$\begin{aligned} \sum_{m=1}^{nr} \tilde{\mathbb{Z}}_m(q, t) &\geq \sum_{\substack{m_1,\ldots,m_n \in \{0,2\}^* \\ |m_1|,\ldots,|m_n| < r}} \frac{\nu'[m_1 1]^q}{|[m_1 1]|^t} \cdots \frac{\nu'[m_n 1]^q}{|[m_n 1]|^t} \\ &= \left(\sum_{m \in \{0,2\}^*, |m| < r} \frac{\nu'[m1]^q}{|[m1]|^t} \right)^n = \left(\sum_{i=0}^{r-1} \sum_{m \in \{0,2\}^i} \frac{\nu'[m1]^q}{|[m1]|^t} \right)^n, \end{aligned}$$

which yields

$$\sum_{k=1}^{nr} \tilde{\mathbb{Z}}_k(q, \mathtt{t}_q(x)) \geq \left(\sum_{i=0}^{r-1} \mathtt{z}_i(q)\, x^{2i+3} \right)^n. \tag{53}$$

Let $x > \mathtt{x}(q)$ and $r_0 > 0$ such that $z_0 := \sum_{i=0}^{r_0-1} \mathtt{z}_i(q) x^{2i+3} > 1$; from (53), one deduces that

$$\lim_n \frac{1}{n} \log \sum_{k=1}^{nr_0} \tilde{\mathbb{Z}}_k(q, \mathtt{t}_q(x)) \geq \log z_0 > 0.$$

This is inconsistent with the fact that $\mathbb{Z}_n(q, \mathtt{t}_q(x)) \leq e^{-n\epsilon}$ for $\epsilon > 0$ and any n large enough and thus $\mathbb{P}_\phi(q, \mathtt{t}_q(x)) \geq 0$ and $\mathbb{X}(q) \leq x$: one concludes that $\mathbb{X}(q) \leq \mathtt{x}(q)$, for x can be taken arbitrarily close to $\mathtt{x}(q)$. Finally, since $\tau(q) \leq q \log 2/\log \beta$, for any $q \in \mathbb{R}$, one necessarily has $0 \leq \mathbb{X}(q) \leq 1$, which complete the proof of (i).

(ii) According to Theorem 2.10, one has $q \leq q_c$ if and only if $\beta^{\tau(q)}/2^q = 1$, which leads to (ii) as a straightforward consequence of (i).

(iii) Notice that the map $F : (q, x) \mapsto F(q, x)$ is finite and continuous at any point $(q, x) \in]-\infty, -2.25[\times [0, 1]$, for part (ii) of Proposition A.2 ensures that

$$0 \leq F(q, x) \leq \sum_n \mathtt{z}_n(q) < \infty,$$

whenever $q < -2.25$ and $0 \le x \le 1$. We first assume that $q_c < q < -2.25$ (which is consistent with respect to part (i) of Proposition A.2); it is clear that $F(q, \cdot)$ increases from 0 up to $F(q, 1)$. By (ii), one has $F(q, 1) > 1$ and thus there exists a unique $0 < x_q < 1$ such that $F(q, x_q) = 1$, which also satisfies $x_q = \max\{0 \le x \le 1;\ F(q, x) \le 1\}$: since $0 \le \mathbb{X}(q) \le 1$, one deduces from (i) that $\mathbb{X}(q) = x_q$.

We now consider the case when $q = q_c$; since $\mathbb{X}(q_c) = 1$, we need to prove that $F(q_c, 1) = \sum_n z_n(q_c) = 1$. Part (ii) insures that $F(q, 1) > 1$ for any $q_c < q$, which implies that $F(q_c, 1) \ge 1$, by the fact that F is continuous at $(q_c, 1)$: since (ii) also implies that $F(q_c, 1) \le 1$, one concludes that $F(q_c, 1) = 1$, completing the proof of (iii) and of the proposition as well. □

Proof of Theorem A.1. We shall prove that $\mathbb{X}$ is not differentiable at q_c; since by Proposition A.3, one has $\mathbb{X}(q) = 1$ whenever $q \le q_c$, this will be established if one shows that $\mathbb{X}'(q_c^+) < 0$. According to part (iii) of Proposition A.3, for any $q_c \le q < -2.25$, one has the equation

$$\sum_{n=0}^{\infty} z_n(q_c)\mathbb{X}(q_c)^{2n+3} = \sum_{n=0}^{\infty} z_n(q)\mathbb{X}(q)^{2n+3},$$

or equivalently, using the fact that $\mathbb{X}(q_c) = 1$,

$$\begin{aligned}\sum_{n=0}^{\infty} (z_n(q_c) - z_n(q)) &= \sum_{n=0}^{\infty} z_n(q)(\mathbb{X}(q)^{2n+3} - 1)\\ &= (\mathbb{X}(q) - \mathbb{X}(q_c)) \sum_{n=0}^{\infty} z_n(q) \sum_{i=0}^{2n+2} \mathbb{X}(q)^i.\end{aligned}$$

Finally one can write

$$\frac{\mathbb{X}(q) - \mathbb{X}(q_c)}{q - q_c} = -\frac{\sum_{n=0}^{\infty} (z_n(q) - z_n(q_c))/(q - q_c)}{\sum_{n=0}^{\infty} z_n(q) \sum_{i=0}^{2n+2} \mathbb{X}(q)^i}. \tag{54}$$

If q tends to q_c with $q_c < q < 2.25$, then $\sum_{i=0}^{2n+2} \mathbb{X}(q)^i$ tends to $2n + 3$ and since by Proposition A.2 one has $\sum_{n=0}^{\infty} z_n(q_c)\,(2n + 3) < \infty$, one concludes from (54) that

$$\mathbb{X}'(q_c^+) = -\frac{\sum_{n=0}^{\infty} \sum_{w \in \{0,2\}^n} ({}^tVP_w V)^{q_c} \log({}^tVP_w V)}{\sum_{n=0}^{\infty} z_n(q_c)(2n + 3)} < 0. \qquad □$$

Proof of Proposition A.2. Given $(a_n)_{n=1}^{\infty}$ a sequence of positive integers, it is easily checked by induction that, for any $k \ge 1$,

$${}^tVQ_{a_1} \cdots Q_{a_k} V \le (1 + a_1) \cdots (1 + a_{k-1})(2 + a_k); \tag{55}$$

likewise, if $A(a_1, \ldots, a_{2k}) := (1 + a_1a_2)(1 + a_3a_4) \cdots (1 + a_{2k-1}a_{2k})$ then, for $\varepsilon = 0$ or 1,

$${}^tVQ_{a_1} \cdots Q_{a_{2k+\varepsilon}} V \ge \begin{cases} A(a_1, \ldots, a_{2k}) & \text{if } \varepsilon = 0,\\ A(a_1, \ldots, a_{2k})(1 + a_{2k+1}) & \text{if } \varepsilon = 1.\end{cases} \tag{56}$$

In what follows, ζ will stand for the Riemann zeta function (i.e. $\zeta(x) = \sum_{n=1}^{\infty} 1/n^x$, $x > 1$).

(i) When $q < -1$ one has $\zeta(-q) - 1 < 1$ and the following inequalities hold:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{z}_n(q) &= 2^q + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_k \geq 1} ({}^tVQ_{a_1} \cdots Q_{a_k} V)^q \\ &\geq 2^q + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_k \geq 1} (1+a_1)^q \cdots (1+a_{k-1})^q (2+a_k)^q \\ &= 2^q + 2\bigg(\sum_{n=1}^{\infty}(n+2)^q\bigg) \sum_{k=0}^{\infty} \bigg(\sum_{n=1}^{\infty}(n+1)^q\bigg)^k = 2^q + \frac{2(\zeta(-q) - 1 - 2^q)}{2 - \zeta(-q)}.\end{aligned}$$

Accordingly, one can use numerical computations to get the lower bound $\sum_n \mathrm{z}_n$ $(-2.25) > 1$, proving that $q_c < -2.25$.

(ii) Let $q < 0$; from (56) and the fact that $(1 + a_{2k+1})^q \geq a_{2k+1}^q$, one gets

$$\begin{aligned}\sum_{n=0}^{\infty} n\mathrm{z}_n(q) &\leq 2\sum_{n=0}^{\infty} n(2+n)^q + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_{2k} \geq 1} (a_1 + \cdots + a_{2k}) A(a_1, \ldots, a_{2k})^q \\ &\quad + 2\sum_{k=0}^{\infty} \sum_{a_1,\ldots,a_{2k+1} \geq 1} (a_1 + \cdots + a_{2k+1}) A(a_1, \ldots, a_{2k})^q a_{2k+1}^q \\ &= 2\sum_{n=0}^{\infty} n(2+n)^q + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_{2k} \geq 1} 2ka_1 A(a_1, \ldots, a_{2k})^q \\ &\quad + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_{2k+1} \geq 1} 2ka_1 A(a_1, \ldots, a_{2k})^q a_{2k+1}^q \\ &\quad + 2\sum_{k=1}^{\infty} \sum_{a_1,\ldots,a_{2k+1} \geq 1} a_{2k+1} A(a_1, \ldots, a_{2k})^q a_{2k+1}^q \\ &= 2\sum_{n=0}^{\infty} n(2+n)^q + 4\bigg(\sum_{n,m \geq 1} n(1+nm)^q\bigg) \sum_{k=1}^{\infty} k \bigg(\sum_{n,m \geq 1} (1+nm)^q\bigg)^{k-1} \\ &\quad + 4\bigg(\sum_{n,m \geq 1} n(1+nm)^q\bigg)\bigg(\sum_{n=1}^{\infty} n^q\bigg) \sum_{k=1}^{\infty} k \bigg(\sum_{n,m \geq 1} (1+nm)^q\bigg)^{k-1} \\ &\quad + 2\bigg(\sum_{n=1}^{\infty} n^{q+1}\bigg) \sum_{k=1}^{\infty} \bigg(\sum_{n,m \geq 1} (1+nm)^q\bigg)^k.\end{aligned}$$

On the one hand, for any $q < -2$ one has $\sum_{n=0}^{\infty} n(2+n)^q < \infty$ and

$$\sum_{n,m \geq 1} n(1+nm)^q \leq 2\sum_{n=1}^{\infty} n(1+n)^q + \bigg(\sum_{n=2}^{\infty} nn^q\bigg)\bigg(\sum_{n=2}^{\infty} n^q\bigg) < \infty. \tag{57}$$

Thus, there exist three positive constants C, C' and C'' such that

$$\sum_{n=0}^{\infty} n\mathrm{z}_n(q) \leq C + C' \sum_{k=1}^{\infty} k \bigg(\sum_{n,m \geq 1} (1+nm)^q\bigg)^{k-1} + C'' \sum_{k=1}^{\infty} \bigg(\sum_{n,m \geq 1} (1+nm)^q\bigg)^k. \tag{58}$$

On the other hand, one has for any $q < -1$,

$$\begin{aligned}\sum_{n,m\geq 1}(1+nm)^q &= \sum_{n,m>1}(1+nm)^q + 2\sum_{n=1}^{\infty}(1+n)^q - 2^q\\ &\leq \sum_{n,m>1}^{\infty}(nm)^q + 2\sum_{n=2}^{\infty}n^q - 2^q\\ &= \left(\sum_{n=2}^{\infty}n^q\right)^2 + 2\sum_{n=2}^{\infty}n^q - 2^q = (\zeta(-q)-1)^2 + 2(\zeta(-q)-1) - 2^q.\end{aligned}$$

For $\theta_0 > 0$ such that $\theta_0^2 + 2\theta_0 - 1/8 = 1$, one can check that $\zeta(2.25) - 1 < \theta_0$, so that

$$\sum_{nm\geq 1}(1+nm)^{-2.25} < \theta_0^2 + 2\theta_0 - 1/8 = 1. \tag{59}$$

Therefore, by (58) and (59), one concludes that $\sum_n n\,\mathrm{z}_n(q) < \infty$ when $q < -2.25$. □

B. *Appendix. Box and packing dimensions*

Let M be a subset of $\mathbf{S}^1$. An *ε-packing* of M $(0 < \varepsilon < 1)$ is a collection $\{J_i\}_i$ of mutually disjoint intervals $J_i \in \mathcal{F}$ with $|J_i| \leq \varepsilon$ and which intersect M; for any $0 \leq \rho \leq 1$ we set

$$\mathcal{P}_\rho(M) := \lim_{\varepsilon\to 0}\sup\left\{\sum_i |J_i|^\rho;\ \{J_i\}_i\right\},$$

where $\{J_i\}_i$ runs over the ε-packing of M ($\mathcal{P}_\rho$ is not sigma-additive). The *box dimension* of M is by definition

$$\dim_B M := \inf\{\rho;\ \mathcal{P}_\rho(M) = 0\} = \sup\{\rho;\ \mathcal{P}_\rho(M) = \infty\};$$

in general, $\dim_B \bigcup_n M_n \neq \sup_n \dim_B M_n$, whenever $M_1, M_2, \ldots$ form a countable family of subsets of $\mathbf{S}^1$. There are many other equivalent definitions of the box dimension of M, one of them using a regular net of $\mathbf{S}^1$: suppose that $\mathcal{F} := \bigcup_n \mathcal{F}_n$ is a net of $\mathbf{S}^1$, then, for any $0 \leq \rho \leq 1$, we set

$$\mathcal{P}_\rho(M|\mathcal{F}) := \lim_{\varepsilon\to 0}\sup\left\{\sum_i |J_i|^\rho;\ \{J_i\}_i\right\},$$

where $\{J_i\}_i$ runs over the ε-packing of M by intervals in $\mathcal{F}$. Recall that $\mathcal{F}$ is said to be regular if there exists a constant $c > 1$ such that for any n one has $|J|/|J'| \leq c$, for any $J \in \mathcal{F}_n$, $J' \in \mathcal{F}_{n+1}$ and $J' \subset J$; if $\mathcal{F}$ is regular then

$$\dim_B M := \inf\{\rho;\ \mathcal{P}_\rho(M|\mathcal{F}) = 0\} = \sup\{\rho;\ \mathcal{P}_\rho(M|\mathcal{F}) = \infty\}$$

and one recovers the usual definition of the box dimension by using the dyadic net for example. The notion of *packing dimension* has been introduced by Tricot in [**44**]; the packing dimension of M is

$$\dim_P M := \inf\left\{\sup_n \dim_B M_n;\ M \subset \bigcup_n M_n\right\},$$

with the important property that $\dim_P \bigcup_n M_n = \sup_n \dim_P M_n$; moreover the packing dimension is a useful notion to get an upper bound of the Hausdorff dimension since

$$\dim_H M \leq \dim_P M \leq \dim_B M.$$

A systematic approach and detailed proof about fractal dimensions can be found in [**11, 28**]; we also refer to [**35**] for a point of view related to dynamical systems.

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